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UNIFORMLY EXPONENTIAL DICHOTOMY FOR STRONGLY CONTINUOUS QUASI GROUPS

SUTRIMA SUTRIMA*, MARDIYANA MARDIYANA AND RIRIN SETIYOWATI

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Abstract. A strongly continuous quasi group (C_0-quasi group) is established as an extension of a C_0-quasi semigroup on a Banach space. The fundamental properties of the C_0-quasi groups are derived from the properties of C_0-quasi semigroups. It is identified a sufficient condition for an infinitesimal generator of a C_0-quasi group. The infinitesimal generator of a C_0-quasi group generates a non-autonomous the abstract Cauchy problem that is well-posed. Uniformly exponential stability of the C_0-quasi groups and the C_0-quasi semigroups on a Banach space X can be identified by the associated evolution semigroups on the spaces $L_p(\mathbb{R},X)$ and $L_p(\mathbb{R}^+,X)$, $1 \leqslant p < \infty$, respectively. The sufficient and necessary conditions, called Dichotomy Theorem, for the uniformly exponential dichotomy of the C_0-quasi groups and the C_0-quasi semigroups are characterized by the associated evolution semigroups. The hyperbolicity of the evolution semigroups is used in the characterization. Dichotomy Theorem can also be identified by a Green's function induced by the associated evolution semigroup. Moreover, the infinitesimal generator of the associated evolution semigroup becomes the main subject in establishment of the sufficiency and necessity for the uniformly exponential stability of the C_0-quasi semigroups.

1. Introduction

Let X be a complex Banach space and we consider a non-autonomous abstract Cauchy problem

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \quad t \in \mathbb{R} - \{0\}, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $A(t)$ is a linear closed operator in X with domain $\mathscr{D}(A(t)) = \mathscr{D}$ is independent of t and dense in X. The solution of problem (1) can be given by means of an evolution family, a two-parameter family $\{U(t,s)\}_{t\geqslant s}$ of bounded operators on X [8, 9] including the special case of t nonnegative [32, 33]. For the autonomous case, the solution of problem (1) can be represented by a strongly continuous group (C_0-group) and by a strongly continuous semigroup (C_0-semigroup) when $t \geqslant 0$ [27]. Characterizations of the C_0-groups can be found in [10, 15, 18].

The stabilities of the evolution family corresponding to problem (1) affect directly to the solution $x(t)$. The theory of stabilities of the evolution family developed from

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* Corresponding author.

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the Theorem Datko-Pazy for an exponential stability of the C_0-semigroups introduced by Datko [6] and Pazy [26]. The uniformly exponential stability of the evolution family has wide applications in the evolution equations [2, 3]. However, the hyperbolicity is more general than the uniformly exponential stability for the semigroups. Recall that a semigroup on X is *hyperbolic* if X can be decomposed as a direct sum of two subspaces (stable and unstable) such that the semigroup is uniformly exponentially stable for positive time on the stable subspace and uniformly exponentially stable for negative time on the unstable subspace [4]. The hyperbolicity of the associated evolution family (associated semigroup for autonomous case) is called *uniformly exponential dichotomy*. There are many results of the uniformly exponential dichotomy of the evolution families and the C_0-semigroups [17, 20, 28, 29, 30]. In particular, the uniformly exponential dichotomy of the evolution family can be characterized by Green's function approach [1, 14, 21].

In applications, it frequently forces that the evolution families have to be reduced to the one parameter semigroup on the space of X-valued functions, called evolution semigroup [4, 8, 9]. The evolution semigroups are applicable to solve problems (1) for $t \geqslant 0$ [22, 23, 24, 25]. Moreover, the uniformly exponential dichotomy of the evolution family can be characterized by the spectra or the hyperbolicity of the associated evolution semigroup on the appropriate space [12, 13, 31, 34]. In particular, there exists a relationship between the Green's operator for the evolution family and the infinitesimal generator of the associated evolution semigroup [4].

It was well-known that strongly continuous quasi semigroup (C_0-quasi semigroup) on a Banach space, as an extension of C_0-semigroup, was an alternatively sophisticate tool to solve the non-autonomous equations. Since it was introduced by Leiva and Barcenas [16] in 1991, the progress of the quasi semigroups is very massive. The properties, spectra and stabilities of the C_0-quasi semigroups have been investigated [5, 19, 35, 36, 39, 40]. Even, the C_0-quasi semigroups are applicable to analyze the controllability, observability, stability, and stabilizability of the non-autonomous linear control systems [37, 38]. The fact that the C_0-semigroups can be generalized to the C_0-quasi semigroups, suggests a generalization of C_0-groups to strongly continuous quasi groups (C_0-quasi groups). Furthermore, the uniformly exponential dichotomy of the C_0-quasi groups is also important to be investigated.

In this paper we are concern on establishment C_0-quasi group as an extension of the C_0-quasi semigroup and characterizations to the uniformly exponential dichotomy. The characterizations are based on the associated evolution semigroups and the appropriate Green's function. The organization of this paper is as follows. In Section 2, the establishment of a C_0-quasi group on a Banach space and its consequence to problem (1) are considered. Investigations of the evolution semigroups corresponding to the C_0-quasi group and the exponential dichotomic property are considered in Section 3. The evolution semigroups on the half line corresponding to the C_0-quasi semigroups are studied in Section 4. Section 5 identifies the dichotomy using the Green's function.

2. Strongly continuous quasi groups

The fact in solving the non-autonomous abstract Cauchy problem (1), the evolution family is reduced to be an evolution semigroup on the space of X-valued functions [4, 8, 9]. Is there a straightforward method to solve the problem? However, the solution of problem (1) with $t \geqslant 0$ can be given by means a C_0-quasi semigroup. This encourages to extend the C_0-quasi semigroup to a C_0-quasi group. If this is realized, then the existence of the solution of problem (1) can be identified by the family $\{A(t)\}$. Further, the controllability of the non-autonomous linear control problems (1) can be analyzed. The latter can not be characterize by the evolution families yet.

DEFINITION 1. Let $\mathscr{L}(X)$ be the set of all bounded linear operators on a Banach space X. A two-parameter commutative family $\{R(t,s)\}_{s,t\in\mathbb{R}}$ in $\mathscr{L}(X)$ is called a strongly continuous quasi group (C_0-quasi group) on X if for each $r,s,t \in \mathbb{R}$ and $x \in X$:

(a) $R(t,0) = I$, the identity operator on X,

(b) $R(t,s+r) = R(t+r,s)R(t,r)$,

(c) $\lim_{s\to 0} \|R(t,s)x - x\| = 0$,

(d) there exists a continuous increasing function $M : \mathbb{R} \to [1,\infty)$ such that

$$\|R(t,s)\| \leqslant M(t+s).$$

Let $\mathscr{D}$ be the set of all $x \in X$ such that the following limits exist

$$\lim_{s\to 0} \frac{R(t,s)x - x}{s}, \quad s,t \in \mathbb{R}.$$

For $t \in \mathbb{R}$ we define an operator $A(t)$ on $\mathscr{D}$ as

$$A(t)x = \lim_{s\to 0} \frac{R(t,s)x - x}{s}.$$

The family of operators $\{A(t)\}_{t\in\mathbb{R}}$ is called an infinitesimal generator of the C_0-quasi group $\{R(t,s)\}_{s,t\in\mathbb{R}}$. In sequel for simplicity, we denote the quasi group $\{R(t,s)\}_{s,t\in\mathbb{R}}$ and the family $\{A(t)\}_{t\in\mathbb{R}}$ by $R(t,s)$ and $A(t)$, respectively.

REMARK 1. (a) Condition (c) of Definition 1 implies that for each $t \in \mathbb{R}$, $R(t,\cdot)$ is strongly continuous in X. Similarly, for each $s \in \mathbb{R}$, it can be shown that $R(\cdot,s)$ is also strongly continuous in X.

(b) Definition 1 gives the definition of the C_0-quasi semigroup if the parameters $r,s,t \in \mathbb{R}$ are replaced by $r,s,t \geqslant 0$, respectively, [35].

EXAMPLE 1. (a) Let $T(t)$ be a C_0-group on a Banach space X with the infinitesimal generator A. The family of operators $R(t,s)$ defined by

$$R(t,s)x = T(s)x, \quad t,s \in \mathbb{R}, \quad x \in X,$$

is a C_0-quasi group on X with the infinitesimal generator $A(t) = A$ on $\mathscr{D} = \mathscr{D}(A)$.

(b) Let X be the space of all bounded continuous real functions on $\mathbb{R}$ with the supremum norm. Define $R(t,s)$ on X by

$$(R(t,s)x)(\xi) = x(\xi + s^2 + 2st), \quad \xi, t, s \in \mathbb{R}, \quad x \in X.$$

The family $R(t,s)$ is a C_0-quasi group on X with the infinitesimal generator $A(t)$, where $A(t)x(\xi) = 2t\frac{dx}{d\xi}$ on domain $\mathscr{D} = \{x \in X : \frac{dx}{d\xi} \in X\}$.

(c) Let $T(t)$ be a C_0-group on a Banach space X generated by A. The family $R(t,s)$ defined by

$$R(t,s)x = e^{T(t+s)-T(t)}x, \quad s,t \in \mathbb{R}, \quad x \in X,$$

is a C_0-quasi group on X with the generator $A(t) = AT(t)$ on the domain $\mathscr{D} = \mathscr{D}(A)$.

The fundamental properties of the C_0-quasi groups follow the properties of the C_0-quasi semigroups.

THEOREM 1. *Let $R(t,s)$ be a C_0-quasi group on a Banach space X with the infinitesimal generator $A(t)$. The following statements hold.*

(a) If $x \in \mathscr{D}$, then $R(t,s)x \in \mathscr{D}$ and

$$R(t,s)A(t)x = A(t)R(t,s)x, \quad t,s \in \mathbb{R}.$$

(b) For each $x \in \mathscr{D}$ and $r \in \mathbb{R}$,

$$\frac{\partial}{\partial t}(R(r,t)x) = A(t+r)R(r,t)x = R(r,t)A(t+r)x.$$

(c) If $A(\cdot)$ is locally integrable, then for each $x \in \mathscr{D}$ and $r \in \mathbb{R}$,

$$R(r,t)x = x + \int_0^t A(t+s)R(r,s)x ds.$$

(d) If $f : \mathbb{R} \to X$ is a continuous function, then for each $t \in \mathbb{R}$

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} R(r,s)f(s)ds = R(r,t)f(t).$$

Proof. The proofs are similar to the proofs of Theorem 3.2 of [35]. □

In general, the domain $\mathscr{D}$ of the infinitesimal generator is not dense in X, see Example 3.3 of [35]. In the rest, we assume that the C_0-quasi groups are the quasi groups with the infinitesimal generator on the dense domain in X. By this assumption, each the C_0-quasi group has a unique infinitesimal generator.

LEMMA 1. *Let $R_1(t,s)$ and $R_2(t,s)$ be the C_0-quasi group on a Banach space X with the infinitesimal generator $A_1(t)$ and $A_2(t)$, respectively. If $A_1(t) = A_2(t)$ for all $t \in \mathbb{R}$, then $R_1(t,s) = R_2(t,s)$ for all $t,s \in \mathbb{R}$.*

Proof. Let $x_0 \in \mathscr{D}(A_1(t)) = \mathscr{D}(A_2(t))$. For $t,s \in \mathbb{R}$ fixed, without loss of the generality we assume $s > 0$. Define a continuous function $f(r) = R_1(t+r,s-r)R_2(t,r)x_0$ for $r \in [0,s]$. The part (a) of Theorem 1 gives $R_2(t,r)x_0 \in \mathscr{D}(A_1(t))$. Differentiating f with respect to r, part (b) of Theorem 3.2 gives

$$\dot{f}(r) = -R_1(t+r,s-r)A_1(t+s)R_2(t,r)x_0 + R_1(t+r,s-r)A_2(t+s)R_2(t,r)x_0 = 0.$$

Since $A_1(t) = A_2(t)$ for all $t \in \mathbb{R}$, f is a constant function on $[0,s]$. Therefore,

$$R_1(t,s)x_0 = f(0) = f(s) = R_2(t,s)x_0.$$

Thus, the both quasi groups are identic on the dense set $\mathscr{D}(A_1(t))$. The boundedness of the quasi group operators forces that the both quasi groups are identic for all $x_0 \in X$. $\square$

LEMMA 2. *Let $R(t,s)$ be a C_0-quasi group on a Banach space X. The following statements hold.*

(a) If $R^+(t,s) := R(t,s)$ for $t,s \geqslant 0$ and $R^-(t,s) := R(-t,-s)$ for $t,s \geqslant 0$, then $R^+(t,s)$ and $R^-(t,s)$ are C_0-quasi semigroups on X.

(b) The $A(t)$ is the infinitesimal generator of the C_0-quasi group $R(t,s)$ if and only if $A(t)$ is the infinitesimal generator of $R^+(t,s)$ and $-A(-t)$ is the infinitesimal generator of $R^-(t,s)$. In this case,

$$R(t,s) = \begin{cases} R^+(t,s), & t,s \geqslant 0 \\ R^-(t,s), & t,s \leqslant 0. \end{cases}$$

Proof. (a) Definition 1 implies the assertions.

(b) $(\Rightarrow)$. Let $A(t)$ be the infinitesimal generator of the quasi group $R(t,s)$,

$$A(t)x = \lim_{s\to 0} \frac{R(t,s)x - x}{s}, \quad t,s \in \mathbb{R}, \quad x \in \mathscr{D}.$$

For $t \geqslant 0$ and $s > 0$, the limit above gives

$$\lim_{s\to 0^+} \frac{R^+(t,s)x - x}{s} = \lim_{s\to 0^+} \frac{R(t,s)x - x}{s} = A(t)x,$$
$$\lim_{s\to 0^+} \frac{R^-(t,s)x - x}{s} = \lim_{s\to 0^+} \frac{R(-t,-s)x - x}{s} = -\lim_{s\to 0^-} \frac{R(-t,s)x - x}{s} = -A(-t)x.$$

$(\Leftarrow)$. Conversely, let $A(t)$ and $-A(-t)$ be the infinitesimal generators of $R^+(t,s)$ and $R^-(t,s)$ with $t,s \geqslant 0$, respectively. These give

$$\lim_{s\to 0^+}\frac{R(t,s)x-x}{s} = \lim_{s\to 0^+}\frac{R^+(t,s)x-x}{s} = A(t), \quad t,s\geqslant 0,$$
$$\lim_{s\to 0^-}\frac{R(t,s)x-x}{s} = -\lim_{s\to 0^-}\frac{R^-(-t,-s)x-x}{-s} = -[-A(t)] = A(t), \quad t,s\leqslant 0.$$

These conclude that $A(t)$ is the infinitesimal generator of the quasi group $R(t,s)$. $\square$

Similar to the C_0-quasi semigroup, the following result gives a sufficient condition in order to problem (1) is well-posed.

THEOREM 2. *If $A(t)$ be the infinitesimal generator of a C_0-quasi group $R(t,s)$ on a Banach space X, then for each $x_0 \in \mathscr{D}$, problem (1) admits a unique solution.*

Proof. Part (a) of Theorem 1 implies that $x(t) = R(0,t)x_0$ is a solution of problem (1). Let $y(t)$ be another solution. Without loss of the generality, we assume $t > 0$. We consider a function $f(s) = R(s,t-s)y(s)$ for all $s \in [0,t]$. The simple calculation shows that $\dot{f}(s) = 0$ for all $s \in (0,t)$ i.e f is a constant function. This gives $y(t) = f(t) = f(0) = R(0,t)y(0) = R(0,t)x_0 = x(t)$. The assertion follows. $\square$

Theorem 2 suggests a sufficient condition for $A(t)$ in order to be an infinitesimal generator of a C_0-quasi group. The following theorem is Hille-Yosida's version for the C_0-quasi groups.

THEOREM 3. *For each $t \in \mathbb{R}$, let $A(t)$ be a closed and densely defined operator on $\mathscr{D}$ and the map $t \mapsto A(t)y$ is a continuous function from $\mathbb{R}$ to X for all $y \in \mathscr{D}$. If $\mathscr{R}(\lambda, A(\cdot))$ is locally integrable and there exist constants $N,\omega \geqslant 0$ such that $[\omega,\infty) \subseteq \rho(A(t))$ and*

$$\|\mathscr{R}(\lambda,A(t))^r\| \leqslant \frac{N}{(|\lambda|-\omega)^r},$$

for all $\lambda \in \mathbb{R}$ with $|\lambda| > \omega$ and $r \in \mathbb{N}$, then $A(t)$ is the infinitesimal generator of C_0-quasi group $R(t,s)$.

Proof. By hypothesis, if necessary the proof is adjusted first, Theorem 1 of [39] implies that $A(t)$ and $-A(t)$ for $t \geqslant 0$ are the infinitesimal generator of a C_0-quasi semigroup $R^+(t,s)$ and $R^-(t,s)$, respectively. We see that

$$R^+(t,s)x = \lim_{n\to\infty} e^{G_n(t+s)-G_n(t)}x \quad \text{and} \quad R^-(t,s)x = \lim_{n\to\infty} e^{-G_n(t+s)+G_n(t)}x,$$

where $G_n(t) := \int_0^t A_n(\tau)d\tau$ and $A_n(t)$ be Yosida approximation of $A(t)$,

$$A_n(t) := nA(t)\mathscr{R}(n,A(t)) = n^2\mathscr{R}(n,A(t)) - nI, \quad n > \omega, \quad n \in \mathbb{N}.$$

If $K(t,s) = R^+(t,s)R^-(t,s)$, then $K(t,s)$ is a C_0-quasi semigroup of bounded operators for $t,s \geqslant 0$. For $x \in \mathscr{D} = \mathscr{D}(-A(t))$, we have

$$\frac{K(t,s)x - x}{s} = R^-(t,s)\frac{R^+(t,s)x - x}{s} + \frac{R^-(t,s)x - x}{x}$$
$$\to A(t)x - A(t)x = 0 \quad \text{as} \quad s \to 0.$$

Thus, for $x \in \mathscr{D}$, we have $K(t,s)x = x$. Since $\mathscr{D}$ is dense in X and $K(t,s)$ is bounded, $K(t,s) = I$ or $R^-(t,s) = [R^+(t,s)]^{-1}$ and

$$R(t,s) = \begin{cases} R^+(t,s), & \text{for } t,s \geqslant 0 \\ R^-(-t,-s), & \text{for } t,s \leqslant 0. \end{cases}$$

gives a C_0-quasi group that is desired. □

The following example illustrates that the C_0-quasi group is able to solve the non-autonomous problem.

EXAMPLE 2. Consider the Schrodinger equation

$$\begin{aligned} &x_t(\xi,t) = ix_{\xi\xi}(\xi,t) - V(\xi,t)x(\xi,t), \quad 0 < \xi < 1, \quad t \in \mathbb{R} - \{0\}, \\ &x(0,t) = x(1,t) = 0, \quad t \neq 0 \\ &x(\xi,0) = x_0(\xi). \end{aligned} \tag{2}$$

Set $X = L_2(0,1)$ with the usual complex inner product. Let

$$H_0^1(0,1) = \{h \in H^1(0,1) : h(0) = 0 = h(1)\},$$

where $H^1(0,1)$ denotes the Sobolev space of order 1 on $[0,1]$. Define

$$A_0 h = -h'' \quad \text{on} \quad \mathscr{D}(A_0) = \{h \in H^2(0,1) : h(0) = h(1) = 0\}.$$

Since A_0 is a self-adjoint operator, Theorem 1.10.8 of [27] implies that iA_0 is the infinitesimal generator of the C_0-group $T(t) = e^{iA_0 t}$, $t \in \mathbb{R}$, on X.

Setting $x(t) = x(\cdot,t)$, $V(t) = V(\cdot,t)$, and $A(t) = iA_0 - V(t)$, we can rewrite the problem (2) as

$$\begin{aligned} &\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R} - \{0\}, \\ &x(0) = x_0. \end{aligned} \tag{3}$$

We choose $V(\xi,t) = \alpha(t)Z(\xi)$ with $Z(\xi) = -\xi^2$ is the repulsive oscillator potential on $[0,1]$ and

$$\alpha(t) = \begin{cases} \frac{1}{t+1}, & \text{if } t \geqslant 0 \\ 1, & \text{if } t < 0. \end{cases}$$

For $t \geqslant 0$, Theorem 3 implies that $-V(t)$ is an infinitesimal generator of the C_0-quasi semigroup $R_1(t,s)$ on X given by

$$R_1(t,s)x(\xi) = \left(\frac{t+s+1}{t+1}\right)^{\xi^2} x(\xi), \quad x \in X, \quad 0 \leqslant \xi \leqslant 1.$$

For $t \leqslant 0$, we see that $-V(t)$ generates the C_0-quasi semigroup $R_2(t,s)$ on X given by

$$R_2(t,s)x(\xi) = e^{s\xi^2}x(\xi), \quad x \in X, \quad 0 \leqslant \xi \leqslant 1.$$

Therefore, $-V(t)$ generates the C_0-quasi group $R(t,s)$ given

$$R(t,s) = \begin{cases} R_1(t,s), & \text{if } t,s \geqslant 0 \\ R_2(t,s), & \text{if } t,s \leqslant 0. \end{cases}$$

We see that $K(t,s) = T(s)R(t,s)$ is a C_0-quasi group on X generated by $A(t)$. Thus, for $x_0 \in \mathscr{D}(A_0)$, problem (3) has a unique solution $x(t) = K(0,t)x_0$. The solution of problem (2) follows.

3. Evolution semigroups and Dichotomy Theorem

In some cases, we are forced to reduce the quasi-semigroups (quasi-groups) to be an evolution semigroups. An example, the behavior of the uniformly exponential stability of a quasi-group is difficult identified directly but the behavior can be described easily by the spectrum of the infinitesimal generator of the corresponding evolution semigroup. The evolution semigroup is based on the classical idea of defining ”time” to be a new variable in order to make the non-autonomous Cauchy problem (1) to be the autonomous one [8, 9, 24]. The evolution semigroup is a multiplicative perturbation of the semigroup of translations in which the spectrum of its infinitesimal generator has been identified.

Let X be a Banach space and let $L_p(\mathbb{R},X)$, $1 \leqslant p < \infty$, be a space of all functions $f:\mathbb{R}\to X$ such that $\int_{-\infty}^{\infty}\|f(t)\|_X^p dt < \infty$ with the norm $\|f\|_{L_p(\mathbb{R},X)} = \left(\int_{-\infty}^{\infty}\|f(t)\|_X^p dt\right)^{\frac{1}{p}}$. Throughout this paper we always assume that $L_p(\mathbb{R},X)$ and $L_p(\mathbb{R}^+,X)$ are the spaces with $1 \leqslant p < \infty$. The following we define the evolution semigroup corresponding to the C_0-quasi group $R(t,s)$ on $L_p(\mathbb{R},X)$, a generalization of the same term for C_0-quasi semigroup [37].

We use the uniformly exponential stability in identifying the dichotomic behaviour of the C_0-quasi groups. This term is a generalization of the similar term for C_0-quasi semigroups [38].

DEFINITION 2. A C_0-quasi group $R(t,s)$ is said to be uniformly exponentially stable on a Banach space X if there exist constants $\alpha > 0$ and $N \geqslant 1$ such that

$$\|R(t,s)x\| \leqslant Ne^{-\alpha|s|}\|x\|, \quad t,s \in \mathbb{R}, \quad x \in X. \tag{4}$$

Definition 2 gives the definition of the uniformly exponential stability for a C_0-quasi semigroup if $R(t,s)$ is a C_0-quasi semigroup. In this case, the exponential bound in the right hand of (4) becomes $Ne^{-\alpha s}$ for all $t,s \geqslant 0$.

DEFINITION 3. Let $R(t,s)$ be a C_0-quasi group on a Banach space X. Evolution semigroup associated to $R(t,s)$ on $L_p(\mathbb{R},X)$ is a family of operators $\{E^s\}_{s\geqslant 0}$ given by

$$(E^s f)(t) = R(t-s,s)f(t-s), \quad s \geqslant 0, \quad t \in \mathbb{R}, \quad f \in L_p(\mathbb{R},X). \tag{5}$$

In sequel for simplicity, the evolution semigroup $\{E^s\}_{s\geqslant 0}$ is denoted by E^s. We see that E^s is strongly continuous on $L_p(\mathbb{R},X)$. Moreover, if $A(t)$ is the infinitesimal generator of the C_0-quasi group $R(t,s)$ with domain $\mathscr{D}$, then an operator Γ defined by

$$(\Gamma f)(t) = -\frac{df}{dt} + A(t)f(t), \quad t\in\mathbb{R}, \tag{6}$$

is the infinitesimal generator of E^s with domain

$$\mathscr{D}(\Gamma) = \{f\in L_p(\mathbb{R},X) : f \text{ is absolutely continuous, } f(t)\in\mathscr{D}\}.$$

LEMMA 3. *The spectrum $\sigma(\Gamma)$ is invariant under translations along imaginary axis and the spectrum $\sigma(E^s)$ is invariant under rotations centered at the origin.*

Proof. For each $\xi\in\mathbb{R}$ we define an invertible operator on $L_p(\mathbb{R},X)$ by $(L_\xi f)(t) = e^{i\xi t}f(t)$. For $f\in L_p(\mathbb{R},X)$ we have

$$\begin{aligned}(\Gamma L_\xi f)(t) &= -\frac{d(L_\xi f)}{dt} + A(t)(L_\xi f)(t)\\ &= (-i\xi L_\xi f)(t) + e^{i\xi t}\left[-\frac{df}{dt} + A(t)\right](t)\\ &= [(-i\xi L_\xi + L_\xi\Gamma)f](t).\end{aligned}$$

This implies that

$$\Gamma L_\xi = -i\xi L_\xi + L_\xi\Gamma \quad \text{or} \quad L_\xi^{-1}\Gamma L_\xi = -i\xi + \Gamma.$$

This gives that $\sigma(\Gamma)$ is invariant under translations along imaginary axis.

Again, for $f\in L_p(\mathbb{R},X)$ we obtain

$$\begin{aligned}(E^sL_\xi f)(t) &= R(t-s,s)L_\xi f(t)\\ &= e^{-i\xi s}e^{i\xi t}R(t-s,s)f(t-s) = (e^{-i\xi s}L_\xi E^s f)(t).\end{aligned}$$

Therefore, $E^sL_\xi = e^{-i\xi s}L_\xi E^s$ or $L_\xi^{-1}E^sL_\xi = e^{-i\xi s}E^s$. Thus, the spectrum $\sigma(E^s)$ is invariant under rotations centered at the origin. □

Next, we define the concept of uniformly exponential dichotomy for a C_0-quasi group which is an extension of the C_0-quasi semigroup introduced by Cuc [5]. Let $P:\mathbb{R}\to\mathscr{L}(X)$ be a projection-valued function, the complementary projection is given by $Q(t) = I - P(t)$ for all $t\in\mathbb{R}$. If $P(t+s)R(t,s) = R(t,s)P(t)$, then

$$R_P(t,s) := P(t+s)R(t,s)P(t) \quad \text{and} \quad R_Q(t,s) := Q(t+s)R(t,s)Q(t),$$

are the restrictions of $R(t,s)$ on $\operatorname{ran}P(t)$ and $\operatorname{ran}Q(t)$, respectively. The $R_P(t,s)$ is an operator from $\operatorname{ran}P(t)$ to $\operatorname{ran}P(t+s)$ while $R_Q(t,s)$ maps $\operatorname{ran}Q(t)$ to $\operatorname{ran}Q(t+s)$.

DEFINITION 4. A C_0-quasi group $R(t,s)$ is said to be uniformly exponentially dichotomic on X if there exist constants $N>1$, $\alpha>0$ and a projection-valued function $P:\mathbb{R}\to\mathscr{L}(X)$ such that for each $x\in X$, the function $x\mapsto P(t)x$ is continuous and bounded, and, for all $t,s\in\mathbb{R}$, the following conditions hold:

(a) $P(t+s)R(t,s) = R(t,s)P(t)$,

(b) $R_Q(t,s)$ is invertible as an operator from $\operatorname{ran} P(t)$ to $\operatorname{ran} P(t+s)$,

(c) $\|R_P(t,s)\| \leqslant Ne^{-\alpha|s|}$,

(d) $\|[R_Q(t,s)]^{-1}\| \leqslant Ne^{-\alpha|s|}$.

By Definition 4, if the quasi group $R(t,s)$ is uniformly exponentially dichotomic on X, then $R(t,s)$ is uniformly exponentially stable on $\operatorname{ran} P(t)$ and $R^{-1}(t,s)$ is uniformly exponentially stable on $\operatorname{ran} Q(t)$. Furthermore, for $t,s \in \mathbb{R}$, we have

$$Q(t)R(t-s,s) = R(t-s,s)Q(t-s).$$

In particular, for $s \geqslant 0$ we obtain

$$\|[R_Q(t-s,s)]^{-1}\| \leqslant Ne^{-\alpha s}.$$

We recall the following spaces of functions. Let $C_0(\mathbb{R},X)$ be a space of all continuous functions $f:\mathbb{R} \to X$ such that $\lim_{t\to\pm\infty} f(t) = 0$ with the supremum norm. Let $C_b(\mathbb{R},X)$ be a space of all bounded continuous functions $f:\mathbb{R} \to X$ with the supremum norm. The set $\mathscr{F}_\infty$ denotes the subspace of functions $f \in C_0(\mathbb{R},X)$ such that $f(t) \in \operatorname{ran} Q(t)$ for all $t \in \mathbb{R}$. Therefore, if $f \in \mathscr{F}_\infty$, then $f(t) = Q(t)f(t)$ for all $t \in \mathbb{R}$. Also, for each $s \geqslant 0$ we define an operator R_s on $\mathscr{F}_\infty$ by

$$(R_s f)(t) := [R_Q(t,s)]^{-1} f(t+s), \quad t \in \mathbb{R}.$$

LEMMA 4. *The operator R_s is bounded on the space $\mathscr{F}_\infty$, and*

$$\|R_s f\|_\infty \leqslant Ne^{-\alpha s}\|f\|_\infty,$$

where N and α are the dichotomy constants as in Definition 4.

Proof. By (a) and (d) of Definition 4 and fact $f \in \mathscr{F}_\infty$, for $s \geqslant 0$ and $u,r \in \mathbb{R}$, we have

$$\begin{aligned}
\|(R_s f)(r-s) - (R_s f)(u-s)\| &= \|R_Q^{-1}(r-s,s)f(r) - R_Q^{-1}(u-s,s)f(u)\| \\
&\leqslant \|R_Q^{-1}(r-s,s)f(r) - R_Q^{-1}(r-s,s)R_Q(r-s,s)Q(r-s)R_Q^{-1}(u-s,s)f(u)\| \\
&\quad + \|Q(r-s)R_Q^{-1}(u-s,s)f(u) - Q(u-s)R_Q^{-1}(u-s,s)f(u)\| \\
&\leqslant Ne^{-\alpha s}\|f(r) - Q(r)R_Q(r-s,s)R_Q^{-1}(u-s,s)f(u)\| \\
&\quad + \|(Q(r-s) - Q(u-s))R_Q^{-1}(u-s,s)f(u)\|.
\end{aligned}$$

The strong continuity of $Q(\cdot)$ implies that

$$\lim_{r\to u} f(r) = f(u) = Q(u)f(u) = \lim_{r\to u} Q(r)R_Q(r-s,s)R_Q^{-1}(u-s,s)f(u).$$

Thus, $(R_s f)(\cdot)$ is continuous, and

$$\|R_s f\|_\infty = \sup_{s\geqslant 0}\|R_Q^{-1}(t,s)f(t+s)\| \leqslant Ne^{-\alpha s}\|f\|_\infty.$$

Furthermore, since $R_Q^{-1}(t,s)f(t+s) \in \operatorname{ran} Q(t)$ for all $s \geqslant 0$, we have

$$(R_s f)(t) = R_Q^{-1}(t,s)f(t+s) = Q(t)R_s f(t) \in \operatorname{ran} Q(t),$$

i.e. $R_s f \in \mathscr{F}_\infty$. □

We will prove the sufficient and necessary conditions for the uniformly exponential dichotomy of the C_0-quasi groups. Recall that the unit circle $\mathbb{T}$ and the open unit disk $\mathbb{D}$ are given by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, respectively.

THEOREM 4. (Dichotomy Theorem) *Assume that $R(t,s)$ is a C_0-quasi group on a Banach space X. Let E^s be the corresponding evolution semigroup given by (5) on $L_p(\mathbb{R},X)$ and let Γ denote its infinitesimal generator given by (6). The following statements are equivalent:*

(a) The quasi group $R(t,s)$ has a uniformly exponential dichotomy on X.

(b) For each $s>0$, $\sigma(E^s)\cap\mathbb{T}=\emptyset$.

(c) $0\in\rho(\Gamma)$.

Proof. The equivalence $(b)\Leftrightarrow(c)$ follows the result on page 24 of [7].

$(a)\Rightarrow(b)$. Let $P(\cdot)\in C_b(\mathbb{R},\mathscr{L}_s(X))$ denote the corresponding projection and $\mathscr{L}_s(X)$ denotes the space of bounded operators on X equipped with strong operator topology. Define the multiplication operators $\mathscr{P}$ and $\mathscr{Q}$ on $L_p(\mathbb{R},X)$ by $(\mathscr{P}f)(t)=P(t)f(t)$ and $(\mathscr{Q}f)(t)=Q(t)f(t)$, respectively. By (a) of Definition 4, $E^s\mathscr{P}=\mathscr{P}E^s$ and $E^s\mathscr{Q}=\mathscr{Q}E^s$. Define semigroups E^s_P and E^s_Q on $\operatorname{ran}\mathscr{P}$ and $\operatorname{ran}\mathscr{Q}$ by

$$
\begin{aligned}
(E^s_P f)(t) &= R_P(t-s,s)f(t-s)), \quad f\in\operatorname{ran}\mathscr{P}, \quad t\in\mathbb{R},\\
(E^s_Q f)(t) &= R_Q(t-s,s)f(t-s)), \quad f\in\operatorname{ran}\mathscr{Q}, \quad t\in\mathbb{R}.
\end{aligned}
$$

We note that $E^s=E^S_P+E^s_P$. Part (c) of Definition 4 gives $\|E^s_P\|\leqslant Ne^{-\alpha s}$. This implies that $\sigma(E^s_P)\subset\mathbb{D}$. By part (b) of Definition 4, E^s_Q is invertible on $\operatorname{ran}\mathscr{Q}$ while Lemma 4 gives that $(E^s_Q)^{-1}=R_s$ and $\|(E^s_Q)^{-1}\|\leqslant Ne^{-\alpha s}$. This implies that $\sigma(E^s_Q)\subset\mathbb{C}\setminus\overline{\mathbb{D}}$. These give that $\sigma(E^s)\cap\mathbb{T}=\emptyset$.

$(b)\Leftarrow(a)$. Assume that $\sigma(E(s))\cap\mathbb{T}=\emptyset$. Let $\mathscr{P}$ be the spectral projection corresponding to $E:=E(1)$ and the spectral set $\sigma(E)\cap\mathbb{D}$. By Spectral Projection Theorem (Theorem 3.14 of [4]), there exists a projection-valued function $P(\cdot)\in C_b(\mathbb{R},\mathscr{L}_s(X))$ such that $(\mathscr{P}f)(t)=P(t)f(t)$. We have to prove that P satisfy Definition 4.

Define the operators $E^s_P:=E^s|_{\operatorname{ran}\mathscr{P}}$ and $E^s_Q=E^s|_{\operatorname{ran}\mathscr{Q}}$. For $t\geqslant s\geqslant 0$ we have

$$
\begin{aligned}
[E^s\mathscr{P}f](t) &= R(t-s,s)P(t-s)f(t-s)\\
&= P(t)R(t-s,s)f(t-s)=[\mathscr{P}E^s f](t).
\end{aligned}
$$

This gives condition (a) of Definition 4, $P(t+s)R(t,s)=R(t,s)P(t)$. Moreover, the fact that the spectral radius $r(E^s_P)=\sup_{\lambda\in\sigma(E^s_P}|\lambda|<1$, Corollary 2.11 of [11] guarantees the existences of constants $N>1$ and $\alpha>0$ such that

$$
\|R_P(t,s)\|\leqslant Ne^{-\alpha s}. \tag{7}
$$

Thus, the condition (c) of Definition 4 is satisfied.

Hypothesis $\sigma(E^s)\cap\mathbb{T}=\emptyset$ for all $s>0$ implies that the operator E^s is invertible on $\operatorname{ran}\mathscr{Q}$. Therefore, operator $R_Q(t,s):\operatorname{ran}Q(t)\to\operatorname{ran}Q(t+s)$ is also invertible on $\operatorname{ran}Q(t)$. On other hand, since $\sigma([E^s_Q]^{-1}))\subset\mathbb{D}$, again by Corollary 2.11 of [11] there exist constants $N>1$ and $\alpha>0$ such that $\|R_Q^{-1}(t,s)\|\leqslant Ne^{-\alpha s}$. In this case, the choice N and α can be adjusted according to (7). Therefore, condition (b) and (d) of Definition 4 is satisfied by P. $\square$

The following example illustrates Dichotomy Theorem 4 in an application.

EXAMPLE 3. Let $X = \mathbb{R}^3$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous increasing function such that $\lim_{t\to\pm\infty} \varphi(t) < \infty$. Defined a C_0-quasi group on X by

$$R(t,s)x = \left(e^{-(v(t+s)-v(t))}x_1, e^{v(t+s)-v(t)}x_2, e^{-s\varphi(0)+v(t+s)-v(t)}x_3\right), \quad t,s \in \mathbb{R}.$$

where $v(t) = \int_0^t \varphi(s)ds$ and $x = (x_1,x_2,x_3)$. The quasi group $R(t,s)$ has an uniformly exponential dichotomy on X.

By assumption, the function φ is invertible on $\mathbb{R}$ and local integrability of φ guarantees the existence of v. The infinitesimal generator $A(t)$ of $R(t,s)$ is given by

$$A(t)x = (-\varphi(t)x_1, \varphi(t)x_2, (-\varphi(0) + \varphi(t))x_3).$$

The evolution semigroup E^s in (5) on the space $L_p(\mathbb{R},\mathbb{R}^3)$ is given by

$$(E^s f)(t) = \left(e^{-(v(t)-v(t-s))}f_1(t-s), e^{v(t)-v(t-s)}f_2(t-s), e^{-s\varphi(0)+v(t)-v(t-s)}f_3(t-s)\right),$$

where $f(t) = (f_1(t), f_2(t), f_3(t))$, $s \geqslant 0$, and $t \in \mathbb{R}$ with the infinitesimal generator

$$(\Gamma f)(t) = -\left(\frac{df_1}{dt}, \frac{df_2}{dt}, \frac{df_3}{dt}\right) + (-\varphi(t)f_1(t), \varphi(t)f_2(t), (-\varphi(0) + \varphi(t))f_3(t)).$$

In fact, Γ is invertible and

$$(\Gamma^{-1} f)(t) = -(g_1(t), g_2(t), g_3(t)) + (-\phi(t)f_1(t), \phi(t)f_2(t), (-\phi(0) + \phi(t))f_3(t)),$$

where $g_i(t) = \int_0^t f_i(s)ds, i = 1,2,3$, and $\phi(t) = \varphi^{-1}(t)$ for all $t \in \mathbb{R}$. This proves that $0 \in \rho(\Gamma)$. Dichotomy Theorem 4 gives that $R(t,s)$ has an uniformly exponential dichotomy on X.

4. Evolution semigroups on the half line

In this section we focus on the C_0-quasi semigroups on a Banach space X; that is, the quasi group with parameter $t,s \geqslant 0$. We will specialize the definition of the evolution semigroups in Definition 3 associated with the C_0-quasi semigroups on the space $L_p(\mathbb{R}^+,X)$.

DEFINITION 5. Let $R(t,s)$ be the C_0-quasi semigroup on a Banach space X. Evolution semigroup corresponding to $R(t,s)$ on $L_p(\mathbb{R}^+,X)$ is a family of operators $\{E_+^s\}_{s\geqslant 0}$,

$$(E_+^s f)(t) = \begin{cases} R(t-s,s)f(t-s), & t \geqslant s \\ 0, & 0 \leqslant t < s, \end{cases} \tag{8}$$

for all $f \in L_p(\mathbb{R}^+,X)$.

If $A(t)$ is the infinitesimal generator of C_0-quasi semigroup $R(t,s)$ with domain $\mathscr{D}$, then an operator Γ_+ defined by

$$(\Gamma_+ f)(t) = -\frac{df}{dt} + A(t)f(t), \quad t \geqslant 0 \tag{9}$$

is the infinitesimal generator of E_+^s with domain

$$\mathscr{D}(\Gamma_+) = \{f \in L_p(\mathbb{R}^+, X) : f \text{ is absolutely continuous, } f(t) \in \mathscr{D}\}.$$

THEOREM 5. *Let E_+^s be an evolution semigroup on $L_p(\mathbb{R}^+,X)$ corresponding to the C_0-quasi semigroup $R(t,s)$ defined by* (8) *with the infinitesimal generator Γ_+ given by* (9). *The quasi semigroup $R(t,s)$ is uniformly exponentially stable on X if and only if the spectral bound $s(\Gamma_+)$ is negative.*

Proof. $(\Rightarrow)$. Assume that $R(t,s)$ is uniformly exponentially stable on X. There exist constants $N > 1$ and $\alpha > 0$ such that

$$\|R(t,s)x\| \leqslant Ne^{-\alpha s}\|x\|, \quad x \in X, \quad t,s \geqslant 0.$$

For any $f \in L_p(\mathbb{R}^+,X)$ we have

$$\begin{aligned}\|E_+^s f\|^p_{L_p(\mathbb{R}^+,X)} &= \int_0^\infty \|(E_+^s f)(t)\|_X^p dt = \int_s^\infty \|R(t-s,s)f(t-s)\|_X^p dt \\ &\leqslant \int_s^\infty Ne^{-\alpha ps}\|f(t-s)\|_X^p dt \leqslant Ne^{-\alpha ps}\|f\|^p_{L_p(\mathbb{R}^+,X)}.\end{aligned}$$

This gives that $\omega(\Gamma_+)$ is negative. The assertion follows of the Corollary 1.13 of [7].

$(\Leftarrow)$. Assume that there exist constants $N > 1$ and $\alpha > 0$ such that

$$\|E_+^s\|_{L_p(\mathbb{R}^+,X)} \leqslant Ne^{-\alpha s}, \tag{10}$$

for all $s \geqslant 0$. We choose $z \in X$ such that $\|z\| = 1$. For any $t \geqslant 0$, we choose $f \in L_p(\mathbb{R}^+,X)$ such that $f(s_0) = z$ for some $s_0 \in [0,t]$. Let $s_1 = t - s_0$, then

$$\|R(t-s_1,s_1)z\| = \|R(t-s_1,s_1)f(t-s_1)\| \leqslant \|(E_+^{s_1} f)(t)\| \leqslant Ne^{-\alpha s_1}.$$

Thus, we can choose $s_2 \geqslant 0$ such that $\|R(t,s_2)\| < 1$, for all $t \geqslant 0$. Theorem 6 of [38] gives that $R(t,s)$ is uniformly exponentially stable on X. $\square$

There exist the sufficiency and necessity for the uniformly exponential stability of the C_0-quasi semigroups. Let $R(t,s)$ denote the C_0-quasi semigroup on a Banach space X. We define a linear operator $\mathbf{G}$ on $L_p(\mathbb{R}^+,X)$ by

$$(\mathbf{G}f)(t) = \int_0^t R(t-s,s)f(t-s)ds = \int_0^t R(s,t+s)f(s)ds, \quad t \geqslant 0. \tag{11}$$

According to the definition of E_+^s in (8), the operator $\mathbf{G}$ can be written by

$$(\mathbf{G}f)(t) = \int_0^\infty (E_+^s f)(t)ds, \quad t \geqslant 0. \tag{12}$$

THEOREM 6. *A C_0-quasi semigroup $R(t,s)$ is uniformly exponentially stable on a Banach space X if and only if $\mathbf{G}$ is bounded on $L_p(\mathbb{R}^+,X)$.*

Proof. ($\Rightarrow$). By assumption, Theorem 5 guarantees that E^s_+ is exponentially stable on $L_p(\mathbb{R}^+,X)$. There exist constants $N>1$ and $\alpha>0$ such that

$$\|E^s_+f\|_{L_p(\mathbb{R}^+,X)} \leqslant Ne^{-\alpha s}\|f\|_{L_p(\mathbb{R}^+,X)}.$$

By (12), for $f\in L_p(\mathbb{R}^+,X)$ we have

$$\begin{aligned}\|\mathbf{G}f\|^p_{L_p(\mathbb{R}^+,X)} &= \int_0^\infty \left\|\int_0^\infty (E^s_+f)(t)ds\right\|^p_X dt\\ &\leqslant \int_0^\infty\int_0^\infty \|(E^s_+f)(t)\|^p_X dsdt \leqslant \frac{N}{\alpha p}\|f\|^p_{L_p(\mathbb{R}^+,X)}.\end{aligned}$$

Thus $\mathbf{G}$ is bounded on $L_p(\mathbb{R}^+,X)$.

($\Leftarrow$). Since $\mathbf{G}$ is bounded on $L_p(\mathbb{R}^+,X)$, by assumption, and $(\mathbf{G}f)(0)=0$, then Theorem 10 of [38] implies that operator Γ_+ is injective on $L_p(\mathbb{R}^+,X)$ and $\Gamma_+u=-f$ is equivalent to $u=\mathbf{G}f$. Therefore, Γ_+ is invertible with $\Gamma_+^{-1}=-\mathbf{G}$. Theorem mapping spectral for the C_0-semigroup, Theorem 3.1 Chap. IV of [7], implies that $s(\Gamma_+)<0$. Theorem 5 gives that $R(t,s)$ is uniformly exponentially stable on X. ☐

As a consequence of Theorem 5 and Theorem 6, we have the following corollary.

COROLLARY 1. *Let $R(t,s)$ be a C_0-quasi semigroup on a Banach space X and let Γ_+ be the infinitesimal generator of the evolution semigroup E^s_+ corresponding to $R(t,s)$ on $L_p(\mathbb{R}^+,X)$. The following statements are equivalent.*

(1) $R(t,s)$ is uniformly exponentially stable on X.

(2) Γ_+ is invertible, with $\Gamma_+^{-1}=-\mathbf{G}$.

(3) $s(\Gamma_+)<0$.

The following theorem is a version of the Dichotomy Theorem for the C_0-quasi semigroup which is the special case of Theorem 4.

THEOREM 7. *Let $R(t,s)$ be a C_0-quasi semigroup on a Banach space X. Let E^s_+ be the corresponding evolution semigroup given by (8) on $L_p(\mathbb{R}^+,X)$ and let Γ_+ denote its infinitesimal generator given by (9). The following statements are equivalent:*

(a) The quasi semigroup $R(t,s)$ has a uniformly exponential dichotomy on X.

(b) For each $s>0$, $\sigma(E^s_+)\cap\mathbb{T}=\emptyset$.

(c) $0\in\rho(\Gamma_+)$.

Proof. The proof is similar to Theorem 4 replacing E^s and Γ by E^s_+ and Γ_+ on the space $L_p(\mathbb{R}^+,X)$, respectively. ☐

REMARK 2. (a) The uniformly exponential stability of the C_0-quasi semigroup has an important role in identifying the stability, stabilizability, and detectability of the non-autonomous linear control systems [38].

(b) Theorem 7 gives an alternative version of the Dichotomy Theorem for the C_0-quasi semigroup of Theorem 3.2 of [5].

5. Green's function and evolution semigroups

In this section, we will investigate a relation between the existence of a uniformly exponential dichotomy for a C_0-quasi group and the existence of a Green's operator $\mathbb{G}$. The Green's operator also represents a formula of the inverse of the infinitesimal generator Γ of the evolution semigroup E^s on $L_p(\mathbb{R},X)$.

First, we recall the definition of the Green's operator for a C_0-semigroup. Let T^t be a C_0-semigroup on a Banach space X. A projection $P \in \mathscr{L}(X)$ is a splitting operator for T^t if $PT^t = T^tP$ and for the corresponding restrictions $T_P^t : \operatorname{ran} P \to \operatorname{ran} P$ and $T_Q^t : \operatorname{ran} Q \to \operatorname{ran} Q$ where $Q = I - P$, the operator T_Q^t is invertible as an operator on $\operatorname{ran} P$. For the splitting operator P, define a function $G_P : \mathbb{R} \setminus \{0\} \to \mathscr{L}(X)$ by

$$G_P(t) = T_P^t \quad \text{for} \quad t > 0 \qquad \text{and} \qquad G_Q(t) = -T_Q^t \quad \text{for} \quad t < 0.$$

DEFINITION 6. The function G_P is called the Green's function for a C_0-semigroup T^t if the operator $\mathbb{G}$, called the Green's operator, defined by

$$(\mathbb{G}f)(t) = \int_{-\infty}^{\infty} G_P(t-s)f(s)ds, \quad f \in L_p(\mathbb{R},X) \tag{13}$$

is bounded on $L_p(\mathbb{R},X)$.

In particular, we will discuss the Green's function for the evolution semigroup E^s defined by (5) on $L_p(\mathbb{R},X)$. Let $\mathscr{P} \in \mathscr{L}(L_p(\mathbb{R},X))$ denote the splitting operator for E^s with the complementary projection $\mathscr{Q} = I - \mathscr{P}$. Define the operator G on $L_p(\mathbb{R}, L_p(\mathbb{R},X))$ by

$$Gf = \int_0^{\infty} [E_Q^s]^{-1} f ds - \int_0^{\infty} E_P^s f ds, \tag{14}$$

where $E_P^s = \mathscr{P}E^s\mathscr{P}$ and $E_Q^s = \mathscr{Q}E^s\mathscr{Q}$ are the corresponding restrictions.

LEMMA 5. *Let E^s be an evolution semigroup on $L_p(\mathbb{R},X)$ defined by (5) with the infinitesimal generator Γ given by (6) and $\mathscr{P}$ is the splitting projection. If the operator G defined by (14) is bounded on $L_p(\mathbb{R}, L_p(\mathbb{R},X))$, then $0 \in \rho(\Gamma)$ and $\Gamma^{-1} = G$.*

Proof. We have to prove that G is a right inverse for Γ and $\ker \Gamma = \{0\}$. For $f \in L_p(\mathbb{R}, L_p(\mathbb{R},X))$, we have

$$\begin{aligned} E^sGf - Gf &= \int_0^{\infty} E_Q^{s-t} f dt - \int_0^{\infty} E_P^{s+t} dt - \int_0^{\infty} E_Q^{-t} f dt + \int_0^{\infty} E_P^t dt \\ &= \int_{-\infty}^{s} E_Q^t f dt - \int_s^{\infty} E_P^t dt - \int_{-\infty}^{0} E_Q^t f dt + \int_0^{\infty} E_P^t dt \\ &= \int_0^s E_Q^t f dt + \int_0^s E_P^t f dt. \end{aligned}$$

This implies that $\frac{1}{s}(E^sGf - Gf)$ converges to $\mathscr{Q}f + \mathscr{P}f = f$ as $s \to 0$. Therefore, $Gf \in \mathscr{D}(\Gamma)$ and $\Gamma Gf = f$.

Next, for $f \in \mathscr{D}(\Gamma)$ and $\Gamma f = 0$, we have

$$\frac{d}{ds}E_P^s f = \frac{d}{ds}\mathscr{P}E^s f = \mathscr{P}E^s\Gamma f = 0$$
$$\frac{d}{ds}[E_Q^s]^{-1} f = \frac{d}{ds}([\mathscr{Q}E^s\mathscr{Q}]^{-1} f) = [\mathscr{Q}E^s\mathscr{Q}]^{-1}\mathscr{Q}\Gamma f = 0.$$

This implies that the functions $s \mapsto E_P^s f$ and $s \mapsto [E_Q^s]^{-1} f$, defined on $\mathbb{R}^+$, are both constant. Thus, for $s > 0$ we have $E_P^s f \equiv E_P^0 f = \mathscr{P} f$ and $[E_Q^s]^{-1} f = \mathscr{Q} f$, and so

$$Gf = \int_0^\infty [E_Q^s]^{-1} f ds - \int_0^\infty E_P^s f ds = \int_0^\infty (\mathscr{Q} f - \mathscr{P} f) ds.$$

The assumption that G is bounded on $L_p(\mathbb{R}, L_p(\mathbb{R}, X))$ gives $Gf \in L_p(\mathbb{R}, L_p(\mathbb{R}, X))$. On other hand, we have shown that $Gf \in \mathscr{D}(\Gamma)$. Consequently, $\mathscr{P} f = \mathscr{Q} f$ or $f = 0$. Therefore, $\ker\Gamma = \{0\}$ and Γ is invertible with $\Gamma^{-1} = G$. $\square$

THEOREM 8. *Let E^s be an evolution semigroup on $L_p(\mathbb{R}, X)$ defined by (5) with the infinitesimal generator Γ given by (6) and $\mathscr{P}$ is the splitting projection. The C_0-semigroup E^s is hyperbolic on $L_p(\mathbb{R}, X)$ if and only if there exists a unique Green's function $G_{\mathscr{P}}$ for E^s. In this case, if the related Green's operator given by*

$$(\mathscr{G} f)(t) = \int_{-\infty}^\infty G_{\mathscr{P}}(t-s) f(s) ds, \quad f \in L_p(\mathbb{R}, L_p(\mathbb{R}, X)), \tag{15}$$

then $\mathscr{G} = -\Gamma^{-1}$.

Proof. $(\Leftarrow)$. Let $G_{\mathscr{P}}$ be a unique Green's function for E^s with the Green's operator $\mathscr{G}$ given by (15). Let V^s be the translation operator $(V^s f)(t) = f(t-s)$ on $L_p(\mathbb{R}, X)$. Using the notations as in the proof of the Theorem 4, for each $s > 0$, we have

$$[E_Q^s]^{-1} = [R_Q(\cdot - s, s)V^s]^{-1} = V^{-s} R_Q^{-1}(\cdot - s, s) = R_Q^{-1}(\cdot - s, s) V^{-s}$$

and $E_P^s = R_P(\cdot - s, s)V^s$. The assumption (b) of Definition 4 guarantees the existence of $[E_Q^s]^{-1}$ for each $s > 0$. Furthermore, from (14), we have

$$\begin{aligned}(Gf)(t) &= \int_0^\infty R_Q^{-1}(t-s, s) f(t+s) ds - \int_0^\infty R_P(t-s, s) f(t-s) ds \\ &= \int_t^\infty E_Q^{t-s} f(s) ds - \int_{-\infty}^t E_P^{t-s} f(s) ds \\ &= -\int_{-\infty}^\infty G_{\mathscr{P}}(t-s) f(s) ds = -(\mathscr{G} f)(t), \quad t \in \mathbb{R}.\end{aligned}$$

By assumption, $\mathscr{G}$ is bounded on $L_p(\mathbb{R}, L_p(\mathbb{R}, X))$, it implies that G is also bounded on $L_p(\mathbb{R}, L_p(\mathbb{R}, X))$. Lemma 5 concludes that Γ is invertible with $\Gamma^{-1} = G = -\mathscr{G}$. Finally, by Theorem 2.39 of [4] the evolution semigroup E^s is hyperbolic on $L_p(\mathbb{R}, X)$ with the hyperbolic projection $\mathscr{P}$ equal to the Riesz projection corresponding to E^1 and the spectral set $\sigma(E^1) \cap \mathbb{D}$.

$(\Rightarrow)$. Assume that E^s is hyperbolic on $L_p(\mathbb{R}, X)$ i.e. $\sigma(E^s) \cap \mathbb{T} = \emptyset$. Let $\mathscr{P}$ be the Riesz projection corresponding to E^1 and the spectral set $\sigma(E^1) \cap \mathbb{D}$. Recall that

$E^s\mathscr{P} = \mathscr{P}E^s$, $\sigma(E^s_P) \subset \mathbb{D}$, and $\sigma([E^s_Q]^{-1}) \subset \mathbb{D}$. These implies that there exist $N > 1$ and $\alpha > 0$ such that $\|E^s_P\| \leqslant Ne^{-\alpha s}$ and $\|[E^s_Q]^{-1}\| \leqslant Ne^{-\alpha s}$ for all $s > 0$. Thus,

$$\|G_{\mathscr{P}}(t)\|_{L_p(\mathbb{R},L_p(\mathbb{R},X))} \leqslant Ne^{-\alpha|t|}, \quad t \in \mathbb{R}$$

and

$$\|\mathscr{G}f\|_{\mathscr{L}(L_p(\mathbb{R},X))} \leqslant \int_{-\infty}^{\infty} \|G_{\mathscr{P}}(t)\|_{L_p(\mathbb{R},L_p(\mathbb{R},X))}\|f\|_{L_p(\mathbb{R},X)}dt \leqslant \frac{2N}{\alpha}\|f\|_{L_p(\mathbb{R},X)}.$$

This shows the existence of the Green's function.

Let $G_{\mathscr{P}'}$ be also the Green's function for E^s. In this case, $\mathscr{P}'$ is the hyperbolic projection for E^s. Therefore, $\mathscr{P}'$ is the Riesz projection corresponding to E^1 and the spectral set $\sigma(E^1) \cap \mathbb{D}$. Consequently, $\mathscr{P}' = \mathscr{P}$. This prove the uniqueness of $G_{\mathscr{P}}$. □

Following Definition 6, we will construct the Green's function for a C_0-quasi group. A projection $P(\cdot) \in C_b(\mathbb{R}, \mathscr{L}_s(X))$ is called a splitting projection for a C_0-quasi group $R(t,s)$ if the (a) and (b) of Definition 4 hold; that is, $P(t+s)R(t,s) = R(t,s)P(t)$ for $t,s \in \mathbb{R}$ and the restriction $R_Q(t,s) = R(t,s)|_{\operatorname{ran} Q(t)}$ is an invertible operator from $\operatorname{ran} Q(t)$ to $\operatorname{ran} Q(t+s)$. Next, we define an operator $G_P : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathscr{L}_s(X)$ by

$$G_P(t,s) = R_P(t,s)P(t) \quad \text{for} \quad t > s,$$
$$G_P(t,s) = -[R_Q(s,t)]^{-1}Q(t) \quad \text{for} \quad t < s.$$

DEFINITION 7. For a C_0-quasi group $R(t,s)$ on a Banach space X, the function G_P is called Green's function for $R(t,s)$ if the operator $\mathbb{G}$ defined on $L_p(\mathbb{R},X)$ by

$$(\mathbb{G}f)(t) = \int_{-\infty}^{\infty} G_P(t,s)f(s)ds, \quad f \in L_p(\mathbb{R},X) \tag{16}$$

is bounded. The operator $\mathbb{G}$ is called Green's operator.

THEOREM 9. *Let Γ be the infinitesimal generator of the evolution semigroup E^s corresponding to a C_0-quasi group $R(t,s)$ defined by (5) on $L_p(\mathbb{R},X)$. The quasi group $R(t,s)$ has a uniformly exponential dichotomy on X if and only if there exists a unique Green's function G_P for $R(t,s)$. Moreover, if the associated Green's operator given by (16), then $\mathbb{G} = -\Gamma^{-1}$ on $L_p(\mathbb{R},X)$.*

Proof. Dichotomy Theorem 4 gives that the C_0-quasi group $R(t,s)$ has a uniformly exponential dichotomy if and only if the evolution semigroup E^s is hyperbolic on $L_p(\mathbb{R},X)$. By Theorem 8, E^s is hyperbolic on $L_p(\mathbb{R},X)$ if and only if there exists a unique Green's function $G_{\mathscr{P}}$ for E^s such that the Green's operator given by (15)

$$(\mathscr{G}f)(t) = \int_{-\infty}^{\infty} G_{\mathscr{P}}(t-s)f(s)ds, \quad f \in L_p(\mathbb{R},L_p(\mathbb{R},X)),$$

is bounded on $L_p(\mathbb{R},L_p(\mathbb{R},X))$. We claim that the existence of the Green's function $G_{\mathscr{P}}$ for E^s is equivalent to the existence of the Green's function G_P for $R(t,s)$.

To prove the equivalence, we consider the isometry

$$\mathscr{L}(L_p(\mathbb{R},X)) \to \mathscr{L}(L_p(\mathbb{R},L_p(\mathbb{R},X))) : H \mapsto I \otimes H,$$

where $[(I\otimes H)h](t,s)=Hh(t,s)$ for $f(t)=h(t,\cdot)\in L_p(\mathbb{R},X)$ for almost all $t\in\mathbb{R}$. In this manner, we have

$$(\mathscr{G}h)(t,\cdot)=\int_{-\infty}^{\infty}G_{\mathscr{P}}(t-s)h(s,\cdot)ds.$$

In fact, we have

$$(\mathscr{G}h)(t,s)=\int_{-\infty}^{t}R_P(s-(t-r),t-r)P(s-(t-r))h(r,s-(t-r))dr$$
$$-\int_{t}^{\infty}R_Q(s-(t-r,t-r)h(r,s-(t-r))dr.$$

Define a map $J: L_p(\mathbb{R},L_p(\mathbb{R},X))\to(L_p(\mathbb{R},L_p(\mathbb{R},X)))$ by

$$(Jh)(t,s)=h(t+s,s).$$

Refers to the proof of Theorem 2.39 of [4], J is an isometric isomorphism such that $J\mathscr{G}J^{-1}=I\otimes\mathbb{G}$. Thus, $\mathscr{G}\in\mathscr{L}(L_p(\mathbb{R},L_p(\mathbb{R},X)))$ if and only if $\mathbb{G}\in\mathscr{L}(L_p(\mathbb{R},X))$. □

COROLLARY 2. *A C_0-quasi semigroup on a Banach space X has a uniformly exponential dichotomy on X if and only if there exists a unique Green's function G_P for the quasi semigroup.*

Proof. For a C_0-quasi semigroup on X, we can construct the evolution semigroup E^s defined by (8) on $L_p(\mathbb{R}^+,X)$ with an infinitesimal generator Γ_+. The result follows the proof of Theorem 9 replacing the quasi group by the quasi semigroup on $L_p(\mathbb{R}^+,X)$. □

REMARK 3. Corollary 2 is also an alternative result of Theorem 3.2 of [5] for the sufficiency and necessity for the uniformly exponential dichotomy of the C_0-quasi semigroups. We see that the sufficient condition of Corollary 2 is more ideal since the way of the construction of the Green's function is more accurate.

In the end of this section, we give an example to confirm the identification of the uniformly exponential dichotomy for a C_0-quasi group via the Green's function.

EXAMPLE 4. Consider the C_0-quasi group $R(t,s)$ in Example 3. It has been shown that the quasi group has an uniformly exponential dichotomy on X. Theorem 9 obliges that the quasi group has a unique Green's function.

Let $P:\mathbb{R}\to X$ be a projection such that $P(t)x=(x_1,x_2,0)$ for all $x=(x_1,x_2,x_3)$. We can verify that P is the splitting projection for $R(t,s)$ satisfying Definition 4. Let $Q(t)=I-P(t)$ be the complementary projection of P, we obtain

$$\operatorname{ran}P(t)=\{(x_1,x_2,0):x_1,x_2\in\mathbb{R}\}$$
$$\operatorname{ran}Q(t)=\{(0,0,x_3):x_3\in\mathbb{R}\}$$
$$R_P(t,s)=R(t,s)|_{\operatorname{ran}P(t)}=\left(e^{-(v(t+s)-v(t))},e^{v(t+s)-v(t)},0\right)$$
$$R_Q(t,s)=R(t,s)|_{\operatorname{ran}Q(t)}=\left(0,0,e^{-s\varphi(0)+v(t+s)-v(t)}\right)$$

$$R_Q^{-1}(t,s) = \left(0, 0, e^{s\varphi(0) - v(t+s) + v(t)}\right).$$

For the splitting projection P, we construct the Green's function G_P by

$$G_P(t,s) = R_P(t,s) = \left(e^{-(v(t+s)-v(t))}, e^{v(t+s)-v(t)}, 0\right) \quad \text{for} \quad t > s,$$

$$G_P(t,s) = -R_Q^{-1}(s,t) = \left(0, 0, -e^{t\varphi(0)-v(t+s)+v(s)}\right) \quad \text{for} \quad t < s,$$

and the corresponding Green's operator $\mathbb{G}$ on the space $L_p(\mathbb{R}, \mathbb{R}^3)$ is

$$(\mathbb{G}f)(t) = \int_{-\infty}^{t} R_P(t,s) f(s) ds - \int_{t}^{\infty} R_Q^{-1}(s,t) f(s) ds.$$

Moreover, if E^s is the evolution semigroup corresponding to the quasi group $R(t,s)$ on the space $L_p(\mathbb{R}, \mathbb{R}^3)$ given in the explanation of Example 3, we have

$$\mathbb{G}f = (g_1, g_2, g_3) - (-\phi f_1, \phi f_2, [-\phi(0) + \phi] f_3) = -\Gamma^{-1} f.$$

6. Conclusions

In this paper, we note that the C_0-quasi semigroups can be extended to be the C_0-quasi groups. The fundamental properties of the C_0-quasi groups are similar with the properties of C_0-quasi semigroups. The sufficiency for the infinitesimal generator of a C_0-quasi group can be identified. The non-autonomous abstract Cauchy problems inducted by the infinitesimal generator of a C_0-quasi group is well-posed. The C_0-quasi groups and the C_0-quasi semigroups can be reduced to the evolution semigroups on the spaces $L_p(\mathbb{R}, X)$ and $L_p(\mathbb{R}^+, X)$, $1 \leqslant p < \infty$, respectively. By the associated evolution semigroups, the uniformly exponential stability of both can be characterized. Dichotomy Theorem declares that the sufficient and necessary conditions for a C_0-quasi group has a uniformly exponential dichotomy are that the associated evolution semigroup is hyperbolic. Moreover, Dichotomy Theorem can also be investigated via the Green's function of the associated evolution semigroup. Dichotomy Theorem for the C_0-quasi semigroups is derived by the similar manner for C_0-quasi groups. In particular, for the C_0-quasi semigroups, the uniformly exponential stability depends on the infinitesimal generator of the associated evolution semigroup.

REFERENCES

[1] H. BART, I. GOHBERG AND M. A. KAASHOEK, *Wiener-Hopf factorization, inverse Fourier transform and exponentially dichotomous operators*, J. Funct. Anal., **68**: 1–42, 1986.

[2] C. BUŞE, *On nonuniform exponential stability of evolutionary process*, Rend. Sem. Mat. Univ. Poi. Torino, **52** (4): 396–406, 1994.

[3] C. BUŞE, M. MEGAN, M. PRAJE AND P. PREDA, *The strong variant of a Barbashin's Theorem on stability of solution for non-autonomous differential equations in Banach spaces*, Integr. Eqns. Oper. Th., **59** (4): 491–500, 2007.

[4] C. CHICONE AND Y. LATUSHKIN, *Evolution semigroups in dyamical systems and differential equations*, Americans Mathematical Society, USA, 1999.

[5] V. CUC, *Generalization of a theorem of Datko and Pazy for C_0-quasi semigroups*, Demonstr. Math., **34** (3): 577–586, 2001.

[6] R. DATKO, *Extending a theorem of A. M. Lyapunov to Hilbert space*, J. Math. Anal. Appl., **32**: 610–616, 1970.

[7] K. J. ENGEL AND R. NAGEL, *One-parameter semigroups for linear evolution equations*, Springer-Verlag, New York, 2000.

[8] D. E. EVANS, *Time dependent perturbations and scattering of strongly continuous groups on Banach spaces*, Math. Ann., **221**: 275–290, 1976.

[9] J. S. HOWLAND, *Stationary Scattering Theory for Time-dependent Hamiltonians*, Math. Ann., **207**: 315–335, 1974.

[10] P. E. T. JORGENSEN, *Spectral theory for infinitesimal generators of one-parameter groups of isometries: the min-max principle and compact perturbations*, J. Math. Anal. Appl., **90**: 343–370, 1982.

[11] C. S. KUBRUSLY, *Spectral theory of operators on Hilbert spaces*, Birkhäuser, New York, 2012.

[12] Y. LATUSHKIN AND S. MONTGOMERY-SMITH, *Lyapunov theorems for Banach spaces*, Bull. Amer. Math. Soc., **31** (1): 44–49, 1994.

[13] Y. LATUSHKIN AND S. MONTGOMERY-SMITH, *Evolutionary semigroups and Lyapunov theorems in Banach spaces*, J. Funct. Anal., **127**: 173–197, 1995.

[14] Y. LATUSHKIN AND T. RANDOLPH, *Dichotomy of differential equations on Banach spaces and an algebra of weighted composition operators*, Integr. Eqns. Oper. Th., **23**: 472–500, 1995.

[15] R. DE LAUBENFEL, *Storngly continuous groups, similarity and numerical range on a a Hilbert space*, Taiwan. J. Math., **1**: 127–133, 1997.

[16] H. LEIVA AND D. BARCENAS, *Quasi-semigroups, evolution equation and controllability*, Notas de Matematicas, **109**: 1–29, 1991.

[17] N. LEVAN AND C. S. KUBRUSLY, *Exponential dichotomy and strongly stable vectors of Hilbert space contraction semigroups*, Mat. Vesnik, **65** (2): 166–177, 2013.

[18] K. LIU, *A charcterizaton of strongly continuous of linear operators on a Hilbert space*, Bull. London Math. Soc., **32**: 54–62, 2000.

[19] M. MEGAN AND V. CUC, *On exponential stability of C_0-quasi semigroups in Banach spaces*, Le Matematiche, **LIV (1999)**: 229–241, 1999.

[20] M. MEGAN, B. SASU AND A. L. SASU, *On non-uniform exponential dichotomy of evolution operators in Banach spaces*, Integr. Eqns. Oper. Th., **44**: 71–78, 2002.

[21] N. V. MINH, F. RABIGER AND R. SCHNAUBELT, *Exponential stability, exponential expansiveness, and exponential dichotomy of evolution equations on the half-line*, Integr. Eqns. Oper. Th., **32**: 332–353, 1998.

[22] R. NAGEL, *Semigroup methods for non-autonomous Cauchy problems, in Evolution Equations*, Lect. Notes Pure Appl. Math., **168**: 301–316, 1995.

[23] H. NEIDHARDT, *Linear non-autonomous Cauchy problems and evolution semigroups*, Adv. Differential Equations, **14** (3–4): 289–340, 2009.

[24] A. G. NICKEL, *On evolution semigroups and wellposedness of non-autonomous Cauchy problems*, Dissertation, The Mathematical Faculty of the Eberhard-Karls-University Tubingen, Berlin, 1996.

[25] A. G. NICKEL, *Evolution semigroups and product formulas for non-autonomous Cauchy problems*, Math. Nach., **212**: 101–116, 2000.

[26] A. PAZY, *On the applicability of Lyapunov's Theorem in Hilbert space*, SIAM J. Math. Anal., **3** (2): 291–294, 1972.

[27] A. PAZY, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[28] C. PÖTZSCHE, *A note on the dichotomy spectrum*, J. Difference Equ. Appl., **15** (10): 1021–1025, 2009.

[29] P. PREDA AND M. MEGAN, *Exponential dichotomy of strongly continuous semigroups*, Bull. Austral. Math. Soc., **30**: 435–448, 1984.

[30] P. PREDA AND M. MEGAN, *Exponential dichotomy of evolutionary processes in Banach spaces*, Czechoslovak Math. J., **35** (110): 312–323, 1985.

[31] R. RAU, *Hyperbolic evolutionary semigroups on vector-valued function spaces*, Semigroup Forum, **48**: 107–118, 1994.

[32] J. SCHMID, *Well-posedness of Non-Autonomous Linear Evolution Equations for Generators whose Commutators are Scalar*, J. Evol. Equ., **16**: 21–50, 2016.

[33] J. SCHMID AND M. GRIESEMER, *Well-posedness of Non-Autonomous Linear Evolution Equations in Uniformly Convex Spaces*, Math. Nachr., **290**: 435–441, 2017.

[34] R. SCHNAUBELT, *Sufficient conditions for exponential stability and dichotomy of evolution equations*, Forum Math., **11**: 543–566, 1999.

[35] S. SUTRIMA, C. R. INDRATI, L. ARYATI AND MARDIYANA, *The fundamental properties of quasi semigroups*, J. Phys. Conf., Article ID 855(2017)012052: 1–9, 2017.

[36] S. SUTRIMA, C. R. INDRATI AND L. ARYATI, *Stability of C_0-quasi semigroups in Banach spaces*, J. Phys. Conf., Article ID 943(2017)012021: 1–14, 2017.

[37] S. SUTRIMA, C. R. INDRATI AND L. ARYATI, *Controllability and observability of non-autonomous Riesz-spectral systems*, Abstr. Appl. Anal., Articel ID 4210135: 1–10, 2018.

[38] S. SUTRIMA, C. R. INDRATI AND L. ARYATI, *Exact null controllability, stabilizability, and detectability of linear non-autonomous control systems: A quasi semigroup approach*, Abstr. Appl. Anal., Article ID 3791609: 1–12, 2018.

[39] S. SUTRIMA, C. R. INDRATI AND L. ARYATI, *Contraction quasi semigroups and their applications in decomposing Hilbert spaces*, Azerb. J. Math., **10** (1): 57–74, 2020.

[40] A. TAJMOUATI, M. KARMOUNI AND Y. ZAHOUAN, *Quasi-Fredholm and Saphar spectra for C_0-quasi-semigroups*, Adv. Oper. Theory, **5** (4): 1325–1339, 2020.

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Sutrima Sutrima
Department of Mathematics
Sebelas Maret University
Ir. Sutami, no. 36 A St, Kentingan, 57126, Surakarta, Indonesia
e-mail: sutrima@mipa.uns.ac.id

Mardiyana Mardiyana
Department of Mathematics Education
Sebelas Maret University
Ir. Sutami, no. 36 A St, Kentingan, 57126, Surakarta, Indonesia
e-mail: mardiyana@staff.uns.ac.id

Ririn Setiyowati
Department of Mathematics
Sebelas Maret University
Ir. Sutami, no. 36 A St, Kentingan, 57126, Surakarta, Indonesia
e-mail: ririnsetiyowati@staff.uns.ac.id