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LINEAR LIE CENTRALIZERS OF THE ALGEBRA OF STRICTLY BLOCK UPPER TRIANGULAR MATRICES

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Abstract. Let $\mathscr{N}$ be the algebra of all $n\times n$ strictly block upper triangular matrices over a field $\mathbb{F}$. In this paper, we describe all linear Lie centralizers of $\mathscr{N}$. We also show that every linear Lie centralizer of $\mathscr{N}$ is a centralizer.

1. Introduction

Let $\mathscr{R}$ be a ring. An additive mapping $\phi:\mathscr{R}\to\mathscr{R}$ is called a *left* (resp. *right*) *centralizer* of $\mathscr{R}$ if $\phi(ab)=\phi(a)b$ (resp. $\phi(ab)=a\phi(b)$) for any $a,b\in\mathscr{R}$. The map ϕ is called a *centralizer* if it is both a left and a right centralizer.

Left and right centralizers were first introduced by G. Hochschild in [8], and later by Johnson in [9] and [10]. In [17] Zalar proved that if $\mathscr{R}$ is a 2-torsion free semiprime ring and $T:\mathscr{R}\to\mathscr{R}$ is an additive mapping such that $T(x^2)=T(x)x$ (or $T(x^2)=xT(x)$) for all $x\in\mathscr{R}$, then T is a centralizer. Vukman also studied centralizers of semiprime rings and proved that if $\mathscr{R}$ is a 2-torsion free semiprime ring and $T:\mathscr{R}\to\mathscr{R}$ is an additive mapping such that $2T(x^2)=T(x)x+xT(x)$ for all $x\in\mathscr{R}$, then T is a centralizer in [12]. Numerous work have been done on the study of centralizers of rings and algebras see [3, 13, 14, 16].

Let $\mathscr{R}$ be a ring. An additive mapping $\delta:\mathscr{R}\to\mathscr{R}$ is called a Lie centralizer of $\mathscr{R}$ if

$$\delta([a,b])=[\delta(a),b]\quad(\text{or } \delta([a,b])=[a,\delta(b)])$$

for all $a,b\in\mathscr{R}$, where $[a,b]=ab-ba$ is the usual Lie product of a and b.

REMARK. The conditions $\delta([a,b])=[\delta(a),b]=[a,\delta(b)]$ are equivalent regardless of the additivity of δ (see [4]).

Fošner and Jing studied the Lie centralizers on triangular rings and nest algebras in [4] and presented characterizations of both centralizers and Lie centralizers on triangular rings and nest algebras. Recently, Ghomanjani and Bahmani dealt with the structure of Lie centralizers of trivial extension algebras in [7]. In [1] hadj and Ahmed characterized the non-additive Lie centralizers of strictly upper triangular matrices over a field of zero characteristic.

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Let $\mathfrak{g}$ be an algebra over a filed $\mathbb{F}$. Then a linear map $f:\mathfrak{g}\to\mathfrak{g}$ is called *linear Lie centralizer* of $\mathfrak{g}$ if

$$f([a,b]) = [f(a),b] \tag{1.1}$$

for all $a,b\in\mathfrak{g}$.

The main goal of this paper is to explicitly describe the linear Lie centralizers of the algebra of strictly block upper triangular matrices over a field $\mathbb{F}$. In recent years, significant progress has been made in studying the algebra of strictly upper triangular matrices over a field or a ring. Some results on the study of the algebra of strictly upper triangular matrices are given in [2, 5, 6, 11, 15].

Fix a field $\mathbb{F}$. Let $M_{m,n}$ be the set of all $m\times n$ matrices over $\mathbb{F}$, and put $M_n = M_{n,n}$. Let $\mathscr{N}$ (resp. $\mathscr{B}$) denote the set of all strictly block upper triangular matrices (resp. block upper triangular matrices) in M_n relative to a given partition. Then $\mathscr{N}$ and $\mathscr{B}$ are subalgebras of $\mathfrak{gl}(n,\mathbb{F})$, i.e. M_n with the usual matrix multiplication. In this paper, we explicitly describe the linear Lie centralizers of $\mathscr{N}$ over $\mathbb{F}$ and will show that in Theorem 2.1: if f is a linear Lie centralizer of $\mathscr{N}$, then

$$f(A) = \lambda A + \mu(A)$$

for all $A\in\mathscr{N}$, where $\lambda\in\mathbb{F}$ and $\mu:\mathscr{N}\to \mathrm{Z}(\mathscr{N})$ is a linear map such that $\mu[\mathscr{N},\mathscr{N}] = 0$ where $\mathrm{Z}(\mathscr{N})$ denotes the center of $\mathscr{N}$.

The main motivation of this work comes from hadj and Ahmed's results on the non-additive Lie centralizers of the algebra of strictly upper triangular matrices over a field [1]. Our work on the linear Lie centralizers of $\mathscr{N}$ not only generalizes the result of hadj and Ahmed, but also use a new approach that is promising to find the linear Lie centralizers of other matrix algebras with appropriate block forms.The essential tools are Lemmas 3.1–3.3, where four types of product preserving linear maps between matrix spaces are determined.

Section 2 gives the basic notations and presents the mail result, i.e., characterizations of the centralizers and linear Lie centralizers of $\mathscr{N}$. Section 3 determines four types of product preserving linear maps between matrix spaces that will play essential roles in finding the linear Lie centralizers of $\mathscr{N}$. Section 4 presents some other lemmas and proves Theorems 2.1.

2. Main results

The linear Lie centralizers of the algebra $\mathscr{N}$ of strictly block upper triangular matrices will be determined in this section. We also characterizes the centralizers of $\mathscr{N}$.

2.1. Notations

Let $[n] = \{1,2,\cdots,n\}$. Fix a field $\mathbb{F}$. Let $M_{m,n}$ (resp. M_n) be the set of $m\times n$ (resp. $n\times n$) matrices over $\mathbb{F}$. Let I_n denote the identity matrix in M_n. A $t\times t$ *block matrix form* in M_n is represented by a sequence $(n_1,n_2,\cdots,n_t)$, where $n_i\in\mathbb{Z}^+$ for $i\in[t]$ and $n_1+\cdots+n_t = n$. Fixing a $t\times t$ block matrix form in M_n represented by a sequence $(n_1, n_2, \cdots, n_t)$, each $A \in M_n$ can be expressed as

$$A = \left[A_{i,j}\right]_{t\times t}$$

where the (i,j) block $A_{i,j} \in M_{n_i,n_j}$. The matrix A can also be expressed as

$$A = \sum_{(i,j)\in[t]\times[t]} A^{i,j}$$

such that each $A^{i,j} \in M_n$ has $A_{i,j}$ on the (i,j) block and 0's elsewhere. A is called

- *block upper triangular* if $A_{i,j} = 0$ for all $1 \leqslant j < i \leqslant t$,
- *strictly block upper triangular* if $A_{i,j} = 0$ for all $1 \leqslant j \leqslant i \leqslant t$,
- *block diagonal* if $A_{i,j} = 0$ for all $i \neq j$.

When A is not given in advance, $A^{i,j}$ and similar expressions may be used to express generic matrices in M_n with 0's outside of the (i,j) block.

Let $\mathscr{B}$ (resp. $\mathscr{N}$, $\mathscr{D}$) denote the set of all block upper triangular matrices (resp. strictly block upper triangular matrices, block diagonal matrices) in M_n. They are subalgebras of the algebra M_n with the usual matrix multiplication.

For $i, j \in [t]$, let $M_n^{i,j}$ denote the set of matrices in M_n with 0's outside of the (i,j) block. Define the *block index set* of $\mathscr{N}$ as

$$\Gamma_{\mathscr{N}} = \{(i,j) \in \mathbb{Z}^2 \mid 1 \leqslant i < j \leqslant t\}. \tag{2.1}$$

For $(i,j) \in \Gamma_{\mathscr{N}}$, denote $\mathscr{N}^{i,j} = M_n^{i,j}$. For $\Delta \subseteq \Gamma_{\mathscr{N}}$, denote

$$\mathscr{N}^{\Delta} = \bigoplus_{(i,j)\in\Delta} \mathscr{N}^{i,j}, \qquad \mathscr{N}^{\Delta^c} = \bigoplus_{(i,j)\in\Gamma_{\mathscr{N}}\setminus\Delta} \mathscr{N}^{i,j}. \tag{2.2}$$

Let $E_{p,q}^{i,j}$ denote the matrix in M_n that takes 1 on the (p,q) entry of the (i,j) block and 0's elsewhere for $(i,j) \in [t]\times[t]$ and $(p,q) \in [n_i]\times[n_j]$.

2.2. Centralizers and linear Lie centralizers of $\mathscr{N}$

We describe the linear Lie centralizers of $\mathscr{N}$ when $t \geqslant 3$. The main result for $t \geqslant 3$ is given in Theorem 2.1 below. We will give a proof of this theorem in section 4. It is well-known that the center $\mathrm{Z}(\mathscr{N})$ of $\mathscr{N}$ is $\mathscr{N}^{1,t}$.

THEOREM 2.1. *Let $t \geqslant 3$ and let $\mathscr{N}$ be the algebra of strictly block upper triangular matrices over a field $\mathbb{F}$. Let $f : \mathscr{N} \to \mathscr{N}$ be a linear Lie centralizer. Then*

$$f(A) = \lambda A + \delta(A) \tag{2.3}$$

for all $A \in \mathscr{N}$, where $\lambda \in \mathbb{F}$ and $\delta : \mathscr{N} \to \mathrm{Z}(\mathscr{N})$ is a linear map such that $\delta([\mathscr{N},\mathscr{N}]) = 0$.

As an application, we consider the centralizers of $\mathscr{N}$. We begin with the following lemma.

LEMMA 2.2. *Let $\mu : \mathscr{N} \to Z(\mathscr{N})$ be a linear mapping satisfying $\mu([\mathscr{N},\mathscr{N}]) = 0$. Then μ is a centralizer of $\mathscr{N}$.*

Proof. Let $A,B \in \mathscr{N}$ and $\Delta' = \{(i,i+1) \mid i \in [t-1]\}$. On one hand, by direct computation, $AB \in \sum_{(i,j)\in\Gamma_{\mathscr{N}}\setminus\Delta'} \mathscr{N}^{i,j}$ and for each $(i,j) \in \Gamma_{\mathscr{N}} \setminus \Delta'$, there exist $(i,k),(k,j) \in \Gamma_{\mathscr{N}}$ such that $\mathscr{N}^{i,j} = \mathscr{N}^{i,k}\mathscr{N}^{k,j} = [\mathscr{N}^{i,k},\mathscr{N}^{k,j}] \subseteq [\mathscr{N},\mathscr{N}]$, so that $\mu(AB) = 0$. On the other hand, $Z(\mathscr{N}) = \mathscr{N}^{1,t}$ and by direct computation $\mathscr{N}^{1,t}\mathscr{N} = 0 = \mathscr{N}\mathscr{N}^{1,t}$, so that $\mu(A)B = 0 = A\mu(B)$. Therefore, μ is a centralizer. $\square$

COROLLARY 2.3. *Every linear Lie centralizer of $\mathscr{N}$ over a field F is a centralizer.*

Proof. It is easy to check that a linear mapping $\phi : \mathscr{N} \to \mathscr{N}$ defined by $\phi(A) = \lambda A$ for $A \in \mathscr{N}$ and $\lambda \in \mathbb{F}$ is a centralizer of $\mathscr{N}$. Hence, by Theorem 2.1 and Lemma 2.2, every linear Lie centralizer of $\mathscr{N}$ over a field $\mathbf{F}$ is a centralizer. $\square$

3. Linear maps preserving matrix products

The linear Lie centralizer property (1.1) over a matrix algebra is closely relative to some matrix product preserving properties. Their relationships are much obvious when the algebra consists of block matrices. Here we will determine linear maps that preserve four different types of matrix products. These maps play essential roles in exploring the linear Lie centralizers of $\mathscr{N}$ as well as other algebras of block matrices. They will also be helpful in studying the centralizers of matrix algebras.

In Lemmas 3.1–3.3, let $E^{p,q}_{m\times n}$ denote the $m \times n$ matrix that has the only nonzero entry 1 in the (p,q) position.

LEMMA 3.1. *Suppose $\mathbb{F}$ is an arbitrary field. If $X \in M_m$ and $Y \in M_n$ satisfy that*

$$XA = AY \tag{3.1}$$

for all $A \in M_{mn}$, then $X = \lambda I_m$ and $Y = \lambda I_n$ for some $\lambda \in \mathbb{F}$.

Proof. Suppose $X = (x_{ip}) \in M_m$ and $Y = (y_{qj}) \in M_n$, where $i,p \in [m]$ and $q,j \in [n]$. For any $(i,j) \in [m]\times[n]$, by (3.1),

$$XE^{i,j}_{m\times n} = E^{i,j}_{m\times n}Y. \tag{3.2}$$

Comparing the (i,j) entry of the matrices in (3.2), we get $x_{ii} = y_{jj}$. Similarly, comparing the (p,j) entry for $p \neq i$, we get $x_{pi} = 0$; and comparing the (i,q) entry for $q \neq j$, we get $0 = y_{jq}$. Therefore, $X = \lambda I_m$ and $Y = \lambda I_n$ for some $\lambda \in \mathbb{F}$. $\square$

LEMMA 3.2. *If linear maps $\phi : M_{m,p} \to M_{m,q}$ and $\varphi : M_{n,p} \to M_{n,q}$ satisfy that*

$$\phi(AB) = A\varphi(B) \qquad \text{for all } A \in M_{m,n},\ B \in M_{n,p},$$

then there is $X \in M_{p,q}$ such that $\phi(C) = CX$ for $C \in M_{m,p}$ and $\varphi(D) = DX$ for $D \in M_{n,p}$.

Proof. For any $j \in [n]$ and $B \in M_{n,p}$,

$$\phi(E^{1,j}_{m\times n}B) = E^{1,j}_{m\times n}\varphi(B).$$

Let $R^1_{m,p}$ denote the subspace of $M_{m,p}$ consisting of matrices with 0's outside of the first row. Similarly for $R^1_{m,q}$. Then for every $j \in [n]$, $R^1_{m,p} = E^{(1,j)}_{m\times n}M_{n,p}$, so that

$$\phi(R^1_{m,p}) = \phi(E^{(1,j)}_{m\times n}M_{n,p}) = E^{(1,j)}_{m\times n}\varphi(M_{n,p}) \subseteq R^1_{m,q}.$$

There exists an $X \in M_{p,q}$ such that the linear transformation $\phi|_{R^1_{m,p}} : R^1_{m,p} \to R^1_{m,q}$ can be expressed as

$$\phi|_{R^1_{m,p}}(T) = TX, \quad \text{for all } T \in R^1_{m,p}.$$

Then for every $B \in M_{n,p}$,

$$E^{(1,j)}_{m\times n}\varphi(B) = \phi(E^{(1,j)}_{m\times n}B) = E^{(1,j)}_{m\times n}BX.$$

Therefore, $\varphi(B) = BX$ for $B \in M_{n,p}$. Hence $\phi(AB) = A\varphi(B) = ABX$ for every $A \in M_{m,n}$ and $B \in M_{n,p}$. The linear combinations of all such AB form $M_{m,p}$. So $\phi(C) = CX$ for all $C \in M_{m,p}$. $\square$

LEMMA 3.3. *If linear maps $\phi : M_{m,p} \to M_{n,p}$ and $\varphi : M_{m,q} \to M_{n,q}$ satisfy that*

$$\phi(BA) = \varphi(B)A \qquad \textit{for all} \quad A \in M_{q,p},\ B \in M_{m,q},$$

then there is $X \in M_{n,m}$ such that $\phi(C) = XC$ for $C \in M_{m,p}$ and $\varphi(D) = XD$ for $D \in M_{m,q}$.

Proof. The proof (omitted) is similar to that of Lemma 3.2. $\square$

4. Proofs of main results

The main goal of this section is to prove Theorem 2.1. We always assume that $t \geqslant 3$ in the following discussion.

4.1. Linear Lie centralizer image locations

First we will give several auxiliary results on the image locations of $f(\mathscr{N}^{i,j})$ for a linear Lie centralizer f and $\mathscr{N}^{i,j} \subseteq \mathscr{N}$. We will observe the following interesting fact: most nonzero blocks of $f(A^{i,j})$ for $A^{i,j} \in \mathscr{N}^{i,j}$ are located on the (i,j)-th and $(1,t)$-th blocks.

The first lemma discusses the linear Lie centralizer image on $\mathscr{N}^{1,2}$ and $\mathscr{N}^{t-1,t}$.

LEMMA 4.1. *Let f be a linear Lie centralizer of $\mathscr{N}$. Then*

$$f(\mathscr{N}^{1,2}) \subseteq \mathscr{N}^{1,2} + \mathrm{Z}(\mathscr{N}), \tag{4.1}$$

$$f(\mathscr{N}^{t-1,t}) \subseteq \mathscr{N}^{t-1,t} + \mathrm{Z}(\mathscr{N}). \tag{4.2}$$

Proof. To prove (4.1), we show that $f(A^{1,2})^{i,j} = 0$ for any $A^{1,2} \in \mathcal{N}^{1,2}$, $(i,j) \in \Gamma_{\mathcal{N}}$ and $(i,j) \notin \{(1,2),(1,t)\}$. Suppose $i > 1$. Then for any $A^{1,i} \in \mathcal{N}^{1,i}$, $[A^{1,2}, A^{1,i}] = 0$, so that

$$0 = f([A^{1,2}, A^{1,i}])^{1,j} = [f(A^{1,2}), A^{1,i}])^{1,j} = -A^{1,i} f(A^{1,2})^{i,j} \tag{4.3}$$

for $A^{1,i} \in \mathcal{N}^{1,i}$. Now we further discuss (4.3) in the following two cases:

- If $i > 2$, $0 = A^{1,i} f(A^{1,2})^{i,j}$ for any $A^{1,i} \in \mathcal{N}^{1,i}$. So $f(A^{1,2})^{i,j} = 0$.
- If $i = 2$, it suffices to show that $f(E^{1,2}_{k,\ell})^{2,j} = 0$ for any $k \in [n_1], \ell \in [n_2]$. Given $E^{1,2}_{k,\ell} \in \mathcal{N}^{1,2}$,

$$0 = E^{1,2}_{k,\ell} f(E^{1,2}_{k,\ell})^{2,j} \tag{4.4}$$

  Comparing the k-th row in the equality (4.4), we see that the ℓ-th row of $f(E^{1,2}_{k,\ell})^{2,j}$ is zero. Since $\ell \in [n_2]$ is arbitrary, $f(E^{1,2}_{k,\ell})^{2,j} = 0$.

Therefore, $f(A^{1,2})^{i,j} = 0$.

Now we show that $f(A^{1,2})^{1,j} = 0$ for $2 < j < t$. Suppose $2 < j < t$. Then for any $A^{j,t} \in \mathcal{N}^{j,t}$,

$$0 = f([A^{1,2}, A^{j,t}])^{1,t} = [f(A^{1,2}), A^{j,t}]^{1,t} = f(A^{1,2})^{1,j} A^{j,t}.$$

Therefore, $f(A^{1,2})^{1,j} = 0$.

The proof of (4.2) is similar. □

Next we consider the linear Lie centralizer image on $\mathcal{N}^{2,3}, \mathcal{N}^{3,4}, \cdots, \mathcal{N}^{t-2,t-1}$.

LEMMA 4.2. *For a linear Lie centralizer f of $\mathcal{N}$ and $1 < i < t-1$,*

$$f(\mathcal{N}^{i,i+1}) \subseteq \mathcal{N}^{i,i+1} + \mathrm{Z}(\mathcal{N}). \tag{4.5}$$

Proof. For any $A^{i,i+1} \in \mathcal{N}^{i,i+1}$, it suffices to prove that the (p,q) block $f(A^{i,i+1})^{p,q} = 0$ for $(p,q) \in \Gamma_{\mathcal{N}}$ and $(p,q) \notin \{(i,i+1),(1,t)\}$. We first show that $f(A^{i,i+1})^{p,q} = 0$ for $(p,q) \in \Gamma_{\mathcal{N}}$, $p \neq i$, $q \neq i+1$, and $(p,q) \neq (1,t)$. Either $p > 1$ or $q < t$. Without loss of generality, suppose $q < t$ (similarly for $p > 1$). Then for any $A^{q,t} \in \mathcal{N}^{q,t}$, $[A^{i,i+1}, A^{q,t}] = 0$ so that

$$0 = f[(A^{i,i+1}, A^{q,t}]^{p,t} = [f(A^{i,i+1}), A^{q,t}]^{p,t} = f(A^{i,i+1})^{p,q} A^{q,t}.$$

Therefore $f(A^{i,i+1})^{p,q} = 0$. Which shows that the possibly nonzero blocks of $f(A^{i,i+1})$ are in the i-th block row and the $(i+1)$-th block column .

Next we show that $f(A^{i,i+1})^{p,i+1} = 0$ for $1 \leqslant p < i$ and $f(A^{i,i+1})^{i,q} = 0$ for $i+1 < q \leqslant t$. Suppose $1 \leqslant p < i$. Then for $A^{i+1,r} \in \mathcal{N}^{i+1,r}$, $i+1 < r \leqslant t$,

$$f(A^{ir})^{pr} = f[A^{i,i+1}, A^{i+1,r}]^{p,r} = [A^{i,i+1}, f(A^{i+1,r})]^{p,r} = 0. \tag{4.6}$$

By (4.6), we get

$$0 = f(A^{ir})^{p,r} = f[A^{i,i+1}, A^{i+1,r}]^{p,r} = [f(A^{i,i+1}), A^{i+1,r}]^{p,r} = f(A^{i,i+1})^{p,i+1} A^{i+1,r}.$$

Therefore $f(A^{i,i+1})^{p,i+1} = 0$.
Similarly, $f(A^{i,i+1})^{i,q} = 0$ for $i+1 < q \leqslant t$. □

Now we consider the linear Lie centralizer image on the other $\mathscr{N}^{i,j}$.

LEMMA 4.3. *For a linear Lie centralizer f of $\mathscr{N}$ and $i,j \in [t]$ and $j > i+1$, the image $f(\mathscr{N}^{i,j})$ satisfies that*

$$f(\mathscr{N}^{i,j}) \subseteq \mathscr{N}^{ij}. \tag{4.7}$$

Proof. Let $j = i+k, k \geqslant 2$. We prove (4.7) by induction on k.

1. $k = 2$: $\mathscr{N}^{i,i+2} = \mathscr{N}^{i,i+1}\mathscr{N}^{i+1,i+2} = [\mathscr{N}^{i,i+1}, \mathscr{N}^{i+1,i+2}]$. For $A^{i,i+1} \in \mathscr{N}^{i,i+1}$ and $A^{i+1,i+2} \in \mathscr{N}^{i+1,i+2}$, according to Lemmas 4.1 and 4.2,
$$f[A^{i,i+1}, A^{i+1,i+2}] = [f(A^{i,i+1}), A^{i+1,i+2}] = f(A^{i,i+1})^{i,i+1}A^{i+1,i+2} \in \mathscr{N}^{i,i+2}.$$
Hence $k = 2$ is done.

2. $k > 2$: Suppose (4.7) holds for all $\ell < k$. Now $\mathscr{N}^{i,i+k} = \mathscr{N}^{i,i+2}\mathscr{N}^{i+2,i+k} = [\mathscr{N}^{i,i+2}, \mathscr{N}^{i+2,i+k}]$. For any $A^{i,i+2} \in \mathscr{N}^{i,i+2}$ and $A^{i+2,i+k} \in \mathscr{N}^{i+2,i+k}$,
$$f[A^{i,i+2}, A^{i+2,i+k}] = [f(A^{i,i+2}), A^{i+2,i+k}] = f(A^{i,i+2})^{i,i+2}A^{i+2,i+k} \in \mathscr{N}^{i,i+k}$$
where the last relation is due to induction hypothesis, Lemma 4.1, and Lemma 4.2. So (4.7) holds for k.

Overall, we have proved (4.7) for all k. □

The above lemmas determine all possibly nonzero blocks of $f(A^{i,j})$ for a linear Lie centralizer f and $A^{i,j} \in \mathscr{N}^{i,j} \subseteq \mathscr{N}$. The next goal is to describe the f-images on these blocks.

LEMMA 4.4. *Let f be a linear Lie centralizer of $\mathscr{N}$. Then there exist $\lambda \in \mathbb{F}$ such that*

$$f(A^{p,r})^{p,r} = \lambda A^{p,r} \qquad \textit{for all } A^{p,r} \in \mathscr{N}^{p,r} \subseteq \mathscr{N}, \tag{4.8}$$

Proof. For any $1 \leqslant p < q < r \leqslant t$, $A^{p,q} \in \mathscr{N}^{p,q}$ and $A^{q,r} \in \mathscr{N}^{q,r}$,

$$f(A^{p,q}A^{q,r})^{p,r} = f([A^{p,q}, A^{q,r}])^{p,r} = [f(A^{p,q}), A^{q,r}]^{p,r} = f(A^{p,q})^{p,q}A^{q,r}.$$

Applying Lemma 3.3, there exist $X^{p,p} \in M_n^{p,p}$ such that

$$\begin{aligned} f(A^{p,q})^{p,q} &= X^{p,p}A^{p,q} \qquad \text{for all } A^{p,q} \in \mathscr{N}^{p,q}, \\ f(A^{p,r})^{p,r} &= X^{p,p}A^{p,r} \qquad \text{for all } A^{p,r} \in \mathscr{N}^{p,r}. \end{aligned} \tag{4.9}$$

Since f is a linear Lie centralizer, for any $1 \leqslant p < q < r \leqslant t$, $A^{p,q} \in \mathscr{N}^{p,q}$ and $A^{q,r} \in \mathscr{N}^{q,r}$,

$$f(A^{p,q}A^{q,r})^{p,r} = f([A^{p,q}, A^{q,r}])^{p,r} = [A^{p,q}, f(A^{q,r})]^{p,r} = A^{p,q}f(A^{q,r})^{q,r}.$$

Applying Lemma 3.2 , there exist $X^{r,r} \in M_n^{r,r}$ such that

$$f(A^{q,r})^{q,r} = A^{q,r}X^{r,r} \qquad \text{for all } A^{q,r} \in \mathcal{N}^{p,q},$$
$$f(A^{p,r})^{p,r} = A^{p,r}X^{r,r} \qquad \text{for all } A^{p,r} \in \mathcal{N}^{p,r}. \tag{4.10}$$

By (4.9) and (4.10), we have

$$X^{p,p}A^{p,r} = A^{p,r}X^{r,r} \qquad \text{for } A^{p,r} \in \mathcal{N}^{p,r}.$$

Applying Lemma 3.1, there exists $\lambda \in \mathbb{F}$ such that $X^{p,p} = \lambda I^{p,p}$ and $X^{r,r} = \lambda I^{r,r}$. Therefore, $f(A^{p,r})^{p,r} = \lambda A^{p,r}$ for $A^{p,r} \in \mathcal{N}^{p,r} \subseteq \mathcal{N}$. □

4.2. Proof of Theorem 2.1

We are ready to prove our main result.

Proof of Theorems 2.1. By Lemma 4.4, there exists $\lambda \in \mathbb{F}$ such that

$$f(A^{i,j})^{i,j} = \lambda A^{i,j} \quad \text{for } A^{i,j} \in \mathcal{N}^{i,j} \subseteq \mathcal{N}.$$

Define $f_0 := f - \lambda I_n$. Thus f_0 is a linear Lie centralizer. Then (4.8) implies that $f_0(A^{i,j})^{i,j} = 0$ for all $A^{i,j} \in \mathcal{N}^{i,j} \subseteq \mathcal{N}$. By Lemma 4.1, 4.2, and 4.3, $f_0(A^{i,j}) = 0$ for $(i,j) \in \Gamma_{\mathcal{N}} \setminus \{(i,i+1) : i \in [t-1]\}$ and the only non-zero block of $f_0(A^{i,i+1})$, $i \in [t-1]$ is the $(1,t)$-th block.

Define a linear map $\delta : \mathcal{N} \to \mathcal{N}$ such that for $A \in \mathcal{N}$,

$$\delta(A) := \sum_{i=1}^{t-1} f_0(A^{i,i+1})^{1,t} = \sum_{i=1}^{t-1} f(A^{i,i+1})^{1,t}.$$

Then $\delta(A) \in \mathrm{Z}(\mathcal{N})$ and $\delta[\mathcal{N},\mathcal{N}] = 0$. Now we get a new linear Lie centralizer

$$f_1 := f_0 - \delta = (f - \lambda I_n) - \delta,$$

where $f_1(A^{i,j}) = 0$ for $A^{i,j} \in \mathcal{N}^{i,j} \subseteq \mathcal{N}$. Therefore

$$f(A) = \lambda A + \delta(A)$$

for $A \in \mathcal{N}$. Hence Theorem 2.1 is proved. □

REFERENCES

[1] DRISS AIAT HADJ AND HADJ AHMED, *Non-additive Lie centralizer of strictly upper triangular matrices*, Extracta Mathematics, pages 77–83, 02, 2019.

[2] JORDAN BOUNDS, *Commuting maps over the ring of strictly upper triangular matrices*, Linear Algebra and its Applications, 507: 132–136, 2016.

[3] M. BRESAR *Centralizing Mappings and Derivations in Prime Rings*, Journal of Algebra, 156 (2): 385–394, 1993.

[4] AJDA FOŠNER AND WU JING, *Lie centralizers on triangular rings and nest algebras*, Advances in Operator Theory, 4, 07, 2018.

[5] PRAKASH GHIMIRE AND HUAJUN HUANG, *Derivations of the Lie algebra of strictly block upper triangular matrices*, 2016.

[6] PRAKASH GHIMIRE AND HUAJUN HUANG, *Lie Triple Derivations of the Lie Algebra of Dominant Block Upper Triangular Matrices*, Algebra Colloquium, 25 (03): 475–492, 2018.

[7] J. GHOMANJANI AND M. A. BAHMANI, *A note on Lie centralizer maps*, Palest. J. Math., 7 (2): 468–471, 2018.

[8] G. HOCHSCHILD, *Cohomology and representations of associative algebras*, Duke Math. J., 14 (4): 921–948, 12, 1947.

[9] B. E. JOHNSON, *Centralisers on Certain Topological Algebras*, Journal of the London Mathematical Society, s1-39 (1): 603–614, 1964.

[10] B. E. JOHNSON, *An Introduction to the Theory of Centralizers*, Proceedings of the London Mathematical Society, s3-14 (2): 299–320, 1964.

[11] SHIKUN OU, DENGYIN WANG AND RUIPING YAO, *Derivations of the Lie algebra of strictly upper triangular matrices over a commutative ring*, Linear Algebra Appl., 424 (2–3): 378–383, 2007.

[12] JOSO VUKMAN, *An identity related to centralizers in semiprime rings*, Commentationes Mathematicae Universitatis Carolinae, 40 (3): 447–456, 1999.

[13] JOSO VUKMAN, *Identities related to derivations and centralizers on standard operator algebras*, Taiwanese Journal of Mathematics, 11 (1): 255–265, 2007.

[14] JOSO VUKMAN AND IRENA KOSI-ULBL, *On centralizers of semiprime rings*, Aequationes Mathematicae, 66 (3): 277–283, Dec 2003.

[15] HENG-TAI WANG AND QING-GUO LI, *Lie triple derivation of the Lie algebra of strictly upper triangular matrix over a commutative ring*, Linear Algebra Appl., 430 (1): 66–77, 2009.

[16] QIONG WEI AND PENGTONG LI, *Centralizers of J-subspace lattice algebras*, Linear Algebra and its Applications, 426 (1): 228–237, 2007.

[17] BORUT ZALAR, *On centralizers of semiprime rings*, Commentationes Mathematicae Universitatis Carolinae, 32 (4): 609–614, 1991

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