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NONLINEAR LIE DERIVATIONS OF INCIDENCE ALGEBRAS

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Abstract. Let $(X,\leqslant)$ be a locally finite preordered set and R a 2-torsion free commutative ring with unity, $I(X,R)$ the incidence algebra of X over R. In this paper, we give an explicit description of the structure of nonlinear Lie derivations of $I(X,R)$. We prove that every nonlinear Lie derivation of $I(X,R)$ is a sum of an inner derivation, a transitive induced derivation and an additive induced Lie derivation.

1. Introduction

Let R be a 2-torsion free commutative ring with unity, A an associative algebra over R. A map $D : A \to A$ is called a *nonlinear* (or *multiplicative*) *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x,y \in A$, and a map $\Psi : A \to A$ is called a *nonlinear* (or *multiplicative*) *Lie derivation* if

$$\Psi([x,y]) = [\Psi(x),y] + [x,\Psi(y)]$$

for all $x,y \in A$. If the maps D and L are also R-linear, then they are called a derivation and a Lie derivation respectively. A (nonlinear) Lie derivation is called *proper* if it can be written as a sum of a (nonlinear) derivation and a central-valued map. In 1961, Herstein [13] proposed many problems concerning the structures of Jordan and Lie mappings on associative simple and prime rings. Roughly speaking, he conjectured that all the involved Lie mappings, including Lie isomorphisms, Lie derivations etc., are proper or of the standard form. The renowned Herstein's Lie-type mappings research program was formulated since then. Based on the techniques introduced by Brešar [4], the study of the Herstein's Lie-type mappings research program finally became one of the foundations of the theory of functional identities (see [5]).

The main motivation of this paper is to study the nonlinear version of the Herstein's Lie-type mappings research program on incidence algebras. Now let us recall the definition of incidence algebras. Let $(X,\leqslant)$ be a locally finite preordered set. This means that $\leqslant$ is a reflexive and transitive binary relation on the set X, and for any $x \leqslant y$ there are only finitely many $z \in X$ such that $x \leqslant z \leqslant y$. The *incidence algebra* $I(X,R)$ of X over R is defined as the set of functions

$$I(X,\mathscr{R}) := \{f : X \times X \longrightarrow \mathscr{R} \mid f(x,y) = 0 \text{ if } x \nleqslant y\}$$

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with multiplication given by the convolution

$$(fg)(x,y) := \sum_{x \leqslant z \leqslant y} f(x,z)g(z,y) \tag{1.1}$$

for all $f,g \in I(X,R)$ and $x,y \in X$. If X is finite, then $I(X,R)$ is also known as the special subring of matrix ring [23]. The full matrix algebra $\mathrm{M}_n(R)$, the upper (or lower) triangular matrix algebra $\mathrm{T}_n(R)$ and the infinite triangular matrix algebras $\mathrm{T}_\infty(R)$ are also particular examples of incidence algebras. In addition, in the theory of operator algebras, the incidence algebra $I(X,R)$ of a finite partially ordered set is referred as a digraph algebra or a finite dimensional commutative subspace lattice (CSL) algebra.

In the past fifty years, isomorphisms and other related algebra maps on incidence algebras have been studied extensively, see [1, 6, 7, 16, 18, 23, 25, 27] and the references therein. In 2015, the Herstein's Lie-type mappings research program on incidence algebras was studied in [30]. Subsequently many other authors have made essential contributions in this topic, for example [15, 17, 29, 34]. On the other hand, the nonlinear version of the Herstein's program on incidence algebra $I(X,R)$ was studied by the author in [32] only when X is finite. The main obstacle for generalizing the results in [32] to the case of X being infinite is that we cannot find an analogue of the linear extension technique introduced in [34].

In this paper we shall study the nonlinear Lie derivations on incidence algebra $I(X,R)$ when the preordered set X is infinite. Notice that there is a long history on the study of nonlinear algebra maps. About fifty years ago, Martindale III [19] proved that every multiplicative bijective map (i.e. nonlinear isomorphism) from A to an arbitrary algebra is additive if A contains a nontrivial idempotent. Since then more and more mathematicians have made essential contributions to the related topics. For examples, nonlinear Lie-type derivations were studied on triangular algebras in [3, 8, 14, 33] and on generalized matrix algebras in [28, 31]. Fošner [10] and Šemrl [24] studied nonlinear commutativity preserving maps. The nonlinear generalized Jordan derivations on prime and semiprime rings were studied in [9, 20]. In the field of operator algebras, the nonlinear Lie-type derivations were studied on von Neumann algebras in [2, 11]. Several kinds of nonlinear operators were considered in standard operator algebras in [12, 21, 22] etc.

This paper is organized as follows. In Section 2, we introduce some basic notions, notations and facts on incidence algebras and several types of nonlinear Lie derivations. In Section 3, an explicit description of the structure of additive induced Lie derivations is obtained on incidence algebras. In Section 4, we prove the main theorem of the paper, which asserts that every nonlinear Lie derivation on $I(X,R)$ is a sum of an inner derivation, a transitive induced derivation and an additive induced Lie derivation.

2. Some canonical nonlinear Lie-derivations

In this section, we will introduce four kinds of nonlinear Lie derivations of incidence algebras. Throughout, $(X,\leqslant)$ is a locally finite preordered set and R is a 2-torsion free commutative ring with unity. Let $X = \bigsqcup_{i \in \mathscr{I}} X_i$ be the decomposition of

X into the union of its distinct connected components, where $\mathscr{J}$ is the index set. It is clear that $I(X,R) = \prod_{i\in\mathscr{J}} I(X_i,R)$.

For a fixed pair $x \leqslant y$ in X, let $e_{xy} : X \times X \longrightarrow R$ be the function given by

$$e_{xy}(u,v) = \begin{cases} 1, & \text{if } (u,v) = (x,y) \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Then the product of $I(X,R)$ satisfies $e_{xy}e_{uv} = \delta_{yu}e_{xv}$, and each element $f \in I(X,R)$ can be written as a formal sum $f = \sum_{x\leqslant y} f(x,y)e_{xy}$. Let $\mathscr{D}(X,R)$ be the diagonal subalgebra of $I(X,R)$, i.e.

$$\mathscr{D}(X,R) := \{\sum_x r_x e_{xx} | r_x \in R\}. \tag{2.2}$$

Let $\mathscr{C}(X_i,R)$ (resp. $\mathscr{C}(X,R)$) be the center of $I(X_i,R)$ (resp. $I(X,R)$), and $I_i := \sum_{x\in X_i} e_{xx}$ for each $i \in \mathscr{J}$. Then $\mathscr{C}(X_i,R)$ is R-linearly spanned by I_i and $\mathscr{C}(X,R) = \prod_{i\in\mathscr{J}} \mathscr{C}(X_i,R)$, see [26, Theorem 1.3.13] for details. It is also useful to point out that $\sum_x r_x e_{xx} \in \mathscr{C}(X,R)$ if and only if $r_x = r_y$ for all $x < y$.

Throughout this paper, for all $x,y \in X$, we define:

- $x \sim y$ if $x \leqslant y$ or $y \leqslant x$,
- $x < y$ if $x \leqslant y$ and $x \neq y$,
- $x \simeq y$ if $x \leqslant y$ and $y \leqslant x$.

It is clear that $\simeq$ is an equivalence relation. So if $x \simeq y$, we say x is *equivalent* to y. Since $(X,\leqslant)$ is locally finite, each equivalence class under $\simeq$ is a finite set.

Now we can introduce the four kinds of canonical nonlinear Lie derivations on $I(X,R)$.

(I) *Inner derivation*. Let $f \in I(X,R)$. Then the map

$$\begin{aligned} \mathrm{ad} f : I(X,R) &\longrightarrow I(X,R) \\ g &\longrightarrow [f,g] \end{aligned}$$

is called an inner derivation.

(II) *Transtive induced derivation*. In what follows, we use the notation $\leqslant$ to present the set $\{(x,y)|x \leqslant y \in X\}$. A map $f :\leqslant\longrightarrow R$ is called *transitive* if

$$f(x,y) + f(y,z) = f(x,z)$$

for all $x,y,z \in X$ such that $x \leqslant y \leqslant z$. Let $\sigma : X \longrightarrow R$ be a map. Then associated to σ there is a transitive map f_σ defined by $f_\sigma(x,y) = \sigma(x) - \sigma(y)$, $x,y \in X$. Transitive maps of this form are called *trivial*. For a fixed transitive map $f :\leqslant\longrightarrow R$, we can define an R-linear map $\Psi_f : I(X,R) \longrightarrow I(X,R)$ determined by

$$\Psi_f(e_{xy}) = f(x,y)e_{xy}, \quad x \leqslant y.$$

According to [23], see also [30, Lemma 2.3], Ψ_f is a derivation of $I(X,R)$, which is called a *transitive induced derivation*. Note that Ψ_f is an inner derivation if and only if f is a trivial transitive map.

(III) *Central-valued map*. Let $\overline{I(X,R)} = \langle [f,g] \mid f,g \in I(X,R) \rangle$, the commutator subalgebra of $I(X,R)$. A *central-valued map* of $I(X,R)$ is a map

$$\chi : I(X,R) \longrightarrow I(X,R)$$

such that

$$\chi(I(X,R)) \subset \mathscr{C}(X,R), \tag{2.3}$$

$$\chi(\overline{I(X,R)}) = 0. \tag{2.4}$$

It is clear that a central-valued map χ is a nonlinear Lie derivation of $I(X,R)$. Conversely, if a nonlinear Lie derivation Φ of $I(X,R)$ satisfies (2.3), then Φ is a central-valued map since $\Phi([e,f]) = [\Phi(e),f] + [e,\Phi(f)] = 0$ for all $e,f \in I(X,R)$.

(IV) *Additive induced derivation*: A map $f : R \longrightarrow R$ is called an *additive derivation* if

$$f(r+s) = f(r) + f(s), \tag{2.5}$$

$$f(rs) = f(r)s + rf(s) \tag{2.6}$$

for all $r,s \in R$.

PROPOSITION 2.1. *Let $F := \{f_i | i \in \mathscr{I}\}$ be a family of additive derivations on R, and $\Psi_F : I(X,R) \longrightarrow I(X,R)$ be the map defined as follows:*

$$\Psi_F(\textstyle\sum_{x \leqslant y} r_{xy}e_{xy}) = \sum_{x \leqslant y} \Psi_F(r_{xy}e_{xy}), \quad r_{xy} \in R, \tag{2.7}$$

$$\Psi_F(r_{xy}e_{xy}) = f_i(r_{xy})e_{xy}, \quad e_{xy} \in I(X_i,R),\ i \in \mathscr{I}. \tag{2.8}$$

Then Ψ_F is a derivation of $I(X,R)$.

Proof. According to (2.7), we only need to show that

$$\Psi_F(re_{xy}se_{uv}) = \Psi_F(re_{xy})se_{uv} + re_{xy}\Psi_F(se_{uv}) \tag{2.9}$$

for all $x \leqslant y, u \leqslant v$, $r,s \in R$. If $y \neq u$, it is obvious that both sides of (2.9) are zero. Now suppose $y = u$, and $e_{xy}, e_{uv} \in I(X_i,R)$ for some $i \in \mathscr{I}$. Then (2.9) follows from

$$\begin{aligned} \Psi_F(re_{xy})se_{uv} + re_{xy}\Psi_F(se_{uv}) &= f_i(r)e_{xy}se_{uv} + re_{xy}f_i(s)e_{uv} \\ &= (f_i(r)s + rf_i(s))e_{xv} \\ &= f_i(rs)e_{xv} \\ &= \Psi_F(re_{xy}se_{uv}). \quad \square \end{aligned}$$

DEFINITION 2.2. Let $F := \{f_i | i \in \mathscr{I}\}$ be a family of additive derivations on R, the derivation $\Psi_F : I(X,R) \rightarrow I(X,R)$ defined by (2.7)–(2.8) is called an additive induced derivation associated to F.

3. Additive induced Lie derivation

In this section, we introduce and study additive induced Lie derivations. In order to introduce the notion of additive induced Lie derivation, we need the following two definitions.

DEFINITION 3.1. Let $n \geqslant 2$ and $x_1, x_2, \cdots x_n$ be n *different* elements in X. Then we say that $\{x_1, x_2, \cdots, x_n\}$ forms a *cycle* if either

(1) $n = 2$ and $x_1 \simeq x_2$, or

(2) $n \geqslant 3$ and $x_1 \sim x_2 \sim x_3 \sim \cdots \sim x_n \sim x_1$.

DEFINITION 3.2. Suppose $x < y$, $u < v$ in X, then we say that e_{xy} is equivalent to e_{uv}, denoted by $e_{xy} \approx e_{uv}$, if either $e_{xy} = e_{uv}$ or there is a cycle containing both $x \sim y$ and $u \sim v$.

LEMMA 3.3. [32, Lemma 2.2] *The binary relation* $\approx$ *is an equivalence relation on the set* $\{e_{xy} | x < y\}$.

Let $\{\mathbf{E}_i |\ i \in \mathscr{I}\}$ be the equivalence classes of $\{e_{xy} | x < y\}$ with respect to the equivalence relation $\approx$, where $\mathscr{I}$ is the index set. If $e_{xy} \approx e_{uv}$, then it is obvious that x, y, u, v must be contained in the same connected component of X. Let $\{\mathbf{E}_i |\ i \in \mathscr{I}_j\}$ be the set of equivalence classes of $\{e_{xy} | x < y \in X_j\}$. Then it is clear that $\mathscr{I} = \cup_{j \in \mathscr{J}} \mathscr{I}_j$.

DEFINITION 3.4. Let $F := \{f_i \mid i \in \mathscr{I}\}$ be a family of additive derivations on R. A nonlinear Lie derivation ψ of $I(X,R)$ is called an *additive induced Lie derivation* associated to F if ψ leaves $\mathscr{D}(X,R)$ invariant, and for all $x < y$ we have

$$\psi(re_{xy}) = f_i(r)e_{xy},\ e_{xy} \in E_i,\ r \in R. \tag{3.1}$$

REMARK 3.5. A central-valued map is an additive induced Lie derivation with $\{f_i = 0 | i \in \mathscr{I}\}$. It is also clear that an additive induced derivation is an additive induced Lie derivation.

The following example gives an additive induced Lie derivation which is neither a central-valued map nor an additive induced derivation.

EXAMPLE 3.6. Let $X = \{x_0 = y_0, x_1, y_1, \cdots, x_i, y_i, \cdots\}$ be a partially ordered set with the order given by

$$x_0 < x_1 < x_2 < \cdots < x_i < \cdots;$$
$$y_0 < y_1 < y_2 < \cdots < y_i < \cdots.$$

It is clear that X is connected, $\{e_{x_i x_j} | 0 \leqslant i < j\}$ and $\{e_{y_k y_l} | 0 \leqslant k < l\}$ are two equivalence classes of the set $\{e_{uv} | u < v\}$. Let $f_1 \neq f_2$ be two additive derivations on R. Define a map $\Psi : I(X,R) \to I(X,R)$ by

$$\Psi(\sum_{u \leqslant v} c_{uv} e_{uv}) = \sum_{u \leqslant v} \Psi(c_{uv} e_{uv}),\ \ c_{uv} \in R, \tag{3.2}$$

$$\Psi(re_{x_ix_j}) = f_1(r)e_{x_ix_j}, \Psi(re_{y_iy_j}) = f_2(r)e_{y_iy_j},\ r \in R,\ 0 \leqslant i \leqslant j \neq 0, \tag{3.3}$$

$$\Psi(ce_{x_0x_0}) = -\sum_{i\geqslant 1}(f_1(c)e_{x_ix_i} + f_2(c)e_{y_iy_i}),\ \ c \in R. \tag{3.4}$$

Then Ψ is an additive induced Lie derivation of $I(X,R)$.

Proof. It is obvious that Ψ satisfies (3.1) and leaves $\mathscr{D}(X,R)$ invariant. So we only need to show that Ψ is a nonlinear Lie derivation. By (3.2), we need to prove

$$\Psi([re_{xy}, se_{uv}]) = [\Psi(re_{xy}), se_{uv}]) + [re_{xy}, \Psi(se_{uv})] \tag{3.5}$$

for all $x \leqslant y, u \leqslant v$ and $r,s \in R$. The proof will be divided into four cases.

(1) $e_{xy} = e_{x_0x_0}, e_{uv} = e_{x_ix_j}$ for $i < j$. If $i = 0$, then (3.5) follows from

$$\begin{aligned}
&[\Psi(re_{xy}), se_{uv}]) + [re_{xy}, \Psi(se_{uv})] \\
&= [-\sum_{i\geqslant 1}(f_1(r)e_{x_ix_i} + f_2(r)e_{y_iy_i}), se_{x_0x_j}] + [re_{x_0x_0}, f_1(s)e_{x_0x_j}] \\
&= f_1(r)se_{x_0x_j} + rf_1(s)e_{x_0x_j} \\
&= f_1(rs)e_{x_0x_j} \\
&= \Psi([re_{xy}, se_{uv}]).
\end{aligned}$$

If $i \neq 0$, then it is obvious that $\Psi([re_{xy}, se_{uv}]) = 0$, and

$$\begin{aligned}
&[\Psi(re_{xy}), se_{uv}]) + [re_{xy}, \Psi(se_{uv})] \\
&= [-\sum_{i\geqslant 1}(f_1(r)e_{x_ix_i} + f_2(r)e_{y_iy_i}), se_{x_ix_j}] + [re_{x_0x_0}, f_1(s)e_{x_ix_j}] \\
&= 0.
\end{aligned}$$

Hence we also have identity (3.5).

(2) $e_{xy} = e_{x_0x_0}, e_{uv} = e_{y_iy_j}$ for $i < j$. Note that $e_{x_0x_0} = e_{y_0y_0}$ since $x_0 = y_0$, hence the proof is similar to (1).

(3) $e_{xy} = e_{x_ix_j}, e_{uv} = e_{x_kx_l}$ for $0 \leqslant i \leqslant j \leqslant k \leqslant l,\ j \neq 0$. If $i = l$, then we have $i = j = k = l$ and the both sides of (3.5) are zero. If $j \neq k$, it is also obvious that the both sides of (3.5) equal to zero. Now suppose that $i \neq l$ and $j = k$, then (3.5) follows from

$$\begin{aligned}
&[\Psi(re_{xy}), se_{uv}]) + [re_{xy}, \Psi(se_{uv})] \\
&= [f_1(r)e_{x_ix_j}, se_{x_kx_l}] + [re_{x_ix_j}, f_1(s)e_{x_kx_l}] \\
&= f_1(rs)e_{x_ix_l} \\
&= \Psi([re_{xy}, se_{uv}]).
\end{aligned}$$

(4) $e_{xy} = e_{y_iy_j}, e_{uv} = e_{y_ky_l}$ for $0 \leqslant i \leqslant j \leqslant k \leqslant l,\ j \neq 0$. The proof is similar to (3). $\square$

PROPOSITION 3.7. *An additive induced Lie derivation associated to $F=\{f_i|i\in I\}$ is uniquely determined by F up to central-valued maps on $I(X,R)$.*

Proof. Suppose ψ_F and ϕ_F are two additive induced Lie derivations associated to F, then we need to prove that $\Psi:=\psi_F-\phi_F$ is a central-valued map.

Firstly, we will show that

$$\Psi(\mathscr{D}(X,R))\in\mathscr{C}(X,R). \tag{3.6}$$

Since ψ_F and ϕ_F leave $\mathscr{D}(X,R)$ invariant, then Ψ also leaves $\mathscr{D}(X,R)$ invariant. Let $\sum_{x\in X}r_xe_{xx}\in\mathscr{D}(X,R)$, where $r_x\in R$ for all $x\in X$. Then we can assume $\Psi(\sum_{x\in X}r_xe_{xx})=\sum_{x\in X}c_xe_{xx}\in\mathscr{D}(X,R)$, $c_x\in R$ for all $x\in X$. In order to prove $\Psi(\sum_{x\in X}r_xe_{xx})\in\mathscr{C}(X,R)$, we only need to show that $c_x=c_y$ if $x<y$. By (3.1), we have

$$\Psi(ce_{xy})=\psi_F(ce_{xy})-\phi_F(ce_{xy})=0,\ x<y,\ c\in R. \tag{3.7}$$

Applying ψ to $[\sum_{x\in X}r_xe_{xx},e_{xy}]=(r_x-r_y)e_{xy}$, by (3.7) we obtain

$$\begin{aligned}0&=\Psi((r_x-r_y)e_{xy})\\&=\Psi([\sum_{x\in X}r_xe_{xx},e_{xy}])\\&=[\Psi(\sum_{x\in X}r_xe_{xx}),e_{xy}]+[\sum_{x\in X}r_xe_{xx},\Psi(e_{xy})]\\&=[\sum_{x\in X}c_xe_{xx},e_{xy}]\\&=(c_x-c_y)e_{xy}.\end{aligned}$$

So $c_x=c_y$ for all $x<y$, and we have proved (3.6).

Next, we will prove that

$$\Psi(I(X,R))\subset\mathscr{D}(X,R). \tag{3.8}$$

Let $H=\sum_{x\leqslant y}h_{xy}e_{xy}$, where $h_{xy}\in R$ for all $x\leqslant y$. Suppose $\Psi(H)=\sum_{x\leqslant y}c_{xy}e_{xy}$. In order to prove $\Psi(H)\in\mathscr{D}(X,R)$, we need to show that $c_{xy}=0$ for all $x<y$. If $x<y$ and x is not equivalent to y, then $[[e_{xx},H],e_{yy}]=h_{xy}e_{xy}$. By (3.6) we have

$$\begin{aligned}\Psi(h_{xy}e_{xy})&=\Psi([[e_{xx},H],e_{yy}])\\&=[[e_{xx},\Psi(H)],e_{yy}]\\&=c_{xy}e_{xy}.\end{aligned}$$

Since $\Psi(h_{xy}e_{xy})=0$ by (3.7), we get $c_{xy}=0$. If $x\simeq y$, then $[[e_{xx},H],e_{yy}]=h_{xy}e_{xy}+h_{yx}e_{yx}$. Applying Ψ to this equality we obtain

$$\begin{aligned}\Psi(h_{xy}e_{xy}+h_{yx}e_{yx})&=\Psi([[e_{xx},H],e_{yy}])\\&=[[e_{xx},\Psi(H)],e_{yy}]\\&=c_{xy}e_{xy}+c_{yx}e_{yx}.\end{aligned} \tag{3.9}$$

By the above equality, we get

$$\Psi([h_{xy}e_{xy}+h_{yx}e_{yx},e_{yx}]) = [\Psi(h_{xy}e_{xy}+h_{yx}e_{yx}),e_{yx}] \\ = [c_{xy}e_{xy}+c_{yx}e_{yx},e_{yx}] \\ = c_{xy}(e_{xx}-e_{yy}). \tag{3.10}$$

On the other hand, $\Psi([h_{xy}e_{xy}+h_{yx}e_{yx},e_{yx}]) = \Psi(h_{xy}(e_{xx}-e_{yy})) \in \mathscr{C}(X,R)$ by (3.6). This implies $c_{xy}=0$, and we have proved (3.8).

Finally, we will prove that $\Psi(I(X,R)) \subset \mathscr{C}(X,R)$. Let $H \in I(X,R)$. According to (3.8), $\Psi(H) = \sum_{x\in X} r_x e_{xx} \in \mathscr{D}(X,R)$, where $r_x \in R$ for all $x \in X$. By (3.7)–(3.8), we have $\Psi([H,e_{xy}]) = [\Psi(H),e_{xy}] = (r_x-r_y)e_{xy} \in \mathscr{D}(X,R)$ for all $x<y$. Hence $r_x-r_y=0$ for all $x<y$. This implies $\Psi(H)\in\mathscr{C}(X,R)$. $\square$

In general, an additive induced Lie derivation may not be proper. So we want to know when an additive induced Lie derivation is proper. We have the following two propositions.

PROPOSITION 3.8. *Let ψ_F be an additive induced Lie derivation associated to $F=\{f_i|i\in\mathscr{I}\}$. If ψ_F is proper, then $f_i=f_l$ for all $j\in\mathscr{J}$ and $i,l\in\mathscr{I}_j$.*

Proof. Suppose ψ_F is proper, then there are a derivation D and a central-valued map C such that $\psi_F = D+C$. Since ψ_F leaves $\mathscr{D}(X,R)$ invariant, we have

$$D(re_{xx}) = \psi_F(re_{xx}) - C(re_{xx}) \in \mathscr{D}(X,R),\ x\in X, r\in R. \tag{3.11}$$

Note that $e_{xy}=[e_{xx},e_{xy}]\in\overline{I(X,R)}$ for $x<y$, so by (2.4) we obtain $C(re_{xy})=0$ and

$$D(re_{xy}) = \psi_F(re_{xy}) - C(re_{xy}) = \psi_F(re_{xy}),\ r\in R. \tag{3.12}$$

According to (3.11), we can assume $D(re_{xx}) = \sum_{u\in X} f_u^x(r)e_{uu}$, where $\{f_u^x|x,u\in X\}$ are functions on R. Then for each pair $x\neq y$ in X, we have

$$0 = D(re_{xx}e_{yy}) = D(re_{xx})e_{yy} + re_{xx}D(e_{yy}) = f_y^x(r)e_{yy} + rf_x^y(1)e_{xx}.$$

This implies $f_y^x=0$ if $x\neq y$. Hence $D(re_{xx}) = f_x^x(r)e_{xx}$ for all $x\in X$, $r\in R$. Note that $f_i(1)=0$ by (2.6), so for $x<y$ we have

$$D(re_{xy}) = D(re_{xx}e_{xy}) = D(re_{xx})e_{xy} + re_{xx}D(e_{xy}) = f_x^x(r)e_{xy}, \tag{3.13}$$

$$D(re_{xy}) = D(e_{xy}re_{yy}) = D(e_{xy})re_{yy} + e_{xy}D(re_{yy}) = f_y^y(r)e_{xy}. \tag{3.14}$$

By the two equalities above, we obtain $f_x^x = f_y^y$ for all $x<y$. This implies

$$f_u^u = f_v^v,\ u,v\in X_j \tag{3.15}$$

since X_j is a connected component of X.

Now suppose $e_{xy}\in\mathbf{E}_i$, $i\in\mathscr{I}_j$, then by (3.12)–(3.13) we have

$$f_x^x(r)e_{xy} = D(re_{xy}) = \psi_F(re_{xy}) = f_i(r)e_{xy}. \tag{3.16}$$

Combining (3.15)–(3.16), we obtain $f_i = f_l$ if $i,l \in \mathscr{I}_j$. $\square$

According to Proposition 3.8, the additive induced Lie derivation given in Example 3.6 is not proper.

PROPOSITION 3.9. *An additive induced Lie derivation is proper if and only if it is a sum of an additive induced derivation and a central-valued map.*

Proof. Let Ψ be an additive induced Lie derivation associated to $F = \{f_i, i \in \mathscr{I}\}$. If Ψ is a sum of an additive induced derivation and a central-valued map, then Ψ is proper by definition. Conversely, if Ψ is proper, then for each $j \in \mathscr{J}$, we have $f_i = f_l$ for all $i,l \in \mathscr{I}_j$ by Proposition 3.8. So there is an additive induced derivation Φ associated to F. By Proposition 3.7, $C := \Psi - \Phi$ is a central-valued map. So $\Psi = \Phi + C$ is a sum of an additive induced derivation and a central-valued map. $\square$

4. The main result

The main result of the paper is the following theorem.

THEOREM 4.1. *Every nonlinear Lie derivation of $I(X,R)$ is a sum of an inner derivation, a transitive induced derivation and an additive induced Lie derivation.*

The proof of Theorem 4.1 will be carried out by a series of lemmas and propositions. Combining Theorem 4.1 with Proposition 3.9, we have the following corollary.

COROLLARY 4.2. *A nonlinear Lie derivation of $I(X,R)$ is proper if and only if it is a sum of an inner derivation, a transitive induced derivation, an additive induced derivation and a central-valued map.*

4.1. The connected case

In this subsection, we assume that $(X,\leqslant)$ is connected and Ψ is a fixed nonlinear Lie derivation of $I(X,R)$. When $(X,\leqslant)$ is finite, the nonlinear Lie derivations of $I(X,R)$ were studied in [32]. So in this subsection, we also assume that $(X,\leqslant)$ is infinite.

DEFINITION 4.3. Let $F : I(X,R) \to I(X,R)$ be a fixed map. For each $x \leqslant y$, let $F_{xy} : I(X,R) \to R$ be the function given by

$$F_{xy}(e) = F(e)(x,y),\ e \in I(X,R). \tag{4.1}$$

By definition the functions $F_{xy}, x \leqslant y$ satisfy

$$F(e) = \sum_{x \leqslant y} F_{xy}(e) e_{xy},\ e \in I(X,R). \tag{4.2}$$

For each pair $x < y$ in X, let $I_{xy} := e_{xx} + 2e_{yy} \in \mathscr{D}(X,R)$ and

$$e_\Psi := \sum_{x<y} \Psi_{xy}(I_{xy}) e_{xy}. \tag{4.3}$$

LEMMA 4.4. *The nonlinear Lie derivation* $\Psi - \mathrm{ad}\, e_\Psi$ *leaves* $\mathscr{D}(X,R)$ *invariant.*

Proof. Let $H = \sum_{x\in X} h_x e_{xx}$ be an element in $\mathscr{D}(X,R)$. Since $I_{xy} \in \mathscr{D}(X,R)$ for each pair $x < y$ in X, we have

$$[H, I_{xy}] = 0. \tag{4.4}$$

Applying Ψ to the both sides of (4.4), we obtain

$$\begin{aligned}[\Psi(H), I_{xy}] + [H, \Psi(I_{xy})] &= \sum_{u\leqslant x} \Psi_{ux}(H)e_{ux} + 2\sum_{u\leqslant y} \Psi_{uy}(H)e_{uy} - \sum_{x\leqslant v} \Psi_{xv}(H)e_{xv} \\ &\quad -2\sum_{y\leqslant v} \Psi_{yv}(H)e_{yv} + \sum_{u\leqslant v}(h_u - h_v)\Psi_{uv}(I_{xy})e_{uv} \\ &= 0.\end{aligned}$$

Considering the coefficient of e_{xy} in the last equality, we get

$$\Psi_{xy}(H) = (h_y - h_x)\Psi_{xy}(I_{xy}). \tag{4.5}$$

By (4.5), we have

$$\begin{aligned}(\Psi - \mathrm{ad}\, e_\Psi)(H) &= \Psi(H) - [e_\Psi, H] \\ &= \sum_{x\leqslant y} \Psi_{xy}(H)e_{xy} - \sum_{x<y}(h_y - h_x)\Psi_{xy}(I_{xy})e_{xy} \\ &= \sum_{x\in X} \Psi_{xx}(H)e_{xx}.\end{aligned}$$

Hence $(\Psi - \mathrm{ad}\, e_\Psi)(H) \in \mathscr{D}(X,R)$ for all $H \in \mathscr{D}(X,R)$. $\square$

From now on, let ${}^1\Psi := \Psi - \mathrm{ad}\, e_\Psi$.

LEMMA 4.5. ${}^1\Psi(e_{xx}) \in \mathscr{C}(X,R)$ *for any* $x \in X$.

Proof. By Lemma 4.4, we can assume ${}^1\Psi(e_{xx}) = \sum_{x\in X} c_x e_{xx}$. In order to prove $\sum_{x\in X} c_x e_{xx} \in \mathscr{C}(X,R)$, we only need to show that $c_r = c_s$ for all $r < s$ in X.

If $r \neq x$ and $s \neq x$, then applying ${}^1\Psi$ to $[e_{xx}, e_{rs}] = 0$ we obtain

$$(c_r - c_s)e_{rs} + \sum_{x\leqslant v} {}^1\Psi_{xv}(e_{rs})e_{xv} - \sum_{u\leqslant x} {}^1\Psi_{ux}(e_{rs})e_{ux} = 0. \tag{4.6}$$

So we get $c_r = c_s$ by considering the coefficient of e_{rs} in (4.6).

If $r = x$, then applying ${}^1\Psi$ to $[e_{xx}, e_{rs}] = e_{rs}$ we have

$$(c_r - c_s)e_{rs} + \sum_{x\leqslant v} {}^1\Psi_{xv}(e_{rs})e_{xv} - \sum_{u\leqslant x} {}^1\Psi_{ux}(e_{rs})e_{ux} = \sum_{u\leqslant v} {}^1\Psi_{uv}(e_{rs})e_{uv}. \tag{4.7}$$

Comparing the coefficients of e_{rs} on the both sides of (4.7), we obtain $c_r = c_s$.

If $s = x$, then applying ${}^1\Psi$ to $[e_{rs}, e_{xx}] = e_{rs}$ we obtain

$$\sum_{u\leqslant x} {}^1\Psi_{ux}(e_{rs})e_{ux} - \sum_{x\leqslant v} {}^1\Psi_{xv}(e_{rs})e_{xv} + (c_s - c_r)e_{rs} = \sum_{u\leqslant v} {}^1\Psi_{uv}(e_{rs})e_{uv}. \tag{4.8}$$

We also get $c_r = c_s$ by comparing the coefficients of e_{rs} on the both sides of (4.8). □

Let $x < y$ be a pair of elements in X and define

$$S_{xy} = \{ce_{xy} | c \in R\}. \tag{4.9}$$

Then we have

LEMMA 4.6. ${}^1\Psi(S_{xy}) \subset S_{xy}$.

Proof. By Lemma 4.5, applying ${}^1\Psi$ to $re_{xy} = [[e_{xx}, re_{xy}], e_{yy}]$ we have

$${}^1\Psi(re_{xy}) = [[e_{xx}, {}^1\Psi(re_{xy})], e_{yy}].$$

If x is not equivalent to y, then we have

$$\begin{aligned} {}^1\Psi(re_{xy}) &= [[e_{xx}, \sum_{u \leqslant v} {}^1\Psi_{uv}(re_{xy})e_{uv}], e_{yy}] \\ &= [\sum_{x \leqslant v} {}^1\Psi_{xv}(re_{xy})e_{xv} - \sum_{u \leqslant x} {}^1\Psi_{ux}(re_{xy})e_{ux}, e_{yy}] \\ &= {}^1\Psi_{xy}(re_{xy})e_{xy}. \end{aligned} \tag{4.10}$$

So we have ${}^1\Psi(S_{xy}) \subset S_{xy}$.

Now suppose $x \simeq y$, then we have

$${}^1\Psi(re_{xy}) = {}^1\Psi_{xy}(re_{xy})e_{xy} + {}^1\Psi_{yx}(re_{xy})e_{yx}. \tag{4.11}$$

Since $(X, \leqslant)$ is locally finite, there are only finitely many elements in X which are equivalent to x. Because X is connected and infinite, there exists an element $z \in X$ such that $z \sim x$ and z is not equivalent to x. If $x < z$, then we have ${}^1\Psi(e_{xz}) = {}^1\Psi_{xz}(e_{xz})e_{xz}$ by (4.10). Applying Ψ^1 to $[re_{xy}, e_{xz}] = 0$ yields

$$0 = [{}^1\Psi_{xy}(re_{xy})e_{xy} + {}^1\Psi_{yx}(re_{xy})e_{yx}, e_{xz}] + [re_{xy}, {}^1\Psi_{xz}(e_{xz})e_{xz}] = {}^1\Psi_{yx}(re_{xy})e_{yz}.$$

By (4.11), we obtain ${}^1\Psi(re_{xy}) = {}^1\Psi_{xy}(re_{xy})e_{xy} \in S_{xy}$. If $z < x$, then $z < y$ since $x \simeq y$. So ${}^1\Psi(e_{zy}) = {}^1\Psi_{zy}(e_{zy})e_{zy}$ by (4.10). Applying ${}^1\Psi$ to $[re_{xy}, e_{zy}] = 0$ we obtain

$$0 = [{}^1\Psi_{xy}(re_{xy})e_{xy} + {}^1\Psi_{yx}(re_{xy})e_{yx}, e_{zy}] + [re_{xy}, {}^1\Psi_{zy}(e_{zy})e_{zy}] = -{}^1\Psi_{yx}(re_{xy})e_{zx}.$$

By (4.11) we also have ${}^1\Psi(re_{xy}) = {}^1\Psi_{xy}(re_{xy})e_{xy} \in S_{xy}$. Hence ${}^1\Psi(S_{xy}) \subset S_{xy}$. □

Let $r_{xy} = {}^1\Psi_{xy}(e_{xy})$ for each pair $x < y$. According to Lemma 4.6, we have

$${}^1\Psi(e_{xy}) = {}^1\Psi_{xy}(e_{xy})e_{xy} = r_{xy}e_{xy},\ x < y. \tag{4.12}$$

LEMMA 4.7. *Let* $\mathbf{f} : \leqslant \to R$ *be the map defined by*

$$\mathbf{f}(x,y) = \begin{cases} r_{xy}, & \text{if } x < y; \\ 0, & \text{if } x = y. \end{cases} \tag{4.13}$$

Then $\mathbf{f}$ *is transitive.*

Proof. By definition, we need to prove

$$\mathbf{f}(x,y)+\mathbf{f}(y,z)=\mathbf{f}(x,z)$$

for all $x,y,z\in X$ such that $x\leqslant y\leqslant z$.

If $x\neq z$, then applying ${}^1\Psi$ to $[e_{xy},e_{yz}]=e_{xz}$ we have

$$r_{xz}e_{xz}={}^1\Psi(e_{xz})={}^1\Psi([e_{xy},e_{yz}])=(r_{xy}+r_{yz})e_{xz}.$$

Thus we get $\mathbf{f}(x,y)+\mathbf{f}(y,z)=\mathbf{f}(x,z)$.

If $x=z$, then we have

$$\begin{aligned}{}^1\Psi([e_{xy},e_{yx}]) =&[{}^1\Psi(e_{xy}),e_{yx}]+[e_{xy},{}^1\Psi(e_{yx})]\\ =&[r_{xy}e_{xy},e_{yx}]+[e_{xy},r_{yx}e_{yx}]\\ =&(r_{xy}+r_{yx})(e_{xx}-e_{yy}).\end{aligned}\tag{4.14}$$

Since $(X,\leqslant)$ is infinite and connected, there exists $w\in X$ such that $w\sim x$ but w is not equivalent to x. Assume that $x\simeq y<w$ (the proof of the case $w<x\simeq y$ is similar). By (4.14), applying ${}^1\Psi$ to $[[e_{xy},e_{yx}],e_{xw}]=[e_{xx}-e_{yy},e_{xw}]=e_{xw}$ we obtain

$$\begin{aligned}r_{xw}e_{xw} &= {}^1\Psi(e_{xw})\\ &= {}^1\Psi([[e_{xy},e_{yx}],e_{xw}])\\ &= [{}^1\Psi([e_{xy},e_{yx}]),e_{xw}]+[[e_{xy},e_{yx}],{}^1\Psi(e_{xw})]\\ &= [(r_{xy}+r_{yx})(e_{xx}-e_{yy}),e_{xw}]+[e_{xx}-e_{yy},r_{xw}e_{xw}]\\ &= (r_{xy}+r_{yx})e_{xw}+r_{xw}e_{xw}.\end{aligned}$$

So we get $r_{xy}+r_{yx}=0$, hence $\mathbf{f}(x,y)+\mathbf{f}(y,x)=0=\mathbf{f}(x,x)$. □

In what follows, let $\Psi_{\mathbf{f}}$ be the transitive induced derivation associated to $\mathbf{f}$ defined by (4.13), and ${}^2\Psi:={}^1\Psi-\Psi_{\mathbf{f}}$. According to (4.12), we have

$${}^2\Psi(e_{xy})=0,\ x<y.\tag{4.15}$$

LEMMA 4.8. *${}^2\Psi$ is an additive induced Lie derivation of $I(X,R)$.*

Proof. By Lemma 4.4 and the definition of transitive induced Lie derivation, it is clear that ${}^2\Psi$ leaves $\mathscr{D}(X,R)$ invariant. Let $\{\mathbf{E}_i|i\in\mathscr{I}\}$ be the equivalence classes of $\{e_{xy}|x<y\}$ under the relation $\approx$. In order to prove that ${}^2\Psi$ is an additive induced Lie derivation, we only need to show that there exists a family of derivations $F=\{f_i|i\in\mathscr{I}\}$ on R such that

$${}^2\Psi(ce_{xy})=f_i(c)e_{xy},\ c\in R,e_{xy}\in\mathbf{E}_i,i\in\mathscr{I}.\tag{4.16}$$

By Lemma 4.6, we have

$$\begin{aligned}{}^2\Psi(ce_{xy}) =&{}^1\Psi(ce_{xy})-\Psi_{\mathbf{f}}(ce_{xy})\\ =&{}^1\Psi_{xy}(ce_{xy})e_{xy}-cr_{xy}e_{xy}\\ \subset&S_{xy}\end{aligned}$$

for all $x<y$ and $c\in R$. So for each pair $x<y$ in X, we can define a function $f_{xy}:R\to R$ such that

$$^{2}\Psi(ce_{xy})=f_{xy}(c)e_{xy},\ c\in R. \tag{4.17}$$

In order to prove (4.16), we need to show the following:

(a) $f_{xy}=f_{uv}$ if $e_{xy},e_{uv}\in\mathbf{E}_i$ for some $i\in\mathscr{I}$.

(b) Let $f_i:=f_{xy}$, where $e_{xy}\in\mathbf{E}_i$. Then f_i is an additive derivation on R.

(a). If $e_{xy}=e_{uv}$, then it is clear that $f_{xy}=f_{uv}$. In what follows, we always assume that $e_{xy}\neq e_{uv}$. Since $e_{xy},e_{uv}\in\mathbf{E}_i$, there exists a cycle $\{x_1\sim x_2\sim x_3\sim\cdots\sim x_n\sim x_1\}$ which contains both $x\sim y$ and $u\sim v$.

If $n=2$, then we have $x\simeq y$ and $u=y,v=x$. Applying $^{2}\Psi$ to $[ce_{xy},e_{yx}]=[e_{xy},ce_{yx}]$ we obtain

$$f_{xy}(c)(e_{xx}-e_{yy})=f_{yx}(c)(e_{xx}-e_{yy}),\ c\in R.$$

So we have

$$f_{xy}=f_{yx},\ x\simeq y. \tag{4.18}$$

Next suppose that $n\geqslant 3$. For a pair $s\sim t$ and $s\neq t$, define

$$f_{\overline{st}}=\begin{cases} f_{st}, & \text{if } s<t;\\ f_{ts}, & \text{if } t<s.\end{cases}$$

Note that the function $f_{\overline{st}}$ is well defined since $f_{st}=f_{ts}$ if $s\simeq t$ by (4.18). So in order to prove $f_{xy}=f_{uv}$, we only need to show

$$f_{\overline{x_{i-1}x_i}}=f_{\overline{x_ix_{i+1}}},\ 2\leqslant i\leqslant n. \tag{4.19}$$

Here x_{n+1} is regarded as x_1. Since $\Psi_{\mathbf{f}}(ce_{xx})=c\mathbf{f}(x,x)e_{xx}=0$ for all $c\in R$ and $x\in X$, we have

$$^{2}\Psi(ce_{xx})={}^{1}\Psi(ce_{xx})\in\mathscr{D}(X,R). \tag{4.20}$$

For a fixed $i\in\{1,2,\cdots,n\}$, let $\{g_z,z\in X\}$ be the set of the functions satisfying

$$^{2}\Psi(ce_{x_ix_i})=\sum_{z\in X}g_z(c)e_{zz}.$$

Then for all $s<t\in X$ such that $s,t\neq x_i$, applying $^{2}\Psi$ to $[ce_{x_ix_i},e_{st}]=0$ we obtain $g_s(c)=g_t(c),\ c\in R$. Since $\{x_1\sim x_2\sim x_3\sim\cdots\sim x_n\sim x_1\}$ is a cycle, we have

$$g_{x_j}=g_{x_k},\ \ 1\leqslant j,k\leqslant n,\ \ j,k\neq i. \tag{4.21}$$

To prove (4.19), we need to consider the following cases: (1) $x_{i-1}<x_i<x_{i+1}$; (2) $x_{i-1}>x_i>x_{i+1}$; (3) $x_{i-1}<x_i,x_i>x_{i+1}$; (4) $x_{i-1}>x_i,x_i<x_{i+1}$.

Case (1). By (4.15), applying $^{2}\Psi$ to

$$[ce_{x_{i-1}x_i},e_{x_ix_{i+1}}]=[e_{x_{i-1}x_i},ce_{x_ix_{i+1}}],\ \forall c\in R$$

we obtain $f_{x_{i-1}x_i} = f_{x_i x_{i+1}}$.

Case (2). The proof is similar to that of case (1).

Case (3). By (4.15), applying ${}^2\Psi$ to $[e_{x_{i-1}x_i}, re_{x_i x_i}] = re_{x_{i-1}x_i}$ and $[re_{x_{i+1}x_i}, re_{x_i x_i}] = re_{x_{i+1}x_i}$, we get

$$g_{x_i}(r) - g_{x_{i-1}}(r) = f_{x_{i-1}x_i}(r), \tag{4.22}$$
$$g_{x_i}(r) - g_{x_{i+1}}(r) = f_{x_{i+1}x_i}(r). \tag{4.23}$$

Combining (4.21)–(4.23), we have $f_{x_{i-1}x_i} = f_{x_{i+1}x_i}$.

Case (4). The proof is similar to that of case (3).

(b). If $e_{xy} \in \mathbf{E}_i$, $i \in \mathscr{I}$, then we have

$$[e_{xx} - re_{xy}, e_{xx} + se_{xy}] = (r+s)e_{xy}, \; r,s \in R. \tag{4.24}$$

Applying ${}^2\Psi$ to both sides of (4.24) we obtain

$$[{}^2\Psi(e_{xx} - re_{xy}), e_{xx} + se_{xy}] + [e_{xx} - re_{xy}, {}^2\Psi(e_{xx} + se_{xy})] = \mathbf{f}_i(r+s)e_{xy}. \tag{4.25}$$

By (4.20) and Lemma 4.5, we have

$${}^2\Psi(e_{xx}) = {}^1\Psi(e_{xx}) \in \mathscr{C}(X,R). \tag{4.26}$$

So

$$\begin{aligned}
&[{}^2\Psi(e_{xx} - re_{xy}), e_{xx} + se_{xy}] \\
&= [{}^2\Psi(e_{xx} - re_{xy}), e_{xx}] + s[{}^2\Psi(e_{xx} - re_{xy}), e_{xy}] \\
&= {}^2\Psi([e_{xx} - re_{xy}, e_{xx}]) + s\,{}^2\Psi([e_{xx} - re_{xy}, e_{xy}]) \\
&= {}^2\Psi(re_{xy}) + s\,{}^2\Psi(e_{xy}) \\
&= \mathbf{f}_i(r)e_{xy}.
\end{aligned} \tag{4.27}$$

Here the second equality follows from (4.15) and (4.26), the fourth equality follows from (4.15). Similarly to (4.27) we have

$$\begin{aligned}
&[e_{xx} - re_{xy}, {}^2\Psi(e_{xx} + se_{xy})] \\
&= [e_{xx}, {}^2\Psi(e_{xx} + se_{xy})] - r[e_{xy}, {}^2\Psi(e_{xx} + se_{xy})] \\
&= {}^2\Psi([e_{xx}, e_{xx} + se_{xy}]) - r\,{}^2\Psi([e_{xy}, e_{xx} + se_{xy}]) \\
&= {}^2\Psi(se_{xy}) - r\,{}^2\Psi(-e_{xy}) \\
&= \mathbf{f}_i(s)e_{xy}.
\end{aligned} \tag{4.28}$$

Here the fourth equality follows from ${}^2\Psi(-e_{xy}) = {}^2\Psi([e_{xy}, e_{xx}]) = 0$. Combining (4.25), (4.27) and (4.28), we obtain

$$\mathbf{f}_i(r+s) = \mathbf{f}_i(r) + \mathbf{f}_i(s), \; r,s \in R. \tag{4.29}$$

On the other hand, aplying ${}^2\Psi$ to $[re_{xx}, se_{xy}] = rse_{xy}$ we obtain

$$\begin{aligned}
\mathbf{f}_i(rs)e_{xy} &= {}^2\Psi(rse_{xy}) \\
&= {}^2\Psi([re_{xx}, se_{xy}]) \\
&= [{}^2\Psi(re_{xx}), se_{xy}] + [re_{xx}, {}^2\Psi(se_{xy})] \\
&= s[{}^2\Psi(re_{xx}), e_{xy}] + r\mathbf{f}_i(s)e_{xy} \\
&= s\,{}^2\Psi([re_{xx}, e_{xy}]) + r\mathbf{f}_i(s)e_{xy} \\
&= s\mathbf{f}_i(r)e_{xy} + r\mathbf{f}_i(s)e_{xy}.
\end{aligned}$$

Hence we have

$$\mathbf{f}_i(rs) = \mathbf{f}_i(r)s + r\mathbf{f}_i(s),\ r, s \in R. \tag{4.30}$$

By (4.29)–(4.30), f_i is an additive derivation on R. □

For a fixed nonlinear Lie derivation Ψ on $I(X,R)$, by Lemmas 4.7–4.8, we have

$$\Psi = \mathrm{ad}_{e_\Psi} + \Psi_{\mathbf{f}} + {}^2\Psi, \tag{4.31}$$

where ad_{e_Ψ} is an inner derivation, $\Psi_{\mathbf{f}}$ is a transitive induced derivation and ${}^2\Psi$ is an additive induced Lie derivation. Hence we have the following proposition.

PROPOSITION 4.9. *Let X be a locally finite preordered set. If X is connected and infinite, then every nonlinear Lie derivation of $I(X,R)$ is a sum of an inner derivation, a transitive induced derivation and an additive induced Lie derivation.*

4.2. General case

In this subsection, we consider the general case where $(X, \leqslant)$ is not connected, and prove Theorem 4.1. Let $X = \bigsqcup_{i\in \mathscr{I}} X_i$ be the decomposition of X into the union of its distinct connected components. For a family of maps $\{\phi_i : I(X,R) \to I(X,R) \mid i \in \mathscr{I}\}$ such that $\mathrm{Im}(\phi_i) \subset I(X_i,R)$ for each $i \in \mathscr{I}$, we can define a direct product

$$\prod_{i\in\mathscr{I}} \phi_i : I(X,R) \to I(X,R) \tag{4.32}$$

by $(\prod_{i\in\mathscr{I}} \phi_i)(f) = \prod_{i\in\mathscr{I}} \phi_i(f)$ for all $f \in I(X,R)$.

Now for each $i \in \mathscr{I}$, let $l_i : I(X_i,R) \to I(X,R)$ and $\pi_i : I(X,R) \to I(X_i,R)$ be the natural injective map and projective map respectively. The following two lemmas are obvious.

LEMMA 4.10. *Let Ψ be a nonlinear Lie derivation of $I(X,R)$. Then $\Psi_i := \pi_i \circ \Psi \circ l_i$ is a nonlinear Lie derivation of $I(X_i,R)$ for each $i \in \mathscr{I}$.*

LEMMA 4.11. *Suppose Φ_i is a nonlinear Lie derivation of $I(X_i,R)$ for each $i \in \mathscr{I}$, then $\prod_{i\in\mathscr{I}} l_i \circ \Phi_i \circ \pi_i$ is a nonlinear Lie derivation of $I(X,R)$.*

With the help of the previous two lemmas, we can prove the following useful lemma.

LEMMA 4.12. *Let Ψ be a nonlinear Lie derivation of $I(X,R)$ and $\Psi_i := \pi_i \circ \Psi \circ l_i$ for each $i \in \mathscr{J}$. Then $C_\Psi := \Psi - \prod_{i\in\mathscr{J}} l_i \circ \Psi_i \circ \pi_i$ is a central-valued map.*

Proof. By Lemma 4.11, $\prod_{i\in\mathscr{J}} l_i \circ \Psi_i \circ \pi_i$ and hence C_Ψ are nonlinear Lie derivations of $I(X,R)$. So in order to prove that C_Ψ is a central-valued map, we only need to show that $C_\Psi(H) \in \mathscr{C}(X,R)$ for all $H \in I(X,R)$.

For any $Y \in I(X_i,R)$, we have

$$\begin{aligned} C_\Psi(Y) &= \Psi(Y) - \prod_{j\in\mathscr{J}} l_j \circ \Psi_j \circ \pi_j(Y) \\ &= \prod_{j\in\mathscr{J}} l_j \circ \pi_j(\Psi(Y)) - l_i \circ \Psi_i \circ \pi_i(Y) \\ &= \prod_{j\in\mathscr{J}} l_j \circ \pi_j(\Psi(Y)) - l_i \circ \pi_i \circ \Psi \circ l_i \circ \pi_i(Y) \\ &= \prod_{j\in\mathscr{J}} l_j \circ \pi_j(\Psi(Y)) - l_i \circ \pi_i \circ \Psi(Y) \\ &= \prod_{j\in\mathscr{J}\setminus\{i\}} l_j \circ \pi_j(\Psi(Y)). \end{aligned}$$

This implies

$$C_\Psi(Y) \in \prod_{j\in\mathscr{J}\setminus\{i\}} I(X_j,R), \ \forall\, Y \in I(X_i,R). \tag{4.33}$$

Let $E \in I(X,R)$ and $H \in I(X_i,R)$, $i \in \mathscr{J}$. Then

$$[C_\Psi(E),H] = C_\Psi([E,H]) - [E,C_\Psi(H)]. \tag{4.34}$$

Since $H \in I(X_i,R)$ and $[E,H] \in I(X_i,R)$, by (4.33) we have

$$\begin{gathered} C_\Psi([E,H]) \in \textstyle\prod_{j\in\mathscr{J}\setminus\{i\}} I(X_j,R), \\ [E,C_\Psi(H)] \in \textstyle\prod_{j\in\mathscr{J}\setminus\{i\}} I(X_j,R). \end{gathered}$$

So the right hand side of (4.34) is contained in $\prod_{j\in\mathscr{J}\setminus\{i\}} I(X_j,R)$, while the left hand side of (4.34) is contained in $I(X_i,R)$ since $H \in I(X_i,R)$. This forces

$$[C_\Psi(E),H] = 0 \tag{4.35}$$

for all $E \in I(X,R)$, $H \in I(X_i,R), i \in \mathscr{J}$. Hence $C_\Psi(E) \in \mathscr{C}(X,R)$ for all $E \in I(X,R)$ and C_Ψ is a central-valued map. $\square$

Now we can prove the main theorem of the paper.

Proof of Theorem 4.1. Let Ψ be a nonlinear Lie derivation of $I(X,R)$. By Lemma 4.10, $\Psi_i = \pi_i \circ \Psi \circ l_i$ is a nonlinear Lie derivation of $I(X_i,R)$ for each $i \in \mathscr{J}$. If X_i is infinte, by Proposition 4.9 we have $\Psi_i = \mathrm{ad}\, e_{\Psi_i} + \Psi_{\mathbf{f}_i} + {}^2\Psi_i$, where e_{Ψ_i} is an element in $I(X_i,R)$ defined by (4.3), $\mathbf{f}_i$ is a transitive map on $(X_i,\leqslant)$ and ${}^2\Psi_i$ is an additive induced Lie derivation of $I(X_i,R)$. If X_i is finite, then since the sum of an additive induced Lie derivation and a central-valued map is also an additive induced Lie derivation, we also have $\Psi_i = \mathrm{ad}\, e_{\Psi_i} + \Psi_{\mathbf{f}_i} + {}^2\Psi_i$ by [32, Theorem 3.1]. It is obvious that

$$\prod_{i\in\mathscr{J}} l_i \circ \mathrm{ad}\, e_{\Psi_i} \circ \pi_i = \mathrm{ad} \prod_{i\in\mathscr{J}} e_{\Psi_i},$$
$$\prod_{i\in\mathscr{J}} l_i \circ \Psi_{\mathbf{f}_i} \circ \pi_i = \Psi_{\mathbf{f}},$$

where $\mathbf{f}$ is the transitive map of $(X,\leqslant)$ defined by $\mathbf{f}(x,y) = \mathbf{f}_i(x,y)$ if $x \leqslant y \in X_i$. It is also clear that $\prod_{i\in\mathscr{J}} l_i \circ {}^2\Psi_i \circ \pi_i$ is an additive induced Lie derivation on $I(X,R)$. So we have $\prod_{i\in\mathscr{J}} l_i \circ \Psi_i \circ \pi_i = \mathrm{ad}\prod_{i\in\mathscr{J}} e_{\Psi_i} + \Psi_{\mathbf{f}} + \prod_{i\in\mathscr{J}} l_i \circ {}^2\Psi_i \circ \pi_i$, i.e. a sum of an inner derivation, a transitve induced derivation and an additive induced Lie derivation. Let $C_\Psi = \Psi - \prod_{i\in\mathscr{J}} l_i \circ \Psi_i \circ \pi_i$. By Lemma 4.12, C_Ψ is a central-valued map, which is a special additive induced Lie derivation. We have proved the theorem. □

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