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C–SELFADJOINTNESS OF THE PRODUCT OF A COMPOSITION OPERATOR AND A MAXIMAL DIFFERENTIATION OPERATOR

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Abstract. Let φ be an automorphism of $\mathbb{D}$. In this paper, we consider the operator $C_\varphi D_{\psi_0,\psi_1}$ on the Hardy space H^2 which is the products of composition and the maximal differential operator. We characterize these operators which are C-selfadjoint with respect to some conjugations C. Moreover, we find all hermitian operators $C_\varphi D_{\psi_0,\psi_1}$, when φ is a rotation.

1. Introduction

The set of real numbers and the set of complex numbers will be denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The Hardy space H^2 is defined as the set of all analytic functions in the unit disk $\mathbb{D}$ for which

$$\|f\|^2 = \sup_{0\leqslant r<1}\left(\int_0^\pi |f(re^{i\theta})|^2\frac{d\theta}{2\pi}\right) < \infty.$$

The Hardy space H^2 is a Hilbert space with the inner product

$$\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})}d\theta.$$

The space H^∞ denotes the set of all bounded analytic functions on $\mathbb{D}$, with $\|f\|_\infty = \sup\{|f(z)| : z\in\mathbb{D}\}$.

For every $w\in\mathbb{D}$ and each non-negative integer $n\geqslant 0$, let $K_w^{[n]}$ denote the unique function in H^2 that $\langle f,K_w^{[n]}\rangle = f^{(n)}(w)$ for each $f\in H^2$; for convenience, we use the notation K_w when $n=0$. The reproducing kernel function K_w in H^2 for a point w in the unit disk is given by $K_w(z) = \frac{1}{1-\overline{w}z}$, with $\|K_w\|^2 = \frac{1}{1-|w|^2}$. We can write $K_w^{[n]}(z) = \frac{d^n}{d\overline{w}^n}k(\overline{w}z)$, where $k(z) = \sum_{j=0}^\infty z^j$.

Let φ be an analytic self-map of $\mathbb{D}$; the *composition operator* with symbol φ is defined by $C_\varphi f = f\circ\varphi$. It is well-known that every composition operator C_φ is

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bounded on H^2 (see [3, Corollary 3.7]). For an analytic function ψ on $\mathbb{D}$, the *weighted composition operator* $C_{\psi,\varphi}$ is defined by the rule $C_{\psi,\varphi}(f) = \psi \cdot f \circ \varphi$.

Let $u \in L^\infty(\partial\mathbb{D})$. The *Toeplitz operator* T_u on H^2 is defined as $T_u f = P(uf)$, where P denotes the orthogonal projection from L^2 onto H^2. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi \in H^\infty$. We have the useful formulas

$$(T_\psi C_\varphi)^* K_w = \overline{\psi(w)} K_{\varphi(w)} \tag{1}$$

and

$$(T_\psi C_\varphi)^* K_w^{[1]} = \overline{\psi(w)\varphi'(w)} K_{\varphi(w)}^{[1]} + \overline{\psi'(w)} K_{\varphi(w)}. \tag{2}$$

Let $\varphi(z) = (az+b)/(cz+d)$ be a linear-fractional self-map of $\mathbb{D}$, where $ad - bc \neq 0$. Then $\sigma(z) = (\overline{a}z - \overline{c})/(-\overline{b}z + \overline{d})$ maps $\mathbb{D}$ into itself, $g(z) = (-\overline{b}z + \overline{d})^{-1}$ and $h(z) = cz + d$ are in H^∞. Cowen in [2] proved that $C_\varphi^* = T_g C_\sigma T_h^*$. The maps σ, g and h are called the Cowen auxiliary functions.

A bounded operator T on a complex Hilbert space H is said to be a *complex symmetric operator* if there exists a conjugation C (an isometric, antilinear and involution) such that $CT^*C = T$. In this paper, we use the symbol J for the special conjugation that $(Jf)(z) = \overline{f(\overline{z})}$ for each analytic function f. The study of complex symmetric operator class was initially addressed by Garcia and Putinar (see [7] and [8]) and has been noticed by many researchers (see also [9]). Many authors have studied complex symmetric composition operators and weighted composition operators (see [1], [4], [6], [12], [13]).

Let H be a Hilbert space. The domain of an unbounded linear operator T is denoted by $\operatorname{dom}(T)$. For two unbounded operators A, B, the notation $A \preceq B$ means that A is a restriction of B on $\operatorname{dom}(A)$, namely $\operatorname{dom}(A) \subseteq \operatorname{dom}(B)$ and $Ax = Bx$ for every $x \in \operatorname{dom}(A)$. Let $T : \operatorname{dom}(T) \subseteq H \to H$ be a closed, densely defined, linear operator. For a conjugation C, we say that T is C-*symmetric* if $T \preceq CT^*C$ and C-*selfadjoint* if $T = CT^*C$ (see [15]). Let us emphasize that $T = CT^*C$ carries with it the requirement that $\operatorname{dom}(T) = \operatorname{dom}(CT^*C)$.

Consider the formal differential expression of the form

$$E(\psi_0, \psi_1) f(z) = \psi_0(z) f(z) + \psi_1(z) f'(z)$$

for each $f \in H^2$, where $\psi_0, \psi_1 \in H^\infty$. We define the *maximal differential operator* D_{ψ_0,ψ_1} as follows

$$\operatorname{dom}(D_{\psi_0,\psi_1}) = \{f \in H^2 : E(\psi_0,\psi_1) f \in H^2\} \qquad D_{\psi_0,\psi_1} f = E(\psi_0,\psi_1) f.$$

The maps ψ_0, ψ_1 are called the symbols of the operator D_{ψ_0,ψ_1}. In particular, if $\psi_0 \equiv 0$ and $\psi_1 \equiv 1$, then D_{ψ_0,ψ_1} is the differentiation operator and it is denoted by D. It is not hard to see that the differentiation operator D is unbounded on the Hardy space. Ohno [14] determined that when $C_\varphi D$ is bounded and compact on the Hardy space. Recently the second author and Hammond [5] have obtained the adjoint, norm and spectrum of some operators $C_\varphi D$ on the Hardy space.

For some conjugation C, C-selfadjoint maximal differential operators have been investigated by the third author and Putinar (see [10] and [11]). In this paper, we will only be considering $C_\varphi D_{\psi_0,\psi_1}$ with $\psi_0,\psi_1 \in H^\infty$ and the map φ is an automorphism of $\mathbb{D}$; that is, $\varphi(z) = \lambda \frac{p-z}{1-\overline{p}z}$, when $p \in \mathbb{D}$ and $|\lambda| = 1$. Note that for $\psi_0,\psi_1 \in H^\infty$, we get

$$C_\varphi D_{\psi_0,\psi_1} = C_\varphi(T_{\psi_0} + T_{\psi_1}D) = T_{\psi_0\circ\varphi}C_\varphi + T_{\psi_1\circ\varphi}C_\varphi D.$$

Since $T_{\psi_0\circ\varphi}C_\varphi$ is a bounded operator and $\psi_1 \circ \varphi \in H^\infty$, one can easily see that $\mathrm{dom}(C_\varphi D_{\psi_0,\psi_1}) \supseteq \mathrm{dom}(C_\varphi D)$. Ohno [14] showed that if φ has a finite angular derivative at any point on $\partial\mathbb{D}$, then $C_\varphi D$ cannot be bounded. Also if φ is an automorphism of $\mathbb{D}$ and ψ_1 is not the zero function, then $C_\varphi D_{\psi_0,\psi_1}$ is an unbounded operator (note that $\|T_{\psi_1\circ\varphi}C_\varphi D(z^n)\| = n\|\psi_1 \circ \varphi\|$ for any positive integer n).

In this paper, we consider the unbounded operator $C_\varphi D_{\psi_0,\psi_1}$, when φ is an automorphism and $\psi_0,\psi_1 \in H^\infty$. The goal of Section 2 is to obtain information about $C_\varphi D_{\psi_0,\psi_1}$ which will be needed in the sequel.

In Section 3, we give a necessary and sufficient condition for $C_\varphi D_{\psi_0,\psi_1}$ to be C-selfadjoint for some conjugation C.

In Section 4, we investigate the action of the adjoint of $C_\varphi D_{\psi_0,\psi_1}$ on the arbitrary element $f \in \mathrm{dom}(C_\varphi D_{\psi_0,\psi_1})^*$. Then we identify what forms ψ_0,ψ_1 and λ must take in order that $C_{\lambda z}D_{\psi_0,\psi_1}$ be hermitian.

2. Some properties

In this section, we state the following basic observations which are necessary for our main results. First, we show that $C_\varphi D_{\psi_0,\psi_1}$ is densely defined.

REMARK 2.1. Let φ be an analytic self-map of $\mathbb{D}$ and $\psi_0,\psi_1 \in H^\infty$. We claim that $K_w \in \mathrm{dom}(C_\varphi D_{\psi_0,\psi_1})$. Since $\mathrm{dom}(C_\varphi D_{\psi_0,\psi_1}) \supseteq \mathrm{dom}(C_\varphi D)$, it suffices to show that $K_w \in \mathrm{dom}(C_\varphi D)$. It is easy to see that $K'_w(z) = \sum_{n=1}^\infty n(\overline{w})^n z^{n-1}$. It is not hard to see that $\sum_{n=1}^\infty n^2|w|^{2n} < \infty$ for each $|w| < 1$. Then $K'_w \in H^2$ and so $C_\varphi K'_w \in H^2$. Hence $K_w \in \mathrm{dom}(C_\varphi D_{\psi_0,\psi_1})$. Since the span of the reproducing kernel functions is dense in H^2, $C_\varphi D_{\psi_0,\psi_1}$ is densely defined.

In the following lemma, we investigate the action of the adjoint of $C_\varphi D_{\psi_0,\psi_1}$ on the reproducing kernel functions.

LEMMA 2.2. *Let φ be an analytic self-map of $\mathbb{D}$. For every $w \in \mathbb{D}$ and non-negative integer m, $K_w^{[m]} \in dom(C_\varphi D_{\psi_0,\psi_1})^*$. Moreover,*

$$(C_\varphi D_{\psi_0,\psi_1})^* K_w = \overline{\psi_0(\varphi(w))}K_{\varphi(w)} + \overline{\psi_1(\varphi(w))}K^{[1]}_{\varphi(w)} \tag{3}$$

and

$$(C_\varphi D_{\psi_0,\psi_1})^* K_w^{[1]} = \overline{\varphi'(w)}\Big(\overline{\psi_0'(\varphi(w))}K_{\varphi(w)} + [\overline{\psi_0(\varphi(w))} + \overline{\psi_1'(\varphi(w))}]K^{[1]}_{\varphi(w)} + \overline{\psi_1(\varphi(w))}K^{[2]}_{\varphi(w)}\Big). \tag{4}$$

Proof. We know that $(C_\varphi D_{\psi_0,\psi_1})^* = D^*_{\psi_0,\psi_1}C^*_\varphi$. For every $w \in \mathbb{D}$ and non-negative integer m, it is easy to see that $C^*_\varphi K_w^{[m]}$ is a linear combination of elements $K_{\varphi(w)}, K^{[1]}_{\varphi(w)}$,

$\dots, K^{[m]}_{\varphi(w)}$. Invoking [10, Lemma 3.1], we obtain that $(C_\varphi D_{\psi_0,\psi_1})^* K^{[m]}_w \in H^2$. As we saw in the first section, we have

$$C_\varphi D_{\psi_0,\psi_1} = T_{\psi_0\circ\varphi} C_\varphi + T_{\psi_1\circ\varphi} C_\varphi D.$$

Then $(C_\varphi D_{\psi_0,\psi_1})^* = C^*_\varphi T^*_{\psi_0\circ\varphi} + (C_\varphi D)^* T^*_{\psi_1\circ\varphi}$. Again we infer from [10, Lemma 3.1], (1) and (2) that

$$(C_\varphi D_{\psi_0,\psi_1})^* K_w = \overline{\psi_0(\varphi(w))} K_{\varphi(w)} + \overline{\psi_1(\varphi(w))} K^{[1]}_{\varphi(w)}$$

and

$$\begin{aligned}(C_\varphi D_{\psi_0,\psi_1})^* K^{[1]}_w &= D^*_{\psi_0,\psi_1}(\overline{\varphi'(w)} K^{[1]}_{\varphi(w)}) \\ &= \overline{\varphi'(w)}\Big(\overline{\psi_0'(\varphi(w))} K_{\varphi(w)} + [\overline{\psi_0(\varphi(w))} + \overline{\psi_1'(\varphi(w))}] K^{[1]}_{\varphi(w)} \\ &\quad + \overline{\psi_1(\varphi(w))} K^{[2]}_{\varphi(w)}\Big). \quad \square\end{aligned}$$

The following observation, which is stated in the case where φ is an automorphism of $\mathbb{D}$, can be generalized to any analytic self-map of $\mathbb{D}$ by an argument similar to that used in [10, Proposition 3.2].

REMARK 2.3. Let φ be an automorphism of $\mathbb{D}$. Suppose that $f, g \in H^2$ and that $f_n \in \operatorname{dom}(D_{\psi_0,\psi_1})$, with $f_n \to f$ and $C_\varphi D_{\psi_0,\psi_1} f_n \to g$ as $n \to \infty$. We know $C_{\varphi^{-1}}$ is bounded. Thus, $C_{\varphi^{-1}} C_\varphi D_{\psi_0,\psi_1} f_n \to C_{\varphi^{-1}} g$ as $n \to \infty$. It states that $D_{\psi_0,\psi_1} f_n \to g \circ \varphi^{-1}$ as $n \to \infty$. Because D_{ψ_0,ψ_1} is closed (see [10, Proposition 3.2]), $D_{\psi_0,\psi_1}(f) = g \circ \varphi^{-1}$. Then $C_\varphi D_{\psi_0,\psi_1}(f) = g$ and so the operator $C_\varphi D_{\psi_0,\psi_1}$ is closed.

3. C-selfadjointness

Suppose that U is unitary; that is, $U^*U = UU^* = I$. Assume that U is complex symmetric with conjugation J. By [4, Lemma 2.2], UJ is a conjugation. An analogue of Lemma 3.1 holds for a complex symmetric operator T (see [4, Proposition 2.3]).

LEMMA 3.1. *Let U be unitary and complex symmetric with conjugation WJ, where W is unitary. Then an operator T is WJ-selfadjoint if and only if UT is UWJ-selfadjoint.*

Proof. It is easy to see that T is a closed and densely defined operator if and only if UT is as well. Let T be WJ-selfadjoint. We have

$$UWJ(UT)^*UWJ = UWJT^*U^*UWJ = UT.$$

Then UT is UWJ-selfadjoint.

Conversely, suppose that UT is UWJ-selfadjoint.

$$WJT^*WJ = U^*UWJ(UT)^*UWJ = U^*UT = T.$$

Therefore, T is WJ-selfadjoint. $\square$

In the next theorem, we characterize all J-selfadjoint operators $C_\varphi D_{\psi_0,\psi_1}$.

THEOREM 3.2. *Suppose that φ is an automorphism of $\mathbb{D}$ and D_{ψ_0,ψ_1} is the maximal differential operator, with symbols ψ_0,ψ_1, where ψ_0 is a polynomial and $\psi_1 \in H^\infty$. Let $a,b,c \in \mathbb{C}$. Then $C_\varphi D_{\psi_0,\psi_1}$ is J-selfadjoint if and only if one of the following occurs.*

(a) $\varphi(z) = \mu z$, $\psi_0(z) = a+bz$ and $\psi_1(z) = b\mu + cz + bz^2$, where $|\mu| = 1$.

(b) $\varphi(z) = \frac{p}{\overline{p}}\frac{\overline{p}-z}{1-pz}$, $\psi_0(z) = (a+bz)(1-\overline{p}z)$ and $\psi_1(z) = (d+cz+bz^2)(1-\overline{p}z)$, where $p \in \mathbb{D}$, $p \neq 0$ and $d = -\frac{p}{\overline{p}}b - pc$.

Proof. Let $C_\varphi D_{\psi_0,\psi_1}$ be J-selfadjoint. For each $w \in \mathbb{D}$, Lemma 2.2 implies that $(C_\varphi D_{\psi_0,\psi_1})^* K_w = \overline{\psi_0(\varphi(w))} K_{\varphi(w)} + \overline{\psi_1(\varphi(w))} K^{[1]}_{\varphi(w)}$. Since $C_\varphi D_{\psi_0,\psi_1}$ is J-selfadjoint,

$$J(C_\varphi D_{\psi_0,\psi_1})^*(1) = (C_\varphi D_{\psi_0,\psi_1})(J(1)).$$

It is easy to see that $\psi_0(\varphi(0))K_{\overline{\varphi(0)}} + \psi_1(\varphi(0))K^{[1]}_{\overline{\varphi(0)}} = \psi_0 \circ \varphi$ and so for each $z \in \mathbb{D}$,

$$\frac{\psi_0(\varphi(0))}{1-\varphi(0)z} + \frac{\psi_1(\varphi(0))z}{(1-\varphi(0)z)^2} = \psi_0(\varphi(z)). \tag{5}$$

We infer from (5) that

$$\psi_0(\varphi(0))(1-\varphi(0)\varphi^{-1}(z)) + \psi_1(\varphi(0))\varphi^{-1}(z) = \psi_0(z)(1-\varphi(0)\varphi^{-1}(z))^2. \tag{6}$$

In (6), set $\varphi^{-1}(z) = \lambda\frac{p-z}{1-\overline{p}z}$, where $|\lambda| = 1$ and $p \in \mathbb{D}$. After some computation, we obtain

$$\begin{aligned}\psi_0(\varphi(0))(1-\overline{p}z)(1-\varphi(0)\lambda p + (-\overline{p}+\lambda\varphi(0))z)\\ +(1-\overline{p}z)(\psi_1(\varphi(0))\lambda p - \psi_1(\varphi(0))\lambda z)\\ = \psi_0(z)(1-\lambda\varphi(0)p + (-\overline{p}+\lambda\varphi(0))z)^2.\end{aligned} \tag{7}$$

Since ψ_0 is a polynomial and the left side of (7) is a polynomial of degree at most 2, we conclude that ψ_0 is constant or $-\overline{p}+\lambda p = 0$. We break the proof into two cases.

(i) Suppose that $-\overline{p}+\lambda p = 0$. It shows that $p = 0$ or $\lambda = \frac{\overline{p}}{p}$. If $p = 0$, then $\varphi(z) = -\overline{\lambda}z$ and we have $C_{-\overline{\lambda}z}D_{\psi_0,\psi_1}$ is J-selfadjoint. By Lemma 3.1, D_{ψ_0,ψ_1} is $C_{-\lambda z}J$-selfadjoint. Thus, [10, Theorem 4.4] implies that $\psi_0(z) = a+bz$ and $\psi_1(z) = -b\overline{\lambda} + cz + bz^2$, where $a,b,c \in \mathbb{C}$. Now assume that $p \neq 0$ and $\lambda = \overline{p}/p$. We have $\varphi(z) = \frac{p}{\overline{p}}\frac{\overline{p}-z}{1-pz}$. Since $C_\varphi D_{\psi_0,\psi_1}$ is J-selfadjoint and $C_{\psi_p,\varphi^{-1}}$ is also J-symmetric, where $\psi_p(z) = \frac{(1-|p|^2)^{1/2}}{1-\overline{p}z}$ (see [4, Proposition 2.1]), Lemma 3.1 states that $C_{\psi_p,\varphi^{-1}}C_\varphi D_{\psi_0,\psi_1}$ is $C_{\psi_p,\varphi^{-1}}J$-selfadjoint. It is not hard to see that $C_{\psi_p,\varphi^{-1}}C_\varphi D_{\psi_0,\psi_1} = D_{\psi_p\cdot\psi_0,\psi_p\cdot\psi_1}$. From [10, Theorem 5.6], we get $\psi_p(z)\psi_0(z) = a+bz$ and $\psi_p(z)\psi_1(z) = d+cz+bz^2$, where $a,b,c \in \mathbb{C}$ and $d = \frac{-pb}{\overline{p}} - pc$. Hence

$$\psi_0(z) = \frac{(a+bz)(1-\overline{p}z)}{(1-|p|^2)^{1/2}}$$

and

$$\psi_1(z) = \frac{(d+cz+bz^2)(1-\overline{p}z)}{(1-|p|^2)^{1/2}}.$$

Therefore, the result follows.

(ii) Suppose that $\psi_0 \equiv \alpha$, where $\alpha \in \mathbb{C}$. By (5), for each $z \in \mathbb{D}$, $\alpha(1-\varphi(0)z) + \psi_1(\varphi(0))z = \alpha(1-\varphi(0)z)^2$. It states that

$$\alpha\varphi(0)^2z^2 + (-2\alpha\varphi(0) - \psi_1(\varphi(0)) + \alpha\varphi(0))z = 0 \tag{8}$$

and so by (8), $\alpha = 0$ or $\varphi(0) = 0$ and therefore, in these two cases again by (8), $\psi_1(\varphi(0)) = 0$. First suppose that $\varphi(0) = 0$. Then $\varphi(z) = \mu z$, where $|\mu| = 1$. Since $C_{\mu z}D_{\alpha,\psi_1}$ is J-selfadjoint, by the similar proof which was seen in the proof of Part (i), D_{α,ψ_1} is $C_{\overline{\mu}z}J$-selfadjoint and so $\psi_0 \equiv \alpha$ and $\psi_1(z) = cz$. Now assume that $\alpha = 0$. Since $C_\varphi D_{0,\psi_1}$ is J-selfadjoint, for each $w \in \mathbb{D}$, $J(C_\varphi D_{0,\psi_1})^*K_w = (C_\varphi D_{0,\psi_1})JK_w$ and (3) dictates that

$$\psi_1(\varphi(w))K^{[1]}_{\overline{\varphi(w)}} = \frac{w\psi_1(\varphi(z))}{(1-w\varphi(z))^2}.$$

It shows that

$$\frac{\psi_1(\varphi(w))\varphi^{-1}(z)}{(1-\varphi(w)\varphi^{-1}(z))^2} = \frac{w\psi_1(z)}{(1-wz)^2}. \tag{9}$$

Since $C_\varphi D_{0,\psi_1}$ is J-selfadjoint, we have $J(C_\varphi D_{0,\psi_1})^*z = C_\varphi D_{0,\psi_1}Jz$. Therefore, by the fact that $\psi_1(\varphi(0)) = 0$ and (4), we see that $\varphi'(0)\psi_1'(\varphi(0))K^{[1]}_{\overline{\varphi(0)}} = \psi_1 \circ \varphi$, which implies that

$$\psi_1(z) = \frac{\varphi'(0)\psi_1'(\varphi(0))\varphi^{-1}(z)}{(1-\varphi(0)\varphi^{-1}(z))^2}. \tag{10}$$

Note that $\varphi'(0) \neq 0$ and $\psi_1'(\varphi(0)) \neq 0$, because φ is an automorphism and ψ_1 is not the zero function (if $\psi_1'(\varphi(0)) = 0$, then by (10), $\psi_1 \equiv 0$ and in this case $C_\varphi D_{\psi_0,\psi_1}$ is the zero operator). By (10), we see that for each $w \in \mathbb{D}$,

$$\psi_1(\varphi(w)) = \frac{w\varphi'(0)\psi_1'(\varphi(0))}{(1-\varphi(0)w)^2}. \tag{11}$$

From (9), (10) and (11), for each $z, w \in \mathbb{D}$, we have

$$\frac{w\varphi'(0)\psi_1'(\varphi(0))\varphi^{-1}(z)}{(1-\varphi(0)w)^2(1-\varphi(w)\varphi^{-1}(z))^2} = \frac{w\varphi'(0)\psi_1'(\varphi(0))\varphi^{-1}(z)}{(1-\varphi(0)\varphi^{-1}(z))^2(1-wz)^2}.$$

For $w \neq 0$ and $z \neq p$, we have

$$(1-\varphi(0)\varphi^{-1}(z))^2(1-wz)^2 = (1-\varphi(0)w)^2(1-\varphi(w)\varphi^{-1}(z))^2. \tag{12}$$

Set $\varphi^{-1}(z) = \lambda\frac{p-z}{1-\overline{p}z}$ in (12), we obtain

$$(1-\varphi(0)\lambda p + (-\overline{p}+\varphi(0)\lambda)z)^2(1-wz)^2 = (1-\varphi(0)w)^2(1-\lambda\varphi(w)p + (\varphi(w)\lambda-\overline{p})z)^2 \tag{13}$$

for each $w \neq 0$ and $z \neq p$. The right side of (13) is a polynomial of degree at most 2. Then $p\lambda = \overline{p}$ and so $p = 0$ or $\lambda = \frac{\overline{p}}{p}$. If $p = 0$, then $\varphi(z) = -\overline{\lambda}z$ and by the similar proof which was stated in (i), $\psi_1(z) = cz$. Now assume that $\lambda = \frac{\overline{p}}{p}$. We have $\varphi(z) = \frac{p}{\overline{p}}\frac{\overline{p}-z}{1-pz}$ and $C_\varphi D_{0,\psi_1}$ is J-selfadjoint. We know that $C_{\psi_p,\varphi^{-1}}$ is also J-symmetric, where $\psi_p(z) = \frac{(1-|p|^2)^{1/2}}{1-\overline{p}z}$. Then by Lemma 3.1, $C_{\psi_p,\varphi^{-1}}C_\varphi D_{0,\psi_1}$ is $C_{\psi_p,\varphi^{-1}}J-$ selfadjoint. By the same argument which was stated in (i), $\psi_p(z)\psi_1(z) = d + cz$, where $d = -pc$. Then $\psi_1(z) = \frac{(d+cz)(1-\overline{p}z)}{(1-|p|^2)^{1/2}}$.

Conversely, first suppose that $\varphi(z) = \mu z$, $\psi_0(z) = a + bz$ and $\psi_1(z) = b\mu + cz + bz^2$ that $|\mu| = 1$. One can see that $C_{\varphi^{-1}}C_\varphi D_{\psi_0,\psi_1} = D_{\psi_0,\psi_1}$ is $C_{\overline{\mu}z}J$-selfadjoint (see [10, Theorem 4.4]). We know that $C_{\overline{\mu}z}$ is J-symmetric (see [4, Proposition 2.1]). Thus, by Lemma 3.1, $C_\varphi D_{\psi_0,\psi_1}$ is J-selfadjoint. Now assume that $\varphi(z) = \frac{p}{\overline{p}}\frac{\overline{p}-z}{1-pz}$, $\psi_0(z) = (a+bz)(1-\overline{p}z)$ and $\psi_1(z) = (d + cz + bz^2)(1-\overline{p}z)$, where $p \in \mathbb{D}$, $p \neq 0$ and $d = -\frac{p}{\overline{p}}b - pc$. We know that $C_{\psi_p,\varphi^{-1}}$ is J-symmetric (see [4, Proposition 2.1]). One can see that $C_{\psi_p,\varphi^{-1}}C_\varphi D_{\psi_0,\psi_1} = D_{\psi_p\cdot\psi_0,\psi_p\cdot\psi_1}$ that $\psi_p(z)\psi_0(z) = (1-|p|^2)^{1/2}(a+bz)$ and $\psi_p(z)\psi_1(z) = (1-|p|^2)^{1/2}(d+cz+bz^2)$. By [10, Theorem 5.6], $D_{\psi_p\psi_0,\psi_p\psi_1}$ is $C_{\psi_p,\varphi^{-1}}J$-selfadjoint and so by Lemma 3.1 and [4, Proposition 2.1], $C_\varphi D_{\psi_0,\psi_1}$ is J-selfadjoint. □

In Theorem 3.3, we find operators $C_\varphi D_{\psi_0,\psi_1}$ which are $C_{\lambda z}J$-selfadjoint.

THEOREM 3.3. *Assume that φ is an automorphism of $\mathbb{D}$ and D_{ψ_0,ψ_1} is the maximal differential operator, with symbols ψ_0 and ψ_1, where ψ_0 is a polynomial and $\psi_1 \in H^\infty$. Let $a,b,c \in \mathbb{C}$. Then for $|\lambda| = 1$, $C_\varphi D_{\psi_0,\psi_1}$ is $C_{\lambda z}J$-selfadjoint if and only if one of the following occurs.*

(a) $\varphi(z) = \lambda\mu z$, $\psi_0(z) = a + bz$ and $\psi_1(z) = b\mu + cz + bz^2$, where $|\mu| = 1$.

(b) $\varphi(z) = \frac{p\lambda}{\overline{p}}\frac{\overline{p\lambda}-z}{1-p\lambda z}$, $\psi_0(z) = (a+bz)(1-\overline{p}z)$ and $\psi_1(z) = (d+cz+bz^2)(1-\overline{p}z)$, where $p \in \mathbb{D}$, $p \neq 0$ and $d = \frac{-p}{\overline{p}}b - pc$.

Proof. Suppose that $C_\varphi D_{\psi_0,\psi_1}$ is $C_{\lambda z}J$-selfadjoint. One can easily see that $C_{\overline{\lambda}z}$ is $C_{\lambda z}J$-symmetric. By Lemma 3.1, $C_{\overline{\lambda}z}C_\varphi D_{\psi_0,\psi_1}$ is J-selfadjoint. Thus, $C_{\varphi(\overline{\lambda}z)}D_{\psi_0,\psi_1}$ is J-selfadjoint. The result follows from Theorem 3.2.

Conversely, let $\varphi(z) = \lambda\mu z$, $\psi_0(z) = a + bz$ and $\psi_1(z) = b\mu + cz + bz^2$. We know that $C_{\overline{\lambda}z}$ is J-symmetric. We get $C_{\overline{\lambda}z}C_\varphi D_{\psi_0,\psi_1} = C_{\mu z}D_{\psi_0,\psi_1}$. By the proceeding theorem $C_{\mu z}D_{\psi_0,\psi_1}$ is J-selfadjoint. Then by Lemma 3.1, $C_\varphi D_{\psi_0,\psi_1}$ is $C_{\lambda z}J$-selfadjoint. Now let φ, ψ_0 and ψ_1 satisfy the hypotheses of Statement (b). It is easy to see that $C_\varphi = C_{\lambda z}C_{\frac{p}{\overline{p}}\frac{\overline{p}-z}{1-pz}}$. We have $C_{\frac{p}{\overline{p}}\frac{\overline{p}-z}{1-pz}}D_{\psi_0,\psi_1}$ is J-selfadjoint by the preceding theorem. Hence by Lemma 3.1, $C_\varphi D_{\psi_0,\psi_1}$ is $C_{\lambda z}J$-selfadjoint. □

In the following theorem, we investigate which symbols ψ_0, ψ_1 and φ give rise the C-selfadjointness of operator $C_\varphi D_{\psi_0,\psi_1}$ with conjugation $C_{\psi_q,\varphi_q}J$, where $q \in \mathbb{D}$ is a nonzero number, $\psi_q(z) = \frac{(1-|q|^2)^{1/2}}{1-\overline{q}z}$ and $\varphi_q(z) = \frac{\overline{q}}{q}\frac{q-z}{1-\overline{q}z}$.

THEOREM 3.4. *Let φ be an automorphism of $\mathbb{D}$ and D_{ψ_0,ψ_1} be the maximal differential operator with symbols ψ_0 and ψ_1, where $\psi_1 \in H^\infty$. Suppose that $\frac{\psi_0}{1-q\varphi_q\circ\varphi^{-1}}$ is a polynomial, where $q \in \mathbb{D}$ is a nonzero number. Then $C_\varphi D_{\psi_0,\psi_1}$ is $C_{\psi_q,\varphi_q}J$-selfadjoint if and only if one of the following occurs.*

(a) $\varphi = \mu\varphi_q$, $\psi_0(z) = (a+bz)(1-q\overline{\mu}z)$ and $\psi_1(z) = (b\mu+cz+bz^2)(1-q\overline{\mu}z)$, where $|\mu| = 1$ and $a,b,c \in \mathbb{C}$.

(b) $\varphi = \varphi_{\overline{p}}\circ\varphi_q$, $\psi_0(z) = (a+bz)(1-\overline{p}z)(1-q\varphi_p(z))$ and $\psi_1(z) = (1-q\varphi_p(z))$ $(d+cz+bz^2)(1-\overline{p}z)$, where $a,b,c \in \mathbb{C}$, $p \in \mathbb{D}$, $p \neq 0$ and $d = \frac{-p}{\overline{p}}b - pc$.

Proof. Let $C_\varphi D_{\psi_0,\psi_1}$ be $C_{\psi_q,\varphi_q}J$-selfadjoint. By [4, Proposition 2.1], C_{ψ_q,φ_q} is J-symmetric. Then it is easy to see that $C^*_{\psi_q,\varphi_q}$ is $C_{\psi_q,\varphi_q}J$-symmetric. Lemma 3.1 implies that $C^*_{\psi_q,\varphi_q}C_\varphi D_{\psi_0,\psi_1}$ is J-selfadjoint. By the Cowen adjoint formula, we have

$$C^*_{\psi_q,\varphi_q}C_\varphi D_{\psi_0,\psi_1} = (1-|q|^2)^{1/2}C_{\varphi\circ\varphi_q^{-1}}D_{\psi_0\cdot(g\circ\varphi_q\circ\varphi^{-1}),\psi_1\cdot(g\circ\varphi_q\circ\varphi^{-1})},$$

where $g(z) = \frac{1}{1-qz}$. Theorem 3.2 implies that one of the following occurs.

(i) $\varphi(\varphi_q^{-1}(z)) = \mu z$, $\psi_0(z)g(\varphi_q(\varphi^{-1}(z))) = a+bz$ and $\psi_1(z)g(\varphi_q(\varphi^{-1}(z))) = b\mu+cz+bz^2$, where $|\mu| = 1$.

(ii) $\varphi(\varphi_q^{-1}(z)) = \frac{p}{\overline{p}}\frac{\overline{p}-z}{1-pz}$, $\psi_0(z)g(\varphi_q(\varphi^{-1}(z))) = (a+bz)(1-\overline{p}z)$ and $\psi_1(z)$ $g(\varphi_q(\varphi^{-1}(z))) = (d+cz+bz^2)(1-\overline{p}z)$, where $p \in \mathbb{D}$, $p \neq 0$ and $d = \frac{-p}{\overline{p}}b - pc$.

If (i) occurs, then $\varphi(z) = \mu\frac{\overline{q}}{q}\frac{q-z}{1-\overline{q}z}$ and so $\psi_0(z)g(\varphi_q(\varphi^{-1}(z))) = \frac{\psi_0(z)}{1-q\overline{\mu}z}$. It shows that $\psi_0(z) = (a+bz)(1-q\overline{\mu}z)$. Also $\psi_1(z)g(\varphi_q(\varphi^{-1}(z))) = \frac{\psi_1(z)}{1-q\overline{\mu}z}$ and thus, $\psi_1(z) = (b\mu+cz+bz^2)(1-q\overline{\mu}z)$. Now assume that (ii) holds. We have $\varphi(\varphi_q^{-1}(z)) = \frac{p}{\overline{p}}\frac{\overline{p}-z}{1-pz}$. Then $\varphi(z) = \frac{p}{\overline{p}}\frac{\overline{p}-\varphi_q(z)}{1-p\varphi_q(z)}$ and $\psi_0(z)g(\varphi_q(\varphi^{-1}(z))) = \frac{\psi_0(z)}{1-q\varphi_p(z)}$. Hence $\psi_0(z) = (1-q\varphi_p(z))(a+bz)(1-\overline{p}z)$. By the same argument, $\psi_1(z) = (1-q\varphi_p(z))(d+cz+bz^2)(1-\overline{p}z)$.

Conversely, the result follows from Theorem 3.2 and the same idea stated in Theorem 3.3. □

In the next, two examples of Theorem 3.4 are given.

Example 3.5. (a) Suppose that $\varphi(z) = i\frac{\frac{1}{2}-z}{1-\frac{1}{2}z}$, $\psi_0(z) = (az+b)(1+\frac{i}{2}z)$ and $\psi_1(z) = (bi+cz+bz^2)(1+\frac{i}{2}z)$, where $a,b,c \in \mathbb{C}$. We have $\varphi = i\varphi_{\frac{1}{2}}$ and

$$1-\frac{1}{2}\varphi_{\frac{1}{2}}\circ\varphi^{-1} = 1-\frac{1}{2}\varphi_{\frac{1}{2}}\circ(i\varphi_{\frac{1}{2}})^{-1} = 1+\frac{i}{2}z.$$

Then $\frac{\psi_0}{1-\frac{1}{2}\varphi_{\frac{1}{2}}\circ\varphi^{-1}}$ is a polynomial. Theorem 3.4(a) dictates that $C_\varphi D_{\psi_0,\psi_1}$ is $C_{\psi_{\frac{1}{2}},\varphi_{\frac{1}{2}}}J$-selfadjoint.

(b) Suppose that $\varphi(z) = \frac{1+\frac{3i}{2}+(\frac{i}{2}-3)z}{3+\frac{i}{2}+(\frac{3i}{2}-1)z}$, $\psi_0(z) = (az+b)(1-\frac{1}{3}z)(1-\frac{i(1-3z)}{6-2z})$ and $\psi_1(z) = (1-\frac{i(1-3z)}{6-2z})(-b-\frac{1}{3}c+cz+bz^2)(1-\frac{1}{3}z)$, where $a,b,c \in \mathbb{C}$. It is easy to see that $\varphi = \varphi_{\frac{1}{3}} \circ \varphi_{\frac{i}{2}}$. We obtain

$$1 - \frac{i}{2}\varphi_{\frac{i}{2}} \circ \varphi^{-1} = 1 - \frac{i}{2}(\varphi_{\frac{i}{2}} \circ \varphi_{\frac{i}{2}}^{-1} \circ \varphi_{\frac{1}{3}}^{-1}) = 1 - \frac{i}{2}\varphi_{\frac{1}{3}} = 1 - \frac{i(1-3z)}{6-2z}.$$

Then $\frac{\psi_0}{1-\frac{i}{2}\varphi_{\frac{i}{2}}\circ\varphi^{-1}}$ is a polynomial. Theorem 3.4(b) implies that $C_\varphi D_{\psi_0,\psi_1}$ is $C_{\psi_{\frac{i}{2}},\varphi_{\frac{i}{2}}}J$-selfadjoint.

4. Hermiticity

A densely defined operator T is *hermitian* if $T^* = T$. In this section, we characterize hermitian operator $C_\varphi D_{\psi_0,\psi_1}$, when φ is a rotation of the unit disk. In the next proposition, for each $f \in \mathrm{dom}(C_{\lambda u} D_{\psi_0,\psi_1})^*$, we find $(C_{\lambda u} D_{\psi_0,\psi_1})^*(f)$.

PROPOSITION 4.1. *Let D_{ψ_0,ψ_1} be the maximal differential operator with symbols*

$$\psi_0(z) = a + bz$$

and

$$\psi_1(z) = d + cz + \alpha b z^2,$$

where $a,b,c,d,\alpha \in \mathbb{C}$. Then for $\lambda \in \partial\mathbb{D}$ and a nonzero point $z \in \mathbb{D}$,

$$\begin{aligned}(C_{\lambda u} D_{\psi_0,\psi_1})^*(f)(z) = &\ (\overline{a} + \overline{d}z) f(\overline{\lambda}z) + \overline{\lambda}(\overline{d}z^2 + \overline{b} + \overline{c}z) f'(\overline{\lambda}z) \\ &+ \overline{b}(\overline{\alpha}-1)\overline{\lambda} f'(\overline{\lambda}z) - \overline{b}(\overline{\alpha}-1)\left(\frac{f(\overline{\lambda}z) - f(0)}{z}\right),\end{aligned}$$

where $f \in dom(C_{\lambda u} D_{\psi_0,\psi_1})^*$.

Proof. Let $f \in \mathrm{dom}(C_{\lambda u} D_{\psi_0,\psi_1})^*$ and z,u be arbitrary points in $\mathbb{D}$ that $z \neq 0$.

$$\begin{aligned} D_{\psi_0,\psi_1} K_z(u) &= \frac{a+bu}{1-\overline{z}u} + \frac{(d+cu+\alpha b u^2)\overline{z}}{(1-\overline{z}u)^2} \\ &= \frac{a + d\overline{z}}{1-\overline{z}u} + \frac{-d\overline{z}(1-\overline{z}u) + d\overline{z}}{(1-\overline{z}u)^2} + \frac{bu(1-\overline{z}u) + \alpha b \overline{z} u^2}{(1-\overline{z}u)^2} + \frac{c\overline{z}u}{(1-\overline{z}u)^2} \\ &= (a+d\overline{z})K_z(u) + d\overline{z}^2 K_z^{[1]}(u) + bK_z^{[1]}(u) + \frac{-b\overline{z}u^2 + \alpha b \overline{z} u^2}{(1-\overline{z}u)^2} + c\overline{z}K_z^{[1]}(u) \\ &= (a+d\overline{z})K_z(u) + (d\overline{z}^2 + b + c\overline{z})K_z^{[1]}(u) + b\overline{z}(\alpha-1)uK_z^{[1]}(u). \end{aligned} \tag{14}$$

We have

$$\begin{aligned} \langle f(\overline{\lambda}u), b\overline{z}(\alpha-1)uK_z^{[1]}(u)\rangle &= \overline{b}z(\overline{\alpha}-1)\langle f(\overline{\lambda}u), uK_z^{[1]}(u)\rangle \\ &= \overline{b}z(\overline{\alpha}-1)\langle T_u^* f(\overline{\lambda}u), K_z^{[1]}(u)\rangle \\ &= \overline{b}z(\overline{\alpha}-1)\langle \frac{f(\overline{\lambda}u) - f(0)}{u}, K_z^{[1]}(u)\rangle \end{aligned}$$

$$= \overline{b}z(\overline{\alpha}-1)\frac{\overline{\lambda}f'(\overline{\lambda}z)z - f(\overline{\lambda}z)+f(0)}{z^2}$$
$$= \overline{b}z(\overline{\alpha}-1)\left(\frac{\overline{\lambda}f'(\overline{\lambda}z)}{z} + \frac{f(0)-f(\overline{\lambda}z)}{z^2}\right)$$
$$= \overline{b}(\overline{\alpha}-1)\left(\overline{\lambda}f'(\overline{\lambda}z) + \frac{f(0)-f(\overline{\lambda}z)}{z}\right). \tag{15}$$

Then (14) and (15) imply that

$$\begin{aligned}(C_{\lambda u}D_{\psi_0,\psi_1})^* f(z) &= \langle C_{\overline{\lambda}u} f, D_{\psi_0,\psi_1}K_z\rangle \\ &= (\overline{a}+\overline{d}z)f(\overline{\lambda}z) + \overline{\lambda}(\overline{d}z^2+\overline{b}+\overline{c}z)f'(\overline{\lambda}z) \\ &\quad + \overline{b}(\overline{\alpha}-1)\overline{\lambda}f'(\overline{\lambda}z) - \overline{b}(\overline{\alpha}-1)\left(\frac{f(\overline{\lambda}z)-f(0)}{z}\right)\end{aligned}$$

and the result follows. □

Let z be a complex number. We have $z=|z|e^{i\theta}$, where $0\leqslant\theta<2\pi$ and we denote θ by $\arg(z)$. In particular, we set $\arg(0)=0$. Now we use Proposition 4.1 to establish the main result of this section.

THEOREM 4.2. *Let $\lambda\in\partial\mathbb{D}$ and D_{ψ_0,ψ_1} be the maximal differential operator with symbols ψ_0 and ψ_1 that $\psi_0,\psi_1\in H^\infty$. The operator $C_{\lambda z}D_{\psi_0,\psi_1}$ is hermitian if and only if one the following occurs.*

(a) $\lambda=1$, $\psi_0(z)=a+bz$ and $\psi_1(z)=\overline{b}+cz+bz^2$, where $b\in\mathbb{C}$ and $a,c\in\mathbb{R}$

(b) $\lambda=-1$, $\psi_0(z)=a+bz$ and $\psi_1(z)=-\overline{b}+cz+bz^2$, where $a,c\in\mathbb{R}$ and $b\in\mathbb{C}$.

Proof. Let $C_{\lambda z}D_{\psi_0,\psi_1}$ be hermitian. Then $C_{\lambda z}D_{\psi_0,\psi_1}K_w=(C_{\lambda z}D_{\psi_0,\psi_1})^*K_w$ for each $z,w\in\mathbb{D}$. Lemma 2.2 implies that

$$\frac{\psi_0(\lambda z)}{1-\overline{w}\lambda z}+\frac{\psi_1(\lambda z)\overline{w}}{(1-\overline{w}\lambda z)^2} = \frac{\overline{\psi_0(\lambda w)}}{1-\overline{\lambda w}z}+\frac{\overline{\psi_1(\lambda w)}z}{(1-\overline{\lambda w}z)^2}. \tag{16}$$

Letting $w=0$ in (16) gives

$$\psi_0(z)=\overline{\psi_0(0)}+\overline{\psi_1(0)}\lambda z$$

and so $\psi_0(0)$ is a real number. Substitute ψ_0 back into (16) to obtain that for each $w\neq 0$,

$$\begin{aligned}&\frac{\psi_0(0)}{\overline{w}(1-\overline{w}\lambda z)}-\frac{\psi_0(0)}{\overline{w}(1-\overline{\lambda w}z)}+\frac{\psi_1(\lambda z)}{(1-\overline{w}\lambda z)^2}-\frac{\psi_1(0)}{1-\overline{\lambda w}z}\\ &\qquad = \frac{1}{\overline{w}}\left(\frac{\overline{\psi_1(\lambda w)}z}{(1-\overline{\lambda w}z)^2}-\frac{\overline{\psi_1(0)}z}{1-\overline{w}\lambda z}\right).\end{aligned} \tag{17}$$

First, we consider the right side of (17). We obtain

$$\frac{1}{\overline{w}}\left(\frac{\overline{\psi_1(\lambda w)}z}{(1-\overline{\lambda w}z)^2}-\frac{\overline{\psi_1(0)}z}{1-\overline{w}\lambda z}\right)=\frac{z}{\overline{w}}\left(\frac{\overline{\psi_1(\lambda w)}(1-\overline{w}\lambda z)-\overline{\psi_1(0)}(1-\overline{\lambda w}z)^2}{(1-\overline{\lambda w}z)^2(1-\overline{w}\lambda z)}\right)$$
$$=z\left(\overline{\lambda}\frac{\overline{\psi_1(\lambda w)}-\overline{\psi_1(0)}}{\overline{w\lambda}}\frac{1}{(1-\overline{\lambda w}z)^2(1-\overline{w}\lambda z)}\right)$$
$$+z\left(\frac{-\overline{\psi_1(\lambda w)}\overline{w}\lambda z-\overline{\psi_1(0)}(\overline{\lambda w}z)^2+2\overline{\psi_1(0)\lambda w}z}{\overline{w}(1-\overline{\lambda w}z)^2(1-\overline{w}\lambda z)}\right).$$

In the above equality, let $w\to 0$. Hence we have

$$\lim_{w\to 0}\frac{1}{\overline{w}}\left(\frac{\overline{\psi_1(\lambda w)}z}{(1-\overline{\lambda w}z)^2}-\frac{\overline{\psi_1(0)}z}{1-\overline{w}\lambda z}\right)=\overline{\lambda\psi_1'(0)}z+(2\overline{\psi_1(0)\lambda}-\overline{\psi_1(0)}\lambda)z^2.$$

After some computation on the left side of (17) and letting $w\to 0$, we get

$$\lim_{w\to 0}\frac{\psi_0(0)}{\overline{w}(1-\overline{w}\lambda z)}-\frac{\psi_0(0)}{\overline{w}(1-\overline{\lambda w}z)}+\frac{\psi_1(\lambda z)}{(1-\overline{w}\lambda z)^2}-\frac{\psi_1(0)}{1-\overline{\lambda w}z}$$
$$=(\psi_0(0)\lambda-\psi_0(0)\overline{\lambda})z+\psi_1(\lambda z)-\psi_1(0).$$

Since $C_{\lambda z}D_{\psi_0,\psi_1}$ is hermitian, we have

$$(\psi_0(0)\lambda-\psi_0(0)\overline{\lambda})z+\psi_1(\lambda z)-\psi_1(0)=\overline{\lambda\psi_1'(0)}z+(2\overline{\psi_1(0)\lambda}-\overline{\psi_1(0)}\lambda)z^2.$$

Then

$$\psi_1(z)=\psi_1(0)+(\psi_0(0)\overline{\lambda}-\psi_0(0)\lambda+\overline{\lambda\psi_1'(0)})\overline{\lambda}z+(2\overline{\psi_1(0)\lambda}-\overline{\psi_1(0)}\lambda)(\overline{\lambda}z)^2. \quad (18)$$

Let $\psi_0(z)=a+bz$, where $a=\psi_0(0)\in\mathbb{R}$ and $b=\overline{\lambda\psi_1(0)}$. Therefore, by (18), we get

$$\psi_1(z)=\overline{b\lambda}+(a\overline{\lambda}^2-a+\overline{c}\overline{\lambda}^2)z+(2b-b\lambda^2)\overline{\lambda}^2z^2,$$

where $c=\psi_1'(0)$. It states that $c=a\overline{\lambda}^2-a+\overline{c}\overline{\lambda}^2$. For convenience, let $\psi_1(z)=\overline{b\lambda}+cz+(2b-b\lambda^2)\overline{\lambda}^2z^2$. It is not hard to see that $z^2\in\operatorname{dom}(C_{\lambda z}D_{\psi_0,\psi_1})$ and $z^2\in\operatorname{dom}((C_{\lambda z}D_{\psi_0,\psi_1})^*)$ (see [10, Lemma 3.1]). Since $C_{\lambda z}D_{\psi_0,\psi_1}$ is hermitian, $C_{\lambda z}D_{\psi_0,\psi_1}z^2=(C_{\lambda z}D_{\psi_0,\psi_1})^*z^2$. One can see that

$$C_{\lambda z}D_{\psi_0,\psi_1}z^2=(b\lambda^3+4b\lambda-2b\lambda^3)z^3+(a\lambda^2+2c\lambda^2)z^2+2\overline{b}z \quad (19)$$

and by Proposition 4.1,

$$(C_{\lambda z}D_{\psi_0,\psi_1})^*z^2=(3b\overline{\lambda})z^3+(a\overline{\lambda}^2+2\overline{\lambda}^2\overline{c})z^2+(2\overline{b}\overline{\lambda}^2+\overline{b}(2-2\overline{\lambda}^2))z^3. \quad (20)$$

Then (19) and (20) state that $b\lambda^3+4b\lambda-2b\lambda^3=3b\overline{\lambda}$. It shows that $b=0$ or $\lambda^2=1$. First suppose that $\lambda^2=1$. The trivial case $\lambda=1$ was described in [10, Theorem 6.3].

Now assume that $\lambda = -1$. Again by (19) and (20), $a+2c = a+2\overline{c}$ and so $c \in \mathbb{R}$ and the result follows. Now let $b = 0$. We have $\psi_0 \equiv a$ and $\psi_1(z) = cz$. From [10, Theorem 3.3], we have

$$(C_{\lambda z}D_{a,cz})^* = D_{a,\overline{c}z}C_{\overline{\lambda}z} = (aC_{\overline{\lambda}z} + \overline{\lambda}T_{\overline{c}z}C_{\overline{\lambda}z}D) = C_{\overline{\lambda}z}D_{a,\overline{c}z}.$$

Since $C_{\lambda z}D_{a,cz}$ is hermitian, we get for each $f \in \mathrm{dom}(C_{\lambda z}D_{a,cz})$,

$$D_{a,cz}(f) = C_{\overline{\lambda}^2 z}D_{a,\overline{c}z}(f). \tag{21}$$

We break the proof in to two cases. First, suppose that λ is not a root of 1. By (21), for $f(z) = z^n$, where n is a non-negative integer, we have $(a+nc)z^n = \overline{\lambda}^{2n}(a+n\overline{c})z^n$. The limit of the above equality as $z \to 1$ shows that

$$a+nc = \overline{\lambda}^{2n}(a+n\overline{c}) \tag{22}$$

for every non-negative integer n. From (22), we have $a+c = \overline{\lambda}^2(a+\overline{c})$ and

$$a+2c = \overline{\lambda}^4(a+2\overline{c}). \tag{23}$$

Hence $(\frac{a+c}{a+\overline{c}})^2 = \frac{a+2c}{a+2\overline{c}}$ and so

$$c^2(a+2\overline{c}) \in \mathbb{R}. \tag{24}$$

Invoking (23) and (24), we see that $\overline{\lambda}^4\frac{\overline{c}^2}{c^2} \in \mathbb{R}$. Let $c = |c|e^{i\theta}$, where $\theta = \arg(c)$. Then $\overline{\lambda}^4 = e^{4i\theta}$ or $\overline{\lambda}^4 = -e^{4i\theta}$. Let $\arg(c^2) = \tilde{\theta}$. Because λ is not a root of unity, $e^{i\theta}$ and $e^{i\tilde{\theta}}$ are not roots of unity. Moreover, (22) shows that the set $\{\frac{a+nc}{a+n\overline{c}} : n = 0,1,\ldots\}$ is dense in $\partial\mathbb{D}$. For arbitrary $\varepsilon > 0$, it is not hard to see that there is an integer N such that for each $n \geqslant N$, $|\arg(a+2nc) - \theta| < \varepsilon$ (note that $a+2nc$ is the major axis of a parallelogram). Then for $n \geqslant N$, $a+2nc+n^2c^2$ lies in the parallelogram that one side is the line segment with endpoints 0 and n^2c^2 and the other side is the line segment with endpoints 0 and $a+2nc$. Since $\theta - \varepsilon \leqslant \arg(a+2nc) \leqslant \theta + \varepsilon$ (note that $e^{i\theta}$ is not a root of unity and so $\theta \neq 0$) and $\arg(n^2c^2) = \tilde{\theta}$, the set $\{\frac{(a+nc)^2}{|a+n\overline{c}|^2}\}$ is not dense in $\partial\mathbb{D}$ which is a contraction. In the other case, assume that there is an integer n_0 such that $\lambda^{n_0} = 1$. Applying (22), we have $a+n_0c = \overline{\lambda}^{2n_0}(a+n_0\overline{c})$ and so $c \in \mathbb{R}$. By setting $n = 1$ in (22), $\lambda^2 = 1$ or $c = -a$. We considered the case $\lambda^2 = 1$. If $c = -a$, then again by (22), $\overline{\lambda}^{2n} = 1$ for every integer $n > 1$. Then λ^2 must be 1 and the result follows.

Conversely, if λ, ψ_0, ψ_1 satisfy the hypotheses of Part (a), the result follows obviously by [10, Theorem 6.3]. Now suppose that $\lambda = -1$, $\psi_0(z) = a+bz$ and $\psi_1(z) = -\overline{b}+cz+bz^2$, where $a,c \in \mathbb{R}$ and $b \in \mathbb{C}$. We infer from [10, Theorem 3.3] that

$$\begin{aligned} C_{-z}D_{\psi_0,\psi_1} &= C_{-z}(T_{\psi_0} + T_{\psi_1}D) \\ &= (T_{\psi_0(-z)} + T_{-\psi_1(-z)}D)C_{-z} \\ &= D_{\psi_0(-z),-\psi_1(-z)}C_{-z} \\ &= D^*_{\psi_0,\psi_1}C_{-z} \\ &= (C_{-z}D_{\psi_0,\psi_1})^*. \end{aligned} \tag{25}$$

Then by (25), $C_{-z}D_{\psi_0,\psi_1}$ is hermitian. □

If $C_{\lambda z}D_{\psi_0,\psi_1}$ is hermitian, then by Theorem 4.2, either $\lambda = 1$ or $\lambda = -1$. In the case that $\lambda = 1$, [10, Corollary 6.5] implies that $C_{\lambda z}D_{\psi_0,\psi_1}$ is $C_{\overline{\beta} z}J$-selfadjoint, where β was defined in [10, Corollary 6.5]. In the next result, for $\lambda = -1$, we show that hermitian operators $C_{\lambda z}D_{\psi_0,\psi_1}$ are C-selfadjoint.

COROLLARY 4.3. *Let D_{ψ_0,ψ_1} be the maximal differential operator with symbols ψ_0 and ψ_1 that $\psi_0,\psi_1 \in H^\infty$. Suppose that $C_{-z}D_{\psi_0,\psi_1}$ is hermitian. Then $C_{-z}D_{\psi_0,\psi_1}$ is $C_{e^{2i\theta}z}J$-selfadjoint, where $\theta = arg(-\overline{\psi_1(0)})$.*

Proof. Suppose that $C_{-z}D_{\psi_0,\psi_1}$ is hermitian. Applying Theorem 4.2, we have $\psi_0(z) = a+bz$ and $\psi_1(z) = -\overline{b}+cz+bz^2$, where $a,c \in \mathbb{R}$ and $b \in \mathbb{C}$. Suppose that $\theta = arg(b)$. Invoking Theorem 3.3 and putting $\mu = -e^{-2i\theta}$, we conclude that $C_{-z}D_{\psi_0,\psi_1}$ is $C_{e^{2i\theta}}J$-selfadjoint. □

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