Operators
and
Matrices
Volume 15, Number 1 (2021), 293–302
doi:10.7153/oam-2021-15-20

ON THE COMMUTATOR AND FREDHOLMNESS OF ISOMETRIC PAIR

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(*Communicated by G. Misra*)

Abstract. In this paper, we characterize the Fredholmness and compactness of commutators of the commuting isometric pair $W = (W_0, W_1)$ by using of their canonical model [3].

1. Introduction

The von Neumann-Wold decomposition on the structure of isometric operator is the milestone of operator theory on Hilbert space. In 1960s, Sz.-Nagy and C. Foias developed the canonical model theory for the contraction [10]. The study of commuting pair (or tuple) of isometries is not so simple, and the characterization of the structure of a pair of isometries would have profound impact in the multivariable operator theory. Many researchers have devoted to this subject with varying success, see [1, 2, 3, 5, 6, 8, 9, 11, 12] and references therein. C. Berger, L. Coburn and A. Lebow classified the commuting tuples of isometries on a Hilbert spaces by the parameters of the pairs (U,P), where U is a unitary operator and P is an orthogonal projection on some Hilbert space [1, 2]. In [8], the authors calculated some numerical invariants of completely non-unitary commuting isometric pairs by using these parameters. In [3], H. Bercovici, R. G. Douglas and C. Foias gave a very concrete canonical model for bi-isometries $W = (W_0, W_1)$, and this new model is related to the canonical functional model and characteristic function Θ of a contraction. In some cases, the characteristic function is more tractable and transparent. Therefore, the canonical model introduced in [3] allows, in principle, many explicit calculations. This paper aims to study the Fredholmness and compactness of commutators of bi-isometries in terms of the characteristic function.

Let $H^2(\mathbb{D}^2)$ be the Hardy space over the bidsik, and R_z, R_w be the restriction of the coordinate shifts T_z, T_w to a submodule. (R_z, R_w) is a pair of commuting isometries, and their properties were deeply explored in the previous work [7, 13, 14, 15]. The goal of this paper is to study to what extend these properties can be generalized to abstract bi-isometries by using their canonical model. Interestingly, it will be shown that most of the conjectures for (R_z, R_w) will be no longer true for general bi-isometries.

The paper is organized as follows. In section 2, we introduce the canonical model for bi-isometries $W = (W_0, W_1)$ developed by H. Bercovici, R. G. Douglas and C. Foias

Mathematics subject classification (2010): Primary 47A45; Secondary 47A13.
Keywords and phrases: Bi-isometries, canonical model, commutator, Fredholmness.
This research is supported by National Natural Science Foundation of China, Number: 11971086, 12031002.

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[3]. In [3], the proof that $W=(W_0,W_1)$ is $\{0\}$-cnu has a small gap and we will complete the proof. Then we characterize the compactness of commutators in terms of the characteristic function. In section 3, we will study the Fredholmness of $W=(W_0,W_1)$ and give an explicit example which is quit different from the case of Hardy space over the bidisk.

2. Compactness of commutators

Let $\mathbb{D}$ be the unit disk and $\mathbb{T}$ be the unit circle in the complex plane. For a separable complex Hilbert space $\mathscr{E}$, we denote by $L^2(\mathscr{E})$ the Hilbert space of all square integrable $\mathscr{E}$-valued functions $f:\mathbb{T}\to\mathscr{E}$. Let $H^2(\mathscr{E})$ denote the $\mathscr{E}$-valued Hardy space on $\mathbb{D}$, and it can be regarded as a subspace of $L^2(\mathscr{E})$ by taking the radial limits. Let $\mathscr{L}(\mathscr{E})$ denote the set of bounded linear operators on $\mathscr{E}$, and given a contractive analytic function $\Theta:\mathbb{D}\to\mathscr{L}(\mathscr{E})$, it has a power series expansion

$$\Theta(z)=\sum_{k=0}^{\infty} z^k\Theta_k,$$

where $\Theta_k\in\mathscr{L}(\mathscr{E})$, and the series is convergent in $\mathbb{D}$ strongly. The strong operator topology limit

$$\Theta(\zeta)=\lim_{r\uparrow 1}\Theta(r\zeta)$$

exists for almost every $\zeta\in\mathbb{T}$. The analytic Toeplitz operator T_Θ on $\mathscr{L}(H^2(\mathscr{E}))$ is defined by

$$T_\Theta f(z)=\Theta(z)f(z),\ \ f\in H^2(\mathscr{E}).$$

In particular, if $\Theta(z)=zI$, T_Θ is the unilateral shift with multiplicity $dim\mathscr{E}$, which is denoted by T_z. Define the Hilbert space

$$\mathscr{H}=H^2(\mathscr{E})\oplus H^2(\overline{\Delta L^2(\mathscr{E})}),$$

where $\Delta(\zeta)=(I-\Theta(\zeta)^*\Theta(\zeta))^{1/2}$. The operator $W_0,\ W_1\in\mathscr{L}(\mathscr{H})$ is defined by

$$\begin{aligned} W_0(f\oplus g)&=zf(z)\oplus\zeta g(w,\zeta),\\ W_1(f\oplus g)&=\Theta(z)f(z)\oplus(\Delta(\zeta)f(\zeta)+wg(w,\zeta)), \end{aligned} \tag{2.1}$$

where $z,w\in\mathbb{D}$ and $\zeta\in\mathbb{T}$. To avoid confusion, we always put the coordinates in the vector-valued functions.

PROPOSITION 2.1. For $f\oplus g\in\mathscr{H}$, we have

1. $W_0^*(f\oplus g)=T_z^*f\oplus\overline{\zeta}g(w,\zeta)$;
2. $W_1^*(f\oplus g)=P_{H^2(\mathscr{E})}(\Theta^*f+\Delta(\zeta)g_0(\zeta))\oplus T_w^*g(w,\zeta)$,

where $P_{H^2(\mathscr{E})}$ is the orthogonal projection on $L^2(\mathscr{E})$ with range $H^2(\mathscr{E})$, $g(w,\zeta)=\sum_{k=0}^{\infty}w^k g_k(\zeta)$, and $g_k\in\overline{\Delta L^2(\mathscr{E})}$.

Proof. By the definition of W_0, (1) is clear. To get item (2), for $u \in H^2(\mathscr{E})$, $v \in H^2(\overline{\Delta L^2(\mathscr{E})})$, we have

$$\begin{aligned}\langle W_1^*(f \oplus g), u \oplus v\rangle &= \langle f \oplus g, \Theta(z)u(z) \oplus (\Delta(\zeta)u(\zeta) + wv(w,\zeta))\rangle \\ &= \langle f, \Theta u\rangle + \langle g, \Delta(\zeta)u(\zeta)\rangle + \langle g, T_w v\rangle \\ &= \langle T_\Theta^* f, u\rangle + \int_{\mathbb{T}} \langle \Delta(\zeta)g(w,\zeta), u(\zeta)\rangle |dw| + \langle T_w^* g, v\rangle \\ &= \langle T_\Theta^* f, u\rangle + \int_{\mathbb{T}} \langle \Delta(\zeta)g_0(\zeta), u(\zeta)\rangle |dw| + \langle T_w^* g, v\rangle,\end{aligned}$$

and this gives (2). □

It is not hard to see that

$$\bigcap_{n=0}^{\infty} W_0^n \mathscr{H} = H^2(\overline{\Delta L^2(\mathscr{E})}),$$

that is, $W_0|_{H^2(\overline{\Delta L^2(\mathscr{E})})}$ is the unitary part of W_0. Hence

$$kerW_0^* = H^2(\mathscr{E}) \ominus zH^2(\mathscr{E}) = \mathscr{E}.$$

The bi-isometries $W = (W_0, W_1)$ is said to be $\{0\}$-cnu if $\mathscr{H}$ contains no direct summand on which W_0 acts as a unitary operator. In [3], it is shown that (2.1) gives a canonical model for $\{0\}$-cnu bi-isometry. The proof of Theorem 3.1 in [3] that $W = (W_0, W_1)$ is $\{0\}$-cnu missed the projection in the formula of W_1^* and we will make up the small gap for the reader's convenience.

THEOREM 2.2. ([3], Proposition 5.2) *The bi-isometry $W = (W_0, W_1)$ defined by (2.1) is $\{0\}$-cnu.*

Proof. Suppose that $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$, where $\mathscr{H}_1$ is the reducing subspace for (W_0, W_1) and $W_0|_{\mathscr{H}_1}$ is a unitary. Since $W_0|_{H^2(\overline{\Delta L^2(\mathscr{E})})}$ is the unitary part of W_0, we have that

$$\mathscr{H}_1 \subset H^2(\overline{\Delta L^2(\mathscr{E})}).$$

For $g \in \mathscr{H}_1$, where $g(w,\zeta) = g_0(\zeta) + wg_1(\zeta) + \cdots$, by Proposition 2.1, we have

$$W_1^*(g) = P_{H^2(\mathscr{E})}(\Delta(\zeta)g_0(\zeta)) \oplus g_1(\zeta) + wg_2(\zeta) + \cdots) \in \mathscr{H}_1,$$

which means that

$$\Delta(\zeta)g_0(\zeta) \in L^2(\mathscr{E}) \ominus H^2(\mathscr{E}),$$

hence

$$g_0 \in \overline{\Delta L^2(\mathscr{E})} \ominus \overline{\Delta H^2(\mathscr{E})}.$$

By induction, we have

$$g_k \in \overline{\Delta L^2(\mathscr{E})} \ominus \overline{\Delta H^2(\mathscr{E})}$$

for $k = 0, 1, 2, \cdots$. It follows that

$$H^2(\mathscr{E}) \oplus H^2(\overline{\Delta H^2(\mathscr{E})}) \subseteq \mathscr{H}_0.$$

Let $f \in H^2(\mathscr{E})$, $g \in H^2(\overline{\Delta L^2(\mathscr{E})})$ and $f \oplus g \perp \mathscr{H}_0$, it is clear that $f = 0$, and then

$$\begin{aligned} 0 &= \langle g, W_0^{*n}(H^2(\overline{\Delta H^2(\mathscr{E})}))\rangle \\ &= \langle g, (H^2(\overline{\Delta \overline{\zeta}^n H^2(\mathscr{E})}))\rangle \end{aligned}$$

for any $n = 0, 1, 2, \cdots$. Since the closed linear span $\bigvee\{\overline{\zeta}^n H^2(\mathscr{E}) | n = 0, 1, 2, \cdots\} = L^2(\mathscr{E})$, we obtain that $g = 0$. This proves that W is $\{0\}$-cnu. $\square$

In the following, we will study compactness of the commutator and cross-commutator of bi-isometries. Firstly, let's look at an example.

EXAMPLE 2.3. Let $\mathscr{E}$ be an infinite dimensional separable Hilbert space and $H^2(\mathbb{D}^2) \otimes \mathscr{E}$ be $\mathscr{E}$-valued Hardy space over $\mathbb{D}^2$ with coordinates z_0 and z_1. Define shift operators

$$W_0 f = z_0 f, \quad W_1 f = z_1 f$$

for $f \in H^2(\mathbb{D}^2) \otimes \mathscr{E}$. It is easy to check that $[W_1^*, W_1][W_0^*, W_0]$ is not compact and $[W_1^*, W_0] = 0$, which is compact. One can check that (see [3])

$$\Theta = \Theta(0) = T_{z_1}|_{H^2_{z_1} \otimes \mathscr{E}}.$$

In [15], if M is a submodule of Hardy space on the bidisk $H^2(\mathbb{D}^2)$, we conjectured that $[R_z^*, R_w]$ is compact if and only if $[R_w^*, R_w][R_z^*, R_z]$ is compact. The example 2.3 shows that this is no longer true for general bi-isometry. In the following, $W = (W_0, W_1)$ always denotes the bi-isometries defined by (2.1). We ask the following question.

QUESTION 1. When is the compactness of $[W_1^*, W_1][W_0^*, W_0]$ is equivalent to the compactness of $[W_0^*, W_1]$?

We will study the commutators $[W_1^*, W_1][W_0^*, W_0]$ and $[W_0^*, W_1]$ in terms of Θ, and then give a sufficient and necessary condition to Question 1.

PROPOSITION 2.4. $[W_1^*, W_1][W_0^*, W_0] = 0$ if and only if Θ is a constant co-isometry.

Proof. For $f \in H^2(\mathscr{E})$, $g \in H^2(\overline{\Delta L^2(\mathscr{E})})$, where $f(z) = f_0 + z f_1 + \cdots$, we have

$$\begin{aligned} [W_1^*, W_1][W_0^*, W_0](f \oplus g) &= (1 - W_1 W_1^*)(f_0 \oplus 0) \\ &= f_0 - W_1 P_{H^2(\mathscr{E})} \Theta^* f_0 \\ &= f_0 - W_1 \Theta_0^* f_0 \\ &= (I - \Theta(\zeta)\Theta_0^*) f_0 \oplus (-\Delta(\zeta)\Theta_0^* f_0) \\ &= (I - \Theta_0 \Theta_0^*) f_0 + \zeta \Theta_1 \Theta_0^* f_0 + \cdots \oplus (-\Delta(\zeta)\Theta_0^* f_0) \end{aligned} \tag{2.2}$$

When $f\oplus g$ varies over $\mathscr{H}$, the corresponding elements f_0 vary over $\mathscr{E}$, therefore if $[W_1^*,W_1][W_0^*,W_0]=0$, then $I-\Theta_0\Theta_0^*=0$, which implies that $\|\Theta_0\|=\|\Theta_0^*\|=1$. So Θ is a constant co-isometry.

Conversely, if Θ is a constant co-isometry, then $(I-\Theta_0^*\Theta_0)\Theta_0^*=0$ and thus $\Delta(\zeta)\Theta_0^*=0$. □

THEOREM 2.5. $[W_1^*,W_1][W_0^*,W_0]$ *is compact on* $\mathscr{H}$ *if and only if* $I-\Theta_0\Theta_0^*$ *is compact on* $\mathscr{E}$.

Proof. For $f\in H^2(\mathscr{E})$, $g\in H^2(\overline{\Delta L^2(\mathscr{E})})$, since

$$\begin{aligned}\|\Delta(\zeta)\Theta_0^*f_0\|^2&=\langle(1-\Theta(\zeta)^*\Theta(\zeta))\Theta_0^*f_0,\Theta_0^*f_0\rangle\\&=\|\Theta_0^*f_0\|^2-\|\Theta_0\Theta_0^*f_0\|^2-\|\Theta_1\Theta_0^*f_0\|^2-\cdots,\end{aligned}$$

and by (2.2), we obtain that

$$\begin{aligned}\|[W_1^*,W_1][W_0^*,W_0](f\oplus g)\|^2&=\|(1-\Theta_0\Theta_0^*)f_0\|^2+\|\Theta_1\Theta_0^*f_0\|^2+\cdots+\|\Delta(\zeta)\Theta_0^*f_0\|^2\\&=\|f_0\|^2-\|\Theta_0^*f_0\|^2\\&=\|(1-\Theta_0\Theta_0^*)^{\frac{1}{2}}f_0\|^2.\end{aligned}$$

Therefore, $[W_1^*,W_1][W_0^*,W_0]$ is compact on $\mathscr{H}$ if and only if $1-\Theta_0\Theta_0^*$ is compact on $\mathscr{E}$. □

PROPOSITION 2.6. $[W_0^*,W_1]=0$ if and only if Θ is a constant isometry.

Proof. Let $f\in H^2(\mathscr{E}),g\in H^2(\overline{\Delta L^2(\mathscr{E})})$, we have

$$\begin{aligned}[W_0^*,W_1](f\oplus g)&=W_0^*(T_\Theta f\oplus(\Delta(\zeta)f(\zeta)+wg(w,\zeta)))-W_1(T_z^*f\oplus\overline{\zeta}g(w,\zeta))\\&=T_z^*T_\Theta f\oplus(\overline{\zeta}\Delta(\zeta)f(\zeta)+\overline{\zeta}wg(w,\zeta))\\&\quad-(T_\Theta T_z^*f\oplus(\Delta(\zeta)T_z^*f+w\overline{\zeta}g(w,\zeta)))\\&=\frac{\Theta(z)-\Theta_0}{z}f_0\oplus\Delta(\zeta)\overline{\zeta}f_0.\end{aligned}\tag{2.3}$$

It follows that $[W_0^*,W_1]=0$ if and only if $\Theta=\Theta_0$ and $\Delta=0$, which is equivalent to that Θ is a constant isometry. □

THEOREM 2.7. $[W_0^*,W_1]$ *is compact on* $\mathscr{H}$ *if and only if* $I-\Theta_0^*\Theta_0$ *is compact on* $\mathscr{E}$.

Proof. Let $f\in H^2(\mathscr{E}),g\in H^2(\overline{\Delta L^2(\mathscr{E})})$, by calculation, we have that

$$\begin{aligned}\|\frac{\Theta(z)-\Theta_0}{z}f_0\|^2&=\int_{\mathbb{T}}\|\Theta(z)f_0\|^2|dz|-\|\Theta_0f_0\|^2\\&=\sum_{k=1}^{\infty}\|\Theta_kf_0\|^2,\end{aligned}$$

and

$$\begin{aligned}\|\Delta(\zeta)\overline{\zeta}f_0\|^2 &= \langle (I-\Theta^*\Theta)f_0, f_0\rangle \\ &= \|f_0\|^2 - \|\Theta f_0\|^2.\end{aligned}$$

By (2.3), we get

$$\begin{aligned}\|[W_0^*, W_1](f\oplus g)\|^2 &= \|f_0\|^2 - \|\Theta_0 f_0\|^2 \\ &= \|(1-\Theta_0^*\Theta_0)^{\frac{1}{2}} f_0\|^2.\end{aligned}$$

The proof is completed. □

Theorem 2.5 and Theorem 2.7 allow us construct explicitly bi-isometry with compact cross commutator and product of self-commutators.

COROLLARY 2.8. The followings hold.

1. Suppose that $dimker\Theta_0^* < \infty$, then $[W_0^*, W_1]$ is compact implies that $[W_1^*, W_1][W_0^*, W_0]$ is compact.
2. Suppose that $dimker\Theta_0 < \infty$, then $[W_1^*, W_1][W_0^*, W_0]$ is compact implies that $[W_0^*, W_1]$ is compact.

Proof. To prove (1), by Theorem 2.7, if $[W_0^*, W_1]$ is compact, then $I-\Theta_0^*\Theta_0$ is compact on $\mathscr{E}$, that is Θ_0 is left semi-Fredholm, then $dimkerQ_0 < \infty$ and $ranQ_0$ is closed (see [4], Chapter XI, Theorem 2.3). Therefore, Θ_0 is Fredholm, and we obtain that $I-\Theta_0\Theta_0^*$ is compact. By Theorem 2.2, $[W_1^*, W_1][W_0^*, W_0]$ is compact. The proof of (2) is similar and this completes the proof. □

If the condition in either (1) or (2) of Corollary 2.8 doesn't hold, it is easy to come up with example such that the conclusion would not be true. For instance, given any contractive analytic $\mathscr{L}(\mathscr{E})$-valued function on $\mathbb{D}$ such that $dimkerQ_0^* = \infty$, and $I-\Theta_0^*\Theta_0$ is compact, then Θ_0 is left semi-Fredholm, but it isn't Fredholm, hence $I-\Theta_0\Theta_0^*$ is not compact, that is $[W_0^*, W_1]$ is compact and $[W_1^*, W_1][W_0^*, W_0]$ is not compact.

Another single operator of interest is the defect operator for the commuting isometric pair.

DEFINITION 2.9. The defect operator of $W = (W_0, W_1)$ is defined by

$$C = I - W_0W_0^* - W_1W_1^* + W_0W_1W_0^*W_1^*.$$

The defect operator C was first defined in [7] for the commuting isometric pair (R_z, R_w), and it also was studied in [8] in terms of model introduced in [2]. In some cases, it is more tractable to study the defect operator in terms of canonical model (2.1).

PROPOSITION 2.10. $C = 0$ if and only if Θ is a unitary constant, that is $(W_0, W_1) = (T_z, T_\Theta)$ on $H^2(\mathscr{E})$. C is compact if and only if both $I-\Theta_0^*\Theta_0$ and $I-\Theta_0\Theta_0^*$ are compact on $\mathscr{E}$.

Proof. It is easy to see that C is 0 on $W_0W_1\mathscr{H}$, and with respect to the decomposition $\mathscr{H}\ominus W_0W_1\mathscr{H}=(\mathscr{H}\ominus W_0\mathscr{H})\oplus W_0(\mathscr{H}\ominus W_0\mathscr{H})$, C^2 has the form

$$C^2=\begin{pmatrix}[W_0^*,W_0][W_1^*,W_1][W_0^*,W_0] & 0\\ 0 & W_0[W_1^*,W_0]^*[W_1^*,W_0]W_0^*\end{pmatrix}.$$

Further, since W_0 is a unitary from $\mathscr{H}\ominus W_1\mathscr{H}$ to $W_0(\mathscr{H}\ominus W_1\mathscr{H})$, C^2 is unitarily equivalent to

$$C^2=\begin{pmatrix}[W_0^*,W_0][W_1^*,W_1][W_0^*,W_0] & 0\\ 0 & [W_1^*,W_0]^*[W_1^*,W_0]\end{pmatrix}.$$

It follows that $C=0$ (or compact) if and only if $[W_1^*,W_1][W_0^*,W_0]$ and $[W_1^*,W_0]$ are both zero (or compact) and the results follow. $\square$

3. Fredholmness of $W=(W_0,W_1)$

In this section, we study the Fredholmness of bi-isometries defined by (2.1). For the isometric pair $W=(W_0,W_1)$ on $\mathscr{H}$, there is a short sequence

$$0\longrightarrow\mathscr{H}\xrightarrow{d_1}\mathscr{H}\oplus\mathscr{H}\xrightarrow{d_2}\mathscr{H}\longrightarrow 0,$$

where $d_1x=(-W_1x,W_0x)$, $d_2(x,y)=W_0x+W_1y$, $x,y\in\mathscr{H}$. $W=(W_0,W_1)$ is said to be *Fredholm* if d_1 and d_2 both have closed range and

$$dim(Kerd_1)+dim(Kerd_2\ominus d_1(\mathscr{H}))+dim(\mathscr{H}\ominus d_2(\mathscr{H}\oplus\mathscr{H}))<\infty,$$

and in this case,

$$indW=-dimkerd_1+dim(Kerd_2\ominus d_1(\mathscr{H}))-dim(\mathscr{H}\ominus d_2(\mathscr{H}\oplus\mathscr{H})).$$

LEMMA 3.1. ([6], Remark 1) *$dim(Kerd_2\ominus d_1(\mathscr{H}))=dim(W_0kerW_1^*\cap W_1kerW_0^*)$; $dim(\mathscr{H}\ominus d_2(\mathscr{H}\oplus\mathscr{H}))=dim(kerW_0^*\cap kerW_1^*)$.*

We will compute these dimensions. Let

$$\mathscr{K}=H^2(\mathscr{E})\oplus\overline{\Delta L^2(\mathscr{E})},$$
$$\mathscr{G}=\{\Theta f\oplus\Delta f: f\in H^2(\mathscr{E})\},$$

which can be viewed as subspaces of $\mathscr{H}=H^2(\mathscr{E})\oplus H^2(\overline{\Delta L^2(\mathscr{E})})$. Set

$$\begin{aligned}\mathscr{H}(\Theta)&=\mathscr{K}\ominus\mathscr{G}\\ &=\{f\oplus g\in\mathscr{K}:\Theta^*f+\Delta g=\sum_{n=1}^{\infty}\overline{\zeta}^n e_n, e_n\in\mathscr{E}\}.\end{aligned}$$

LEMMA 3.2. ([3], Proposition 7.1 and Lemma 7.3) *We have*

$$kerW_0^*=\mathscr{E}, kerW_1^*=\mathscr{H}(\Theta).$$

LEMMA 3.3. *$kerW_0^* \cap kerW_1^* = ker\Theta_0^*$.*

Proof. For almost every $\zeta \in \mathbb{T}$, let

$$\Theta(\zeta) = \Theta_0 + \zeta\Theta_1 + \cdots,$$

where $\Theta_k \in \mathscr{L}(\mathscr{E})$. If $e \in kerW_0^* \cap kerW_1^*$, we have

$$\Theta^* e = \Theta_0^* e + \overline{\zeta}\Theta_1^* e + \cdots,$$

hence $W_1^* e = \Theta_0^* e$ and it yields that $\Theta_0^* e = 0$. It is also easy to see the converse part, and this completes the proof. □

LEMMA 3.4. *We have*

$$W_0 kerW_1^* \cap W_1 kerW_0^* = \{W_1 e : e \in ker\Theta_0\}.$$

Hence $dim(W_0 kerW_1^* \cap W_1 kerW_0^*) = dimker\Theta_0$.

Proof. For $e \in ker\Theta_0$, it is clear that $W_1 e \in W_1 kerW_0^*$. Let

$$f(z) = T_z^* \Theta e \in H^2(\mathscr{E}), g(\zeta) = \Delta(\zeta)\overline{\zeta} e \in \overline{\Delta L^2(\mathscr{E})},$$

then $f(z) = \frac{\Theta(z)e - \Theta_0 e}{z} = \frac{\Theta(z)e}{z}$. Therefore

$$\Theta^* f + \Delta g = \Theta^*(\zeta)\Theta(\zeta)\overline{\zeta} e + \Delta^2 \overline{\zeta} e = \overline{\zeta} e,$$

which means that $f \oplus g \in \mathscr{H}(\Theta)$. One can check that

$$\begin{aligned} W_0(f \oplus g) &= zf(z) \oplus \zeta g(\zeta) \\ &= \Theta(z)e \oplus \Delta(\zeta)e = W_1 e, \end{aligned}$$

we get $W_1 e \in W_0 kerW_1^*$.

Conversely, let $x = W_1 e \in W_0 kerW_1^*$, there exist $f \oplus g \in \mathscr{H}(\Theta)$ such that

$$\Theta(z)e \oplus \Delta(\zeta)e = zf(z) \oplus \zeta g(\zeta),$$

hence $\Theta(z)e = zf(z)$, so $\Theta_0 e = 0$.

Since W_1 is an isometry, it is easy to get the equation about dimension and this completes the proof. □

DEFINITION 3.5. The fringe operator F acting on $\mathscr{H} \ominus W_0\mathscr{H}$ is defined by

$$\begin{aligned} F : \mathscr{H} \ominus W_0\mathscr{H} &\to \mathscr{H} \ominus W_0\mathscr{H} \\ e &\mapsto P_{\mathscr{H} \ominus W_0\mathscr{H}} W_1 e. \end{aligned}$$

REMARK 3.6. For $e \in \mathscr{H} \ominus W_0\mathscr{H} = \mathscr{E}$,

$$\begin{aligned} Fe &= P_{\mathscr{H} \ominus W_0\mathscr{H}}(\Theta(z)e \oplus \Delta(\zeta)e) \\ &= \Theta_0 e. \end{aligned}$$

The following proposition comes essentially from [13].

PROPOSITION 3.7. *$ranF = (W_0\mathscr{H} + W_1\mathscr{H}) \ominus W_0\mathscr{H}$.*

Proof. For every $x \in \mathscr{H} \ominus W_0\mathscr{H}$,

$$Fx = (I - W_0W_0^*)W_1x = W_1x - W_0W_0^*W_1x \in W_0\mathscr{H} + W_1\mathscr{H}.$$

Since Fx is orthogonal to $W_0\mathscr{H}$, $Fx \in (W_0\mathscr{H} + W_1\mathscr{H}) \ominus W_0\mathscr{H}$.

In the other direction, if $x = W_0x_0 + W_1x_1 \in (W_0\mathscr{H} + W_1\mathscr{H}) \ominus W_0\mathscr{H}$, the for every $y \in \mathscr{H}$,

$$\begin{aligned} \langle x_0 + W_0^*W_1x_1, y\rangle &= \langle x_0, y\rangle + \langle W_0^*W_1x_1, y\rangle \\ &= \langle W_0x_0, W_0y\rangle + \langle W_1x_1, W_0y\rangle \\ &= \langle x, W_0y\rangle = 0. \end{aligned}$$

This implies that

$$x_0 = -W_0^*W_1x_1,$$

and hence

$$x = -W_0W_0^*W_1x_1 + W_1x_1 = (I - W_0W_0^*)W_1x_1 = Fx_1 = F(P_{kerW_0^*}x_1). \qquad \square$$

COROLLARY 3.8. *F has closed range if and only if $W_0\mathscr{H} + W_1\mathscr{H}$ is closed.*

By Lemma 3.3, Lemma 3.4, Remark 3.6, and Corollary 3.8, we get the following theorem.

THEOREM 3.9. *$W = (W_0, W_1)$ is Fredholm if and only if Θ_0 is Fredholm. In this case, $indW = ind\Theta_0$.*

Let M be a submodule of $H^2(\mathbb{D}^2)$. The characteristic function of the pair (R_z, R_w) is (see, e.g., section 5 in [3])

$$\Theta(\lambda) = P_{M \ominus zM}(I - \lambda R_z^*)^{-1}R_w|_{M \ominus zM}.$$

Recall that the fringe operators F_z and F_w are defined by (see [13])

$$\begin{aligned} F_z : M \ominus zM \to M \ominus zM, F_zf = P_{M \ominus zM}(wf), \\ F_w : M \ominus wM \to M \ominus wM, F_wg = P_{M \ominus wM}(zg). \end{aligned}$$

Then $\Theta_0 = F_z$ and Θ is the characteristic function of F_w. Moreover, R_w is unitarily equivalent to the Toeplitz operator T_Θ. It seems that the operator-theoretical based information of the pair (R_z, R_w) is encoded in F_z and F_w simultaneously. The following example shows the difference between the pair (R_z, R_w) on a submodule of $H^2(\mathbb{D}^2)$ and the abstract bi-isometries $W = (W_0, W_1)$.

EXAMPLE 3.10. Let $\mathscr{E}$ be an infinite dimensional Hilbert space, and for any integer n, let Q_0 be the unilateral shift with multiplicity n. Let $Q(z) = Q_0$ be constant $\mathscr{L}(\mathscr{E})$-valued analytic function on $\mathbb{D}$, and $W = (W_0, W_1)$ is defined as (2.1). It is easy to see that $I - Q_0^* Q_0 = 0$, $I - Q_0 Q_0^*$ is a rank n operator, hence Q_0 is Fredholm and $ind Q_0 = -n$. Therefore, $W = (W_0, W_1)$ is a bi-isometry with compact defect operator C and $ind W = -n$. As is known to experts, if the defect operator C of (R_z, R_w) is compact, then $ind(R_z, R_w) = -1$ (see [13]).

Acknowledgements. The author would like to thank the referee for his/her careful reading of the paper and helpful suggestions.

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(Received February 9, 2020)

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