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THE PRODUCT OF OPERATORS AND THEIR THE MOORE–PENROSE INVERSES ON HILBERT C^*–MODULES

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Abstract. We assure the existence of the Moore–Penrose inverse of a product UTS, under the assumptions that T has a closed range and that there exist U' and S' such that $U'UT = T = TSS'$, and then we characterize the Moore–Penrose inverse of UTS in terms of the corresponding inverses of T. Also, we obtain the block matrix decomposition of operators, which implies that the reverse order law for operators establishes. Finally we achieve some relations between the product of operators and their the Moore-Penrose inverses.

1. Introduction

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing inner products to take values in a C^*-algebra rather than in the field of real or complex numbers. Some fundamental properties of inner product spaces are no longer valid in inner product C^*-modules in their complete generality. Consequently, when we are studying inner product C^*-modules, it is always of interest under which conditions as well as which more general, situations might appear. The book [4] is used as a standard reference source.

The Moore-Penrose inverse is a topic of considerable research in matrix theory, ring theory, operator algebra with a variety of applications including control theory, signal processing and estimation theory. The existence of the Moore-Penrose inverse is of interest in the study of the structure of a non commutative algebra.

Xu and Sheng [8] showed that a bounded adjointable operator between two Hilbert C^*-modules admits a bounded the Moore–Penrose inverse if and only if that operator has closed range. Ensuring of the existence of the Moore-Penrose inverse of product operators and its computing is not an easy task in general.

Gouveia and Puystjens introduced an equation on finite matrices and applied it for several familiar factorizations of matrices such as the polar, the Schur, and the singular-value decompositions [2]. Patrìcio in [7] gave necessary and sufficient conditions in order to product of known operators be the Moore–Penrose invertible.

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In this paper, the existence of $(TS)^\dagger$, $(UT)^\dagger$ and $(UTS)^\dagger$, under the assumption that T has a closed range and the existence of U' and S' such that $U'UT = T = TSS'$, is guaranteed, and then we characterize the Moore-Penrose inverse UTS in terms of the corresponding inverses of T, and also by focuses on block matrix decomposition of operators reobtain it, in terms of the corresponding the Moore-Penrose inverse T. The same technique enabling us to find that conditions under which the reverse order law for operators hold and it leads to obtain new results of the product of operators and their the Moore-Penrose inverses in the infinite dimensional settings on the Hilbert C^*-module.

Let us fix our notation and terminology. A Hilbert $\mathfrak{A}$-module $\mathscr{X}$ is a right $\mathfrak{A}$-module equipped with an $\mathfrak{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathscr{X} \times \mathscr{X} \to \mathfrak{A}$ such that $\mathscr{X}$ is complete with respect to the induced norm $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$ $(x \in \mathscr{X})$. Throughout the rest of this paper, $\mathfrak{A}$ denotes a C^*-algebra and $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$ and $\mathscr{K}$ denote Hilbert $\mathfrak{A}$-modules. Let $\mathscr{L}(\mathscr{X},\mathscr{Y})$ be the set of operators $T : \mathscr{X} \to \mathscr{Y}$ for which there is an operator $T^* : \mathscr{Y} \to \mathscr{X}$ such that $\langle Tx,y\rangle = \langle x,T^*y\rangle$ for any $x \in \mathscr{X}$ and $y \in \mathscr{Y}$. It is known that any element $T \in \mathscr{L}(\mathscr{X},\mathscr{Y})$ must be bounded and $\mathfrak{A}$-linear. We call $\mathscr{L}(\mathscr{X},\mathscr{Y})$ the set of adjointable operators from $\mathscr{X}$ to $\mathscr{Y}$. For any $T \in \mathscr{L}(\mathscr{X},\mathscr{Y})$, the range and the null space of T are represented by $\mathscr{R}(T)$ and $\mathscr{N}(T)$, respectively. In case $\mathscr{X} = \mathscr{Y}$, the space $\mathscr{L}(\mathscr{X},\mathscr{X})$, which is abbreviated to $\mathscr{L}(\mathscr{X})$, is a C^*-algebra.

A closed submodule M of $\mathscr{X}$ is said to be *orthogonally complemented* if $\mathscr{X} = M \oplus M^\perp$, where $M^\perp = \{x \in \mathscr{X} : \langle x,y\rangle = 0 \text{ for any } y \in M\}$. If $T \in \mathscr{L}(\mathscr{X},\mathscr{Y})$ does not have closed range, then neither $\mathscr{N}(T)$ nor $\overline{\mathscr{R}(T)}$ needs to be orthogonally complemented. In addition, if $T \in \mathscr{L}(\mathscr{X},\mathscr{Y})$ and $\overline{\mathscr{R}(T^*)}$ is not orthogonally complemented, then it may happen that $\mathscr{N}(T)^\perp \neq \overline{\mathscr{R}(T^*)}$; see [4, 5]. The above facts show that the theory of Hilbert C^*-modules are much different and more complicated than that of Hilbert spaces.

An operator $S \in \mathscr{L}(\mathscr{Y},\mathscr{X})$ is an inner inverse of T, if $TST = T$ holds. In this case T is inner invertible, or relatively regular. It is well known that T is inner invertible if and only if $\mathscr{R}(T)$ is closed in $\mathscr{Y}$. The Moore-Penrose inverse of $T \in \mathscr{L}(\mathscr{X},\mathscr{Y})$ is the operator $X \in \mathscr{L}(\mathscr{Y},\mathscr{X})$ which satisfies the Penrose equations

$$(1)\ TXT = T, \quad (2)\ XTX = X, \quad (3)\ (TX)^* = TX, \quad (4)\ (XT)^* = XT.$$

The Moore–Penrose inverse of T exists if and only if $\mathscr{R}(T)$ is closed in $\mathscr{Y}$. If the Moore–Penrose inverse of T exists, then it is unique, and it is denoted by $T^\dagger$. If $\theta \subseteq \{1,2,3,4\}$ and X satisfies the equations (i) for all $i \in \theta$, then X is a θ-inverse of T. The set of all θ-inverses of T is denoted by $T\{\theta\}$. In particular, $T\{1,2,3,4\} = \{T^\dagger\}$.

The term orthogonal projection will be reserved for T which is self-adjoint and idempotent. From the definition of the Moore–Penrose inverse, it can be proved that the Moore–Penrose inverse of an operator (if it exists) is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections into $\mathscr{R}(T^*)$ and $\mathscr{R}(T)$, respectively. Clearly, T is the Moore–Penrose invertible if and only if T^* is the Moore–Penrose invertible [4, Theorem 3.2], and in this case $(T^*)^\dagger = (T^\dagger)^*$, $(T^*T)^\dagger = T^\dagger(T^*)^\dagger$, $T^* = T^*TT^\dagger$ and $T^\dagger = T^*(TT^*)^\dagger$.

2. The Moore-Penrose inverse of a product

In the following theorems, the existence of $(TS)^{\dagger}$, $(UT)^{\dagger}$ and $(UTS)^{\dagger}$ satisfying the stated conditions that T has a closed range and the existence of U' and S' such that $U'UT = T = TSS'$, is guaranteed by Theorems 1 and 2. In order to compute their the Moore–Penrose inverses, we determine inverse of two operators in terms of the corresponding the Moore–Penrose inverse of T, that they play fundamental roles in the related results in this section.

THEOREM 1. *Let $T \in \mathscr{L}(\mathscr{X},\mathscr{Y})$, $S \in \mathscr{L}(\mathscr{Z},\mathscr{X})$, $U \in \mathscr{L}(\mathscr{Y},\mathscr{K})$ and T have closed range. If there exist operators $U' \in \mathscr{L}(\mathscr{K},\mathscr{Y})$ and $S' \in \mathscr{L}(\mathscr{X},\mathscr{Z})$ such that*

$$U'UT = T = TSS',$$

then

(i) UT and TS have closed ranges and $T^{\dagger}U' \in (UT)\{1,2,4\}$ and $S'T^{\dagger} \in (TS)\{1,2,3\}$.

(ii) $(UT)^*UT + 1 - T^{\dagger}T$ and $(TS)(TS)^* + 1 - TT^{\dagger}$ are invertible operators. In this case,

$$\left((UT)^*UT + 1 - T^{\dagger}T\right)^{-1} = (UT)^{\dagger}((UT)^*)^{\dagger} + 1 - T^{\dagger}T$$

and

$$\left(TS(TS)^* + 1 - TT^{\dagger}\right)^{-1} = ((TS)^*)^{\dagger}(TS)^{\dagger} + 1 - TT^{\dagger}.$$

Proof. (i) Putting $X = T^{\dagger}U'$ implies that

$$\begin{aligned} UTXUT &= UTT^{\dagger}U'UT = UTT^{\dagger}T = UT, \\ XUTX &= T^{\dagger}U'UTT^{\dagger}U' = T^{\dagger}U' = X, \\ (XUT)^* &= (T^{\dagger}U'UT)^* = \left(T^{\dagger}T\right)^* = T^{\dagger}T. \end{aligned}$$

Then $T^{\dagger}U' \in (UT)\{1,2,4\}$. It immediately concludes that UT has closed range.

Also, letting $Y = S'T^{\dagger}$ concludes that

$$\begin{aligned} TSYTS &= TSS'T^{\dagger}TS = TT^{\dagger}TS = TS, \\ YTSY &= S'T^{\dagger}TSS'T^{\dagger} = S'T^{\dagger}TT^{\dagger} = S'T^{\dagger} = Y, \\ (TSY)^* &= (TSS'T^{\dagger})^* = (TT^{\dagger})^* = TT^{\dagger}. \end{aligned}$$

Then TS has closed range and $S'T^{\dagger} \in (TS)\{1,2,3\}$.

(ii) The statement (i) concludes that UT and TS have closed ranges. By [6, Corollary 2.4] $((UT)^*UT)^{\dagger}$ exists and $((UT)^*UT)^{\dagger} = (UT)^{\dagger}((UT)^*)^{\dagger}$. Taking adjoint of $U'UT = T$ we get $T^*U^*(U')^* = T^*$. This implies that $\mathscr{R}(T^*) = \mathscr{R}((UT)^*)$, therefore $(UT)^{\dagger}UT = T^{\dagger}T$. Now, putting $C = (UT)^*UT + 1 - T^{\dagger}T$ and $D = (UT)^{\dagger}((UT)^*)^{\dagger} +$

$1-T^\dagger T$ implies that

$$\begin{aligned}
CD &= \Big((UT)^*UT+1-T^\dagger T\Big)\Big((UT)^\dagger((UT)^*)^\dagger+1-T^\dagger T\Big)\\
&= (UT)^*UT(UT)^\dagger((UT)^*)^\dagger+(UT)^*UT-(UT)^*UTT^\dagger T\\
&\quad +(UT)^\dagger((UT)^*)^\dagger+1-T^\dagger T\\
&\quad - T^\dagger T(UT)^\dagger((UT)^*)^\dagger-T^\dagger T+T^\dagger TT^\dagger T\\
&= (UT)^*((UT)^*)^\dagger+(UT)^*UT-(UT)^*UT\\
&\quad +(UT)^\dagger((UT)^*)^\dagger+1-T^\dagger T\\
&\quad - (UT)^\dagger UT(UT)^\dagger((UT)^*)^\dagger-T^\dagger T+T^\dagger T\\
&= (UT)^*((UT)^*)^\dagger+(UT)^\dagger((UT)^*)^\dagger+1-T^\dagger T-(UT)^\dagger((UT)^*)^\dagger\\
&= (UT)^*((UT)^\dagger)^*+1-T^\dagger T\\
&= \Big((UT)^\dagger UT\Big)^*+1-T^\dagger T\\
&= (UT)^\dagger UT+1-T^\dagger T\\
&= T^\dagger T+1-T^\dagger T\\
&= 1.
\end{aligned}$$

Since $(UT)^\dagger UT=T^\dagger T$, therefore

$$\begin{aligned}
(UT)^\dagger((UT)^*)^\dagger T^\dagger T &= (UT)^\dagger((UT)^*)^\dagger(UT)^\dagger(UT)\\
&= (UT)^\dagger((UT)^*)^\dagger. \qquad (1)
\end{aligned}$$

Also we obtain

$$\begin{aligned}
DC &= \Big((UT)^\dagger((UT)^*)^\dagger+1-T^\dagger T\Big)\Big((UT)^*UT+1-T^\dagger T\Big)\\
&= (UT)^\dagger((UT)^*)^\dagger(UT)^*UT+(UT)^\dagger((UT)^*)^\dagger-(UT)^\dagger((UT)^*)^\dagger T^\dagger T\\
&\quad +(UT)^*UT+1-T^\dagger T\\
&\quad - T^\dagger T(UT)^*UT-T^\dagger T+T^\dagger TT^\dagger T\\
\text{(by 1)} &= (UT)^\dagger UT+(UT)^\dagger((UT)^*)^\dagger-(UT)^\dagger((UT)^*)^\dagger\\
&\quad +(UT)^*UT+1-T^\dagger T-(UT)^\dagger UT(UT)^*UT\\
&= (UT)^\dagger UT+(UT)^*UT+1-T^\dagger T-(UT)^*UT\\
&= (UT)^\dagger UT+1-T^\dagger T\\
&= 1.
\end{aligned}$$

With similar argument, we prove that $((TS)^*)^\dagger(TS)^\dagger+1-TT^\dagger$ is invertible. Since TS has closed range then [6, Corollary 2.4] $(TS(TS)^*)^\dagger$ exists and $(TS(TS)^*)^\dagger = ((TS)^*)^\dagger(TS)^\dagger$. On the other hand, from $T=TSS'$ it follows that $TS(TS)^\dagger=TT^\dagger$ and $\mathscr{R}(T)=\mathscr{R}(TS)$.

Now, we put $G = TS(TS)^* + 1 - TT^\dagger$ and $H = ((TS)^*)^\dagger (TS)^\dagger + 1 - TT^\dagger$. Then

$$\begin{aligned} GH &= \left(TS(TS)^* + 1 - TT^\dagger\right)\left(((TS)^*)^\dagger (TS)^\dagger + 1 - TT^\dagger\right) \\ &= TS(TS)^*((TS)^*)^\dagger (TS)^\dagger + TS(TS)^* - TS(TS)^* TT^\dagger \\ &\quad + ((TS)^*)^\dagger (TS)^\dagger + 1 - TT^\dagger \\ &\quad - TT^\dagger ((TS)^*)^\dagger (TS)^\dagger - TT^\dagger + TT^\dagger TT^\dagger \\ &= TS(TS)^\dagger + TS(TS)^* - TS(TS)^* TS(TS)^\dagger \\ &\quad + ((TS)^*)^\dagger (TS)^\dagger + 1 - TT^\dagger \\ &\quad - TS(TS)^\dagger ((TS)^*)^\dagger (TS)^\dagger - TT^\dagger + TT^\dagger \\ &= TS(TS)^\dagger + TS(TS)^* - TS(TS)^* \\ &\quad + ((TS)^*)^\dagger (TS)^\dagger + 1 - TS(TS)^\dagger - ((TS)^*)^\dagger (TS)^\dagger \\ &= 1. \end{aligned}$$

Also,

$$\begin{aligned} HG &= \left(((TS)^*)^\dagger (TS)^\dagger + 1 - TT^\dagger\right)\left(TS(TS)^* + 1 - TT^\dagger\right) \\ &= ((TS)^*)^\dagger (TS)^\dagger TS(TS)^* + ((TS)^*)^\dagger (TS)^\dagger - ((TS)^*)^\dagger (TS)^\dagger TT^\dagger \\ &\quad + TS(TS)^* + 1 - TT^\dagger \\ &\quad - TT^\dagger TS(TS)^* - TT^\dagger + TT^\dagger TT^\dagger \\ &= ((TS)^*)^\dagger (TS)^* + ((TS)^*)^\dagger (TS)^\dagger - ((TS)^*)^\dagger (TS)^\dagger TS(TS)^\dagger \\ &\quad + TS(TS)^* + 1 - TT^\dagger \\ &\quad - TS(TS)^\dagger TS(TS)^* - TT^\dagger + TT^\dagger \\ &= ((TS)^*)^\dagger (TS)^* + ((TS)^*)^\dagger (TS)^\dagger - ((TS)^*)^\dagger (TS)^\dagger \\ &\quad + TS(TS)^* + 1 - TT^\dagger - TS(TS)^* \\ &= ((TS)(TS)^\dagger)^* + 1 - TT^\dagger \\ &= (TS)(TS)^\dagger + 1 - TT^\dagger \\ &= 1. \end{aligned}$$

This completes the proof. □

We notice that $T, S \in \mathcal{L}(\mathcal{X})$, then $[T,S] = TS - ST$ denotes the commutator of T and S.

THEOREM 2. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{K}$ be Hilbert $\mathfrak{A}$-modules and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range and $S \in \mathcal{L}(\mathcal{Z}, \mathcal{X}), U \in \mathcal{L}(\mathcal{Y}, \mathcal{K})$. If there exist operators $U' \in \mathcal{L}(\mathcal{K}, \mathcal{Y})$ and $S' \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ such that*

$$U'UT = T = TSS'.$$

Then

(i) $(TS)^\dagger = (TS)^* G^{-1}$ *and* $(UT)^\dagger = C^{-1}(UT)^*$,

(ii) $[G^{-1}, TS(TS)^\dagger] = 0$ *and* $[C^{-1}, (UT)^\dagger UT] = 0$,

(iii) UTS *has closed range and* $(UTS)^\dagger = (TS)^* G^{-1} T C^{-1} (UT)^*$,

(iv) $S^* C(UT)^\dagger = (TS)^\dagger G U^*$,

where $C = (UT)^* UT + 1 - T^\dagger T$ *and* $G = (TS)(TS)^* + 1 - TT^\dagger$.

Proof. (i) From $U'UT = T = TSS'$ we get the following: $\mathscr{R}(T) = \mathscr{R}(TS)$ and $\mathscr{R}(T^*) = \mathscr{R}((UT)^*)$. The first equiality implies that $TT^\dagger = TS(TS)^\dagger$. From the second equiality it follows that $T^\dagger T = (UT)^\dagger UT$. By details which are shown in the proof of Theorem 2, we conclude that C and G are invertible and theirs inverses are D and H, respectively. Hence we have

$$\begin{aligned} (TS)^* G^{-1} &= (TS)^* \Big(((TS)^*)^\dagger (TS)^\dagger + 1 - TT^\dagger \Big) \\ &= (TS)^* ((TS)^*)^\dagger (TS)^\dagger + (TS)^* - (TS)^* (TS)(TS)^\dagger \\ &= (TS)^* ((TS)^*)^\dagger (TS)^\dagger + (TS)^* - (TS)^* \\ &= (TS)^* ((TS)^*)^\dagger (TS)^\dagger \\ &= (TS)^\dagger. \end{aligned}$$

and

$$\begin{aligned} C^{-1}(UT)^* &= \Big((UT)^\dagger ((UT)^*)^\dagger + 1 - T^\dagger T \Big) (UT)^* \\ &= (UT)^\dagger ((UT)^*)^\dagger (UT)^* + (UT)^* - (UT)^\dagger UT (UT)^* \\ &= (UT)^\dagger ((UT)^*)^\dagger (UT)^* + (UT)^* - (UT)^* \\ &= (UT)^\dagger ((UT)^*)^\dagger (UT)^* \\ &= (UT)^\dagger. \end{aligned}$$

(ii) From statement (i) we have G is invertible and $(TS)^* G^{-1} = (TS)^\dagger$, also G is self adjoint. Taking adjoint, we obtain $G^{-1}(TS) = ((TS)^*)^\dagger$, then $\mathscr{R}(G^{-1}(TS)) = \mathscr{R}(((TS)^*)^\dagger) = \mathscr{R}(TS)$. Hence by [1, Lemma 2.1] the desired result follows.

Analogously, we can prove that C is invertible and $C^{-1}(UT)^* = (UT)^\dagger$, also C is self adjoint. Then $\mathscr{R}(C^{-1}(UT)^*) = \mathscr{R}((UT)^\dagger) = \mathscr{R}((UT)^*)$. Reuse by [1, Lemma 2.1] concludes that $[C^{-1}, (UT)^\dagger UT] = 0$.

(iii) The proof of the statement (i) can be used to see that C and G are invertible and $(TS)^* G^{-1} = (TS)^\dagger$ and $C^{-1}(UT)^* = (UT)^\dagger$. Letting $B = UTS$ and $X = (TS)^* G^{-1} T C^{-1} (UT)^*$ conclude that

$$\begin{aligned} BXB &= UTS(TS)^* G^{-1} T C^{-1} (UT)^* UTS \\ &= UTS(TS)^\dagger T (UT)^\dagger UTS \\ &= U \Big(TS(TS)^\dagger \Big) T \Big((UT)^\dagger UT \Big) S \\ &= UTT^\dagger TT^\dagger TS \\ &= UTS. \end{aligned}$$

and in the same way we reach

$$\begin{aligned} XBX &= (TS)^*G^{-1}TC^{-1}(UT)^*UTS(TS)^*G^{-1}TC^{-1}(UT)^* \\ &= (TS)^\dagger T(UT)^\dagger UTS(TS)^\dagger T(UT)^\dagger \\ &= (TS)^\dagger TT^\dagger TS(TS)^\dagger T(UT)^\dagger \\ &= (TS)^\dagger TS(TS)^\dagger T(UT)^\dagger \\ &= (TS)^\dagger TT^\dagger T(UT)^\dagger \\ &= (TS)^\dagger T(UT)^\dagger \\ &= X. \end{aligned}$$

Also BX and XB are orthogonal projections, since

$$\begin{aligned} BX &= UTS(TS)^*G^{-1}TC^{-1}(UT)^* \\ &= UTS(TS)^\dagger T(UT)^\dagger \\ &= UTT^\dagger T(UT)^\dagger \\ &= UT(UT)^\dagger, \end{aligned}$$

and

$$\begin{aligned} XB &= (TS)^*G^{-1}TC^{-1}(UT)^*UTS \\ &= (TS)^\dagger\Big(T(UT)^\dagger UT\Big)S \\ &= (TS)^\dagger\Big(TT^\dagger T\Big)S \\ &= (TS)^\dagger TS. \end{aligned}$$

Then UTS has closed range and the uniqueness of the Moore-Penrose inverse implies that $(UTS)^\dagger = (TS)^*G^{-1}TC^{-1}(UT)^*$.

(iv) Multiplying the equality $(TS)^\dagger = (TS)^*G^{-1}$ by GU^* on the right side and the equality $(UT)^\dagger = C^{-1}(UT)^*$ by S^*C on the left side, the desired result follows. □

3. Matrix representation for $U'UT = T = TSS'$

In this section, we obtain the block matrix decomposition of operators, which implies that the reverse order law for operators establishes. Moreover, we achieve some relations between the product of operators and their the Moore-Penrose inverses.

The following theorem provides some conditions in order to $U'U$ and SS' are orthogonal projections.

THEOREM 3. *Suppose that $T \in \mathscr{L}(\mathscr{X},\mathscr{Y})$ has closed range, $S \in \mathscr{L}(\mathscr{Z},\mathscr{X})$ and $U \in \mathscr{L}(\mathscr{Y},\mathscr{K})$. If there exist operators $U' \in \mathscr{L}(\mathscr{K},\mathscr{Y})$ and $S' \in \mathscr{L}(\mathscr{X},\mathscr{Z})$ such that*

$$U'UT = T = TSS', \quad \mathscr{R}(S) = \mathscr{R}(T^*), \quad \mathscr{R}(U^*) = \mathscr{R}(T),$$

then

(i) $U'U$ *and* SS' *are orthogonal projections,*

(ii) $S(TS)^\dagger = (UT)^\dagger U$.

Proof. (i) Using [3, Lemma 2.3] and [3, Lemma 2.4], the orthogonal sums $\mathscr{X} = \mathscr{R}(T^*) \oplus \mathscr{N}(T)$, $\mathscr{Y} = \mathscr{R}(T) \oplus \mathscr{N}(T^*)$, $\mathscr{Z} = \mathscr{R}(S^*) \oplus \mathscr{N}(S)$ and $\mathscr{K} = \mathscr{R}(U) \oplus \mathscr{N}(U^*)$ imply that the matrix representation of T has the form $T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix}$ where T_1 is invertible and $T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix}$. Also $S = \begin{bmatrix} S_1 & 0 \\ S_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(S^*) \\ \mathscr{N}(S) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix}$ and $S' = \begin{bmatrix} S'_1 & S'_2 \\ S'_3 & S'_4 \end{bmatrix}$, $S^\dagger = \begin{bmatrix} D^{-1}S_1^* & D^{-1}S_3^* \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(S^*) \\ \mathscr{N}(S) \end{bmatrix}$, where $D = S_1^*S_1 + S_3^*S_3$ is invertible. Also, we have $U = \begin{bmatrix} U_1 & U_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(U) \\ \mathscr{N}(U^*) \end{bmatrix}$ and $U' = \begin{bmatrix} U'_1 & U'_2 \\ U'_3 & U'_4 \end{bmatrix}$, $U^\dagger = \begin{bmatrix} U_1^*E^{-1} & 0 \\ U_2^*E^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(U) \\ \mathscr{N}(U^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix}$, where $E = U_1U_1^* + U_2U_2^*$ is invertible. Since $\mathscr{R}(S) = \mathscr{R}(T^*)$ then

$$\begin{aligned} SS^\dagger = T^\dagger T &\Leftrightarrow \begin{bmatrix} S_1 & 0 \\ S_3 & 0 \end{bmatrix} \begin{bmatrix} D^{-1}S_1^* & D^{-1}S_3^* \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} S_1D^{-1}S_1^* & S_1D^{-1}S_3^* \\ S_3D^{-1}S_1^* & S_3D^{-1}S_3^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{1}$$

Equation (1) implies that

$$S_1D^{-1}S_3^* = 0 \tag{2}$$
$$S_3D^{-1}S_3^* = 0. \tag{3}$$

By multiplication S_1^* on the left of the equation (2) and multiplication S_3^* on the left of equation (3) we conclude that

$$S_1^*S_1D^{-1}S_3^* + S_3^*S_3D^{-1}S_3^* = (S_1^*S_1 + S_3^*S_3)D^{-1}S_3^* = 0,$$

therefore, $S_3 = 0$. Similarly, since $\mathscr{R}(U^*) = \mathscr{R}(T)$ then $U_2 = 0$.

Now consider the following chain of equivalences, which is related to the assumption $U'UT = T$:

$$\begin{aligned} U'UT = T &\Leftrightarrow \begin{bmatrix} U'_1 & U'_2 \\ U'_3 & U'_4 \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} U'_1U_1T_1 & 0 \\ U'_3U_1T_1 & 0 \end{bmatrix} = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &\Leftrightarrow U'_1U_1T_1 = T_1, \quad U'_3U_1T_1 = 0. \end{aligned}$$

Invertibility of T_1 implies that $U'_1U_1 = 1$ and $U'_3U_1 = 0$. So we obtain $U'U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Similar arguments show that $SS' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is clear, $U'U$ and SS' are orthogonal projections.

(ii) Using [3, Lemma 2.3], the orthogonal complemented submodules $\mathscr{X} = \mathscr{R}(T^*) \oplus \mathscr{N}(T)$ and $\mathscr{Y} = \mathscr{R}(T) \oplus \mathscr{N}(T^*)$ and $\mathscr{Z} = \mathscr{Z}_1 \oplus \mathscr{Z}_2$ and $\mathscr{K} = \mathscr{K}_1 \oplus \mathscr{K}_2$, conclude that matrix decompositions $T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix}$ where T_1 is invertible. Also $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{Z}_1 \\ \mathscr{Z}_2 \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(S) \\ \mathscr{N}(S^*) \end{bmatrix}$ and $U = \begin{bmatrix} U_1 & 0 \\ U_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(U^*) \\ \mathscr{N}(U) \end{bmatrix} \to \begin{bmatrix} \mathscr{K}_1 \\ \mathscr{K}_2 \end{bmatrix}$. Reuse [3, Lemma 2.3] derives

$$S(TS)^\dagger = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (T_1S_1)^*E^{-1} & 0 \\ (T_1S_2)^*E^{-1} & 0 \end{bmatrix} = \begin{bmatrix} S_1(T_1S_1)^*E^{-1} + S_2(T_1S_2)^*E^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} (UT)^\dagger U &= \begin{bmatrix} F^{-1}(U_1T_1)^* & F^{-1}(U_2T_1)^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 & 0 \\ U_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} F^{-1}(U_1T_1)^*U_1 + F^{-1}(U_2T_1)^*U_2 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where $E = T_1S_1(T_1S_1)^* + T_1S_2(T_1S_2)^*$ and $F = (U_1T_1)^*U_1T_1 + (U_2T_1)^*U_2T_1$ are invertible. Since

$$(S_1S_1^* + S_2S_2^*)T_1^*E^{-1} = (S_1S_1^* + S_2S_2^*)T_1^*(T_1^*)^{-1}(S_1S_1^* + S_2S_2^*)^{-1}T_1^{-1} = T_1^{-1}$$

and

$$F^{-1}T_1^*(U_1^*U_1 + U_2^*U_2) = T_1^{-1}(U_1^*U_1 + U_2^*U_2)^{-1}(T_1^*)^{-1}T_1^*(U_1^*U_1 + U_2^*U_2) = T_1^{-1}$$

hold, then $\begin{bmatrix} (S_1S_1^* + S_2S_2^*)T_1^*E^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} F^{-1}T_1^*(U_1^*U_1 + U_2^*U_2) & 0 \\ 0 & 0 \end{bmatrix}$ and consequently, $S(TS)^\dagger = (UT)^\dagger U$. □

In the following theorem, by applying the block matrix decomposition trick, we reobtain $(UTS)^\dagger$, in terms of the corresponding the Moore-Penrose inverse T, and we show that the reverse order law holds for product of operators.

THEOREM 4. *Let $T \in \mathscr{L}(\mathscr{X}, \mathscr{Y})$ have closed range and $S \in \mathscr{L}(\mathscr{X})$, $U \in \mathscr{L}(\mathscr{Y})$. If there exist operators $U' \in \mathscr{L}(\mathscr{K}, \mathscr{Y})$ and $S' \in \mathscr{L}(\mathscr{X}, \mathscr{Z})$ such that $U'UT = T = TSS'$, then*

(i) UTS has closed range and $(UTS)^\dagger = (TS)^\dagger T(UT)^\dagger$;

(ii) If S and U have closed ranges then $(UTS)^\dagger = S^\dagger T^\dagger U^\dagger$, under the additiational assumption that $(TS)^\dagger = S^\dagger T^\dagger$ and $(UT)^\dagger = T^\dagger U^\dagger$

(iii) $(UTS(TS)^\dagger)^\dagger = T(UT)^\dagger$;

(iv) $(UTT^\dagger)^\dagger = T(UT)^\dagger$;

(v) $((UT)^\dagger UTS)^\dagger = (TS)^\dagger T$;

(vi) $(T^\dagger TS)^\dagger = (TS)^\dagger T$;

(vii) $(UTS)^\dagger UT(UT)^\dagger = (UTS)^\dagger$;

(viii) $(TS)^\dagger TS(UTS)^\dagger = (UTS)^\dagger$;

(ix) $(UTS)^\dagger = S^\dagger (UT)^\dagger$, *under the additiational assumption that* $\mathscr{R}(T^*) = \mathscr{R}(S)$.

Proof. Since $U'UT = T = TSS'$, it follows that $\mathscr{R}(T) = \mathscr{R}(TS)$ and $\mathscr{R}(T^*) = \mathscr{R}((UT)^*)$. Using [3, Lemma 2.3] , the orthogonal complemented submodules $\mathscr{R}(T^*)$ and $\mathscr{R}(T)$ conclude that matrix decompositions $T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix}$ and $T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix}$. Also $S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix}$. Since $\mathscr{R}(T) = \mathscr{R}(TS)$ by [3, Lemma 2.4] matrix form TS is $TS = \begin{bmatrix} H_1 & H_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix}$ and $(TS)^\dagger = \begin{bmatrix} H_1^* D^{-1} & 0 \\ H_2^* D^{-1} & 0 \end{bmatrix}$, where $D = H_1 H_1^* + H_2 H_2^*$ is invertible. On the other hand, product of matrix forms T and S conclude that $TS = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} = \begin{bmatrix} T_1 S_1 & T_1 S_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix}$. In the same way, by [3, Lemma 2.4] , $UT = \begin{bmatrix} K_1 & 0 \\ K_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix}$ and $(UT)^\dagger = \begin{bmatrix} F^{-1}K_1^* & F^{-1}K_2^* \\ 0 & 0 \end{bmatrix}$, where $F = K_1^* K_1 + K_2^* K_2$ is invertible. The product of matrix forms U and T lead to $UT = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} U_1 T_1 & 0 \\ U_3 T_1 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix}$. With comparing these representations matrix of TS ensure that $T_1 S_1 = H_1$, $T_1 S_2 = H_2$ and $T_1(S_1 S_1^* + S_2 S_2^*)T_1^* = D$. Invertibility D and T_1 imply that $E = S_1 S_1^* + S_2 S_2^*$ is invertible. Also, we compair the representations matrix of UT and conclude that $U_1 T_1 = K_1$, $U_3 T_1 = K_2$ and $T_1^*(U_1^* U_1 + U_3^* U_3)T_1 = F$. Invertibility F and T_1 imply that $J = U_1^* U_1 + U_3^* U_3$ is invertible.

(i) Let $X = (TS)^\dagger T(UT)^\dagger$. We conclude that the operator X has the following matrix form:

$$\begin{aligned} X = (TS)^\dagger T(UT)^\dagger &= \begin{bmatrix} H_1^* D^{-1} & 0 \\ H_2^* D^{-1} & 0 \end{bmatrix} \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F^{-1}K_1^* & F^{-1}K_2^* \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} S_1^* E^{-1} T_1^{-1} J^{-1} U_1^* & S_1^* E^{-1} T_1^{-1} J^{-1} U_3^* \\ S_2^* E^{-1} T_1^{-1} J^{-1} U_1^* & S_2^* E^{-1} T_1^{-1} J^{-1} U_3^* \end{bmatrix}. \end{aligned}$$

Since

$$\begin{aligned} &UTSXUTS \\ &= \begin{bmatrix} U_1 T_1 S_1 & U_1 T_1 S_2 \\ U_3 T_1 S_1 & U_3 T_1 S_2 \end{bmatrix} \begin{bmatrix} S_1^* E^{-1} T_1^{-1} J^{-1} U_1^* & S_1^* E^{-1} T_1^{-1} J^{-1} U_3^* \\ S_2^* E^{-1} T_1^{-1} J^{-1} U_1^* & S_2^* E^{-1} T_1^{-1} J^{-1} U_3^* \end{bmatrix} \begin{bmatrix} U_1 T_1 S_1 & U_1 T_1 S_2 \\ U_3 T_1 S_1 & U_3 T_1 S_2 \end{bmatrix} \\ &= \begin{bmatrix} U_1 T_1 (S_1 S_1^* + S_2 S_2^*) E^{-1} T_1^{-1} J^{-1} U_1^* & U_1 T_1 (S_1 S_1^* + S_2 S_2^*) E^{-1} T_1^{-1} J^{-1} U_3^* \\ U_3 T_1 (S_1 S_1^* + S_2 S_2^*) E^{-1} T_1^{-1} J^{-1} U_1^* & U_3 T_1 (S_1 S_1^* + S_2 S_2^*) E^{-1} T_1^{-1} J^{-1} U_3^* \end{bmatrix} \\ &\quad \begin{bmatrix} U_1 T_1 S_1 & U_1 T_1 S_2 \\ U_3 T_1 S_1 & U_3 T_1 S_2 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} U_1J^{-1}U_1^* & U_1J^{-1}U_3^* \\ U_3J^{-1}U_1^* & U_3J^{-1}U_3^* \end{bmatrix} \begin{bmatrix} U_1T_1S_1 & U_1T_1S_2 \\ U_3T_1S_1 & U_3T_1S_2 \end{bmatrix} \\
&= \begin{bmatrix} U_1J^{-1}(U_1^*U_1+U_3^*U_3)T_1S_1 & U_1J^{-1}(U_1^*U_1+U_3^*U_3)T_1S_2 \\ U_3J^{-1}(U_1^*U_1+U_3^*U_3)T_1S_1 & U_3J^{-1}(U_1^*U_1+U_3^*U_3)T_1S_2 \end{bmatrix} \\
&= \begin{bmatrix} U_1T_1S_1 & U_1T_1S_2 \\ U_3T_1S_1 & U_3T_1S_2 \end{bmatrix} \\
&= UTS,
\end{aligned}
$$

and

$$
\begin{aligned}
&XUTSX \\
&= \begin{bmatrix} S_1^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_1^*E^{-1}T_1^{-1}J^{-1}U_3^* \\ S_2^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_2^*E^{-1}T_1^{-1}J^{-1}U_3^* \end{bmatrix} \begin{bmatrix} U_1T_1S_1 & U_1T_1S_2 \\ U_3T_1S_1 & U_3T_1S_2 \end{bmatrix} \\
&\quad \begin{bmatrix} S_1^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_1^*E^{-1}T_1^{-1}J^{-1}U_3^* \\ S_2^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_2^*E^{-1}T_1^{-1}J^{-1}U_3^* \end{bmatrix} \\
&= \begin{bmatrix} S_1^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_1^*E^{-1}T_1^{-1}J^{-1}U_3^* \\ S_2^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_2^*E^{-1}T_1^{-1}J^{-1}U_3^* \end{bmatrix} \begin{bmatrix} U_1J^{-1}U_1^* & U_1J^{-1}U_3^* \\ U_3J^{-1}U_1^* & U_3J^{-1}U_3^* \end{bmatrix} \\
&= \begin{bmatrix} S_1^*E^{-1}T_1^{-1}J^{-1}(U_1^*U_1+U_3^*U_3)J^{-1}U_1^* & S_1^*E^{-1}T_1^{-1}J^{-1}(U_1^*U_1+U_3^*U_3)J^{-1}U_3^* \\ S_2^*E^{-1}T_1^{-1}J^{-1}(U_1^*U_1+U_3^*U_3)J^{-1}U_1^* & S_2^*E^{-1}T_1^{-1}J^{-1}(U_1^*U_1+U_3^*U_3)J^{-1}U_3^* \end{bmatrix} \\
&= \begin{bmatrix} S_1^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_1^*E^{-1}T_1^{-1}J^{-1}U_3^* \\ S_2^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_2^*E^{-1}T_1^{-1}J^{-1}U_3^* \end{bmatrix} \\
&= X,
\end{aligned}
$$

also, the operators

$$
\begin{aligned}
XUTS &= \begin{bmatrix} S_1^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_1^*E^{-1}T_1^{-1}J^{-1}U_3^* \\ S_2^*E^{-1}T_1^{-1}J^{-1}U_1^* & S_2^*E^{-1}T_1^{-1}J^{-1}U_3^* \end{bmatrix} \begin{bmatrix} U_1T_1S_1 & U_1T_1S_2 \\ U_3T_1S_1 & U_3T_1S_2 \end{bmatrix} \\
&= \begin{bmatrix} S_1^*E^{-1}T_1^{-1}J^{-1}(U_1^*U_1+U_3^*U_3)T_1S_1 & S_1^*E^{-1}T_1^{-1}J^{-1}(U_1^*U_1+U_3^*U_3)T_1S_2 \\ S_2^*E^{-1}T_1^{-1}J^{-1}(U_1^*U_1+U_3^*U_3)T_1S_1 & S_2^*E^{-1}T_1^{-1}J^{-1}(U_1^*U_1+U_3^*U_3)T_1S_2 \end{bmatrix} \\
&= \begin{bmatrix} S_1^*E^{-1}S_1 & S_1^*E^{-1}S_2 \\ S_2^*E^{-1}S_1 & S_2^*E^{-1}S_2 \end{bmatrix}
\end{aligned}
$$

and $UTSX = \begin{bmatrix} U_1J^{-1}U_1^* & U_1J^{-1}U_3^* \\ U_3J^{-1}U_1^* & U_3J^{-1}U_3^* \end{bmatrix}$ are self adjoint, then uniqueness of the Moore–Penrose inverse implies that, $(UTS)^\dagger = (TS)^\dagger T(UT)^\dagger$.

(ii) By previous statement is obvious.

(iii) We compute

$$
\begin{aligned}
(UTS(TS)^\dagger)^\dagger &= \left(\begin{bmatrix} U_1T_1S_1 & U_1T_1S_2 \\ U_3T_1S_1 & U_3T_1S_2 \end{bmatrix} \begin{bmatrix} H_1^*D^{-1} & 0 \\ H_2^*D^{-1} & 0 \end{bmatrix} \right)^\dagger \\
&= \left(\begin{bmatrix} U_1T_1ET_1^*(T_1^*)^{-1}E^{-1}T_1^{-1} & 0 \\ U_3T_1ET_1^*(T_1^*)^{-1}E^{-1}T_1^{-1} & 0 \end{bmatrix} \right)^\dagger \\
&= \left(\begin{bmatrix} U_1 & 0 \\ U_3 & 0 \end{bmatrix} \right)^\dagger \\
&= \begin{bmatrix} J^{-1}U_1^* & J^{-1}U_3^* \\ 0 & 0 \end{bmatrix}. \qquad (4)
\end{aligned}
$$

On the other hand, we obtain

$$\begin{aligned} T(UT)^{\dagger} &= \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F^{-1}K_1^* & F^{-1}K_2^* \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} T_1T_1^{-1}J^{-1}(T_1^*)^{-1}T_1^*U_1^* & T_1T_1^{-1}J^{-1}(T_1^*)^{-1}T_1^*U_3^* \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} J^{-1}U_1^* & J^{-1}U_3^* \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{5}$$

Hence, equations (4) and (5) imply that $(UTS(TS)^{\dagger})^{\dagger} = T(UT)^{\dagger}$.

(iv) By applying the equality $\mathscr{R}(T) = \mathscr{R}(TS)$ we obtain $TT^{\dagger} = TS(TS)^{\dagger}$. According to the previous statement, it is obvious.

(v) This implication can be proved in the same way as the statement (iii).

(vi) The equality $\mathscr{R}(T^*) = \mathscr{R}((UT)^*)$ implies that $T^{\dagger}T = (UT)^{\dagger}UT$. By previous statement is trivial.

(vii) We obtain $(UTS)^{\dagger}UT(UT)^{\dagger} = (TS)^{\dagger}T(UT)^{\dagger}UT(UT)^{\dagger} = (TS)^{\dagger}T(UT)^{\dagger} = (UTS)^{\dagger}$ by according to the statement (i).

(viii) Similarly before, it is obvious.

(ix) Since $\mathscr{R}(T^*) = \mathscr{R}(S)$, then S closed range. Also, we have $\mathscr{R}((UT)^*) = \mathscr{R}(T^*)$. Therefore, $\mathscr{R}((UT)^*) = \mathscr{R}(S)$. Now, by [3, Lemma 2.4] matrix forms S, $S^{\dagger}$, UT and $(UT)^{\dagger}$ are $S = \begin{bmatrix} S_1 & S_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(S) \\ \mathscr{N}(S^*) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(S) \\ \mathscr{N}(S^*) \end{bmatrix}$, $S^{\dagger} = \begin{bmatrix} S_1^*E^{-1} & 0 \\ S_2^*E^{-1} & 0 \end{bmatrix}$, $UT = \begin{bmatrix} K_1 & 0 \\ K_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T^*) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(T^*) \end{bmatrix}$ and $(UT)^{\dagger} = \begin{bmatrix} F^{-1}K_1^* & F^{-1}K_2^* \\ 0 & 0 \end{bmatrix}$ where $E = S_1S_1^* + S_2S_2^*$ and $F = K_1^*K_1 + K_2^*K_2$ are invertible. Let

$$X = S^{\dagger}(UT)^{\dagger} = \begin{bmatrix} S_1^*E^{-1} & 0 \\ S_2^*E^{-1} & 0 \end{bmatrix} \begin{bmatrix} F^{-1}K_1^* & F^{-1}K_2^* \\ 0 & 0 \end{bmatrix},$$

then straightforward computations show that X is Moore-Penrose inverse of UTS. □

THEOREM 5. *Suppose that $T \in \mathscr{L}(\mathscr{X},\mathscr{Y})$ has closed range and $S \in \mathscr{L}(\mathscr{X})$. If there exist operator $S' \in \mathscr{L}(\mathscr{X})$ such that $T = TSS'$, then*

(i) $T^{\dagger}TSS^*T^{\dagger}T$ and $TSS^*T^{\dagger}$ have closed ranges and

$$(T^{\dagger}TSS^*T^{\dagger}T)^{\dagger} = T^*((TS)^*)^{\dagger}(TS)^{\dagger}T$$

and

$$(TSS^*T^{\dagger})^{\dagger} = TT^*((TS)^*)^{\dagger}(TS)^{\dagger};$$

(ii) There is an invertible operator $F \in \mathscr{L}(\mathscr{X})$ such that

$$(T^{\dagger}TSS^*T^{\dagger}T)^{\dagger} = F(T^{\dagger}TSS^*T^{\dagger}T) = (T^{\dagger}TSS^*T^{\dagger}T)F;$$

(iii) $((T^*T)^mSS^*(T^*T)^n)^{\dagger} = (T^{\dagger}(T^*)^{\dagger})^nT^*((TS)^*)^{\dagger}(TS)^{\dagger}T(T^{\dagger}(T^*)^{\dagger})^m$ $(m,n \in \mathbb{N})$;

(iv) $(1 - TT^{\dagger} + (TSS^*T^{\dagger})^{\dagger})^{-1} = 1 - TT^{\dagger} + TT^*((TS)^*)^{\dagger}(TS)^{\dagger}$;

(v) If S has a closed range and SS' is self adjoint, then $TSS^*S(S'-S^\dagger)T^*=0$;

(vi) If $SS^*T^\dagger T = T^\dagger TSS^*T^\dagger T$, then $S(TS)^\dagger = T^\dagger$;

(vii) $(T^\dagger TSS^*T^*)^\dagger = (TS(TS)^*)^\dagger T$.

Proof. Since $T = TSS'$, it follows that $\mathscr{R}(T)=\mathscr{R}(TS)$. Using [3, Lemma 2.3], the orthogonal complemented submodules $\mathscr{R}(T^*)$ and $\mathscr{R}(T)$ conclude that matrix decompositions $T=\begin{bmatrix} T_1 & 0 \\ 0 & 0\end{bmatrix} : \begin{bmatrix}\mathscr{R}(T^*) \\ \mathscr{N}(T)\end{bmatrix} \to \begin{bmatrix}\mathscr{R}(T) \\ \mathscr{N}(T^*)\end{bmatrix}$ and $T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0\end{bmatrix} : \begin{bmatrix}\mathscr{R}(T) \\ \mathscr{N}(T^*)\end{bmatrix} \to \begin{bmatrix}\mathscr{R}(T^*) \\ \mathscr{N}(T)\end{bmatrix}$. Also $S=\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4\end{bmatrix}, S'=\begin{bmatrix} S_1' & S_2' \\ S_3' & S_4'\end{bmatrix} : \begin{bmatrix}\mathscr{R}(T^*) \\ \mathscr{N}(T)\end{bmatrix} \to \begin{bmatrix}\mathscr{R}(T^*) \\ \mathscr{N}(T)\end{bmatrix}$. By [3, Lemma 2.4] matrix form TS is $TS=\begin{bmatrix} H_1 & H_2 \\ 0 & 0\end{bmatrix} : \begin{bmatrix}\mathscr{R}(T^*) \\ \mathscr{N}(T)\end{bmatrix} \to \begin{bmatrix}\mathscr{R}(T) \\ \mathscr{N}(T^*)\end{bmatrix}$ and $(TS)^\dagger = \begin{bmatrix} H_1^*D^{-1} & 0 \\ H_2^*D^{-1} & 0\end{bmatrix}$, where $D = H_1H_1^* + H_2H_2^*$ is invertible. On the other hand, product of matrix forms T and S conclude that $TS=\begin{bmatrix} T_1 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4\end{bmatrix} = \begin{bmatrix} T_1S_1 & T_1S_2 \\ 0 & 0\end{bmatrix} : \begin{bmatrix}\mathscr{R}(T^*) \\ \mathscr{N}(T)\end{bmatrix} \to \begin{bmatrix}\mathscr{R}(T) \\ \mathscr{N}(T^*)\end{bmatrix}$. With comparing these representations matrix of TS ensure that

$$T_1S_1 = H_1, \quad T_1S_2 = H_2 \tag{6}$$

and

$$T_1(S_1S_1^* + S_2S_2^*)T_1^* = D. \tag{7}$$

Invertibility of D and T_1 imply that $E = S_1S_1^* + S_2S_2^*$ is invertible.

Also,

$$\begin{aligned} T &= TSS' \\ \begin{bmatrix} T_1 & 0 \\ 0 & 0\end{bmatrix} &= \begin{bmatrix} T_1(S_1S_1'+S_2S_3') & T_1(S_1S_2'+S_2S_4') \\ 0 & 0\end{bmatrix}. \end{aligned}$$

Invertibility of T_1 implies that

$$S_1S_1' + S_2S_3' = 1, \quad S_1S_2' + S_2S_4' = 0. \tag{8}$$

(i) By (7) we have $E = T_1^{-1}D(T_1^*)^{-1}$ that is $E^{-1} = T_1^*D^{-1}T_1$. Considering block matrices of these operators conclude that

$$\begin{aligned} (T^\dagger TSS^*T^\dagger T)^\dagger &= \begin{bmatrix} E^{-1} & 0 \\ 0 & 0\end{bmatrix} = T^*(TSS^*T^*)^\dagger T = T^*(TS(TS)^*)^\dagger T \\ &= T^*((TS)^*)^\dagger (TS)^\dagger T. \end{aligned}$$

Since $\mathscr{R}(T)=\mathscr{R}(TS)$, then $TT^\dagger = TS(TS)^\dagger$. Hence we have

$$\begin{aligned} (TSS^*T^\dagger)^\dagger &= \begin{bmatrix} (T_1(S_1S_1^*+S_2S_2^*)T_1^{-1})^{-1} & 0 \\ 0 & 0\end{bmatrix} = \begin{bmatrix} T_1(S_1S_1^*+S_2S_2^*)^{-1}T_1^{-1} & 0 \\ 0 & 0\end{bmatrix} \\ &= T(T^\dagger TSS^*T^\dagger T)^\dagger T^\dagger = TT^*((TS)^*)^\dagger (TS)^\dagger TT^\dagger \\ &= TT^*((TS)^*)^\dagger (TS)^\dagger. \end{aligned}$$

(ii) From the proof of the previous implication and [3, Theorem 3.6] is straightforward.

(iii) By (7) and matrix forms, we have

$$\begin{aligned}((T^*T)^m SS^*(T^*T)^n)^\dagger &= \begin{bmatrix} (T_1^*T_1)^m(S_1S_1^*+S_2S_2^*)(T_1^*T_1)^n & 0 \\ 0 & 0 \end{bmatrix}^\dagger \\ &= \begin{bmatrix} ((T_1^*T_1)^m(S_1S_1^*+S_2S_2^*)(T_1^*T_1)^n)^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= (T^\dagger(T^*)^\dagger)^n(T^\dagger TSS^*T^\dagger T)^\dagger(T^\dagger(T^*)^\dagger)^m \\ (\text{By statement (i)}) &= (T^\dagger(T^*)^\dagger)^n T^*((TS)^*)^\dagger(TS)^\dagger T(T^\dagger(T^*)^\dagger)^m.\end{aligned}$$

(iv) Matrix operator $1-TT^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and equality (7) ensure that $1-TT^\dagger + (TSS^*T^\dagger)^\dagger = \begin{bmatrix} T_1ET_1^{-1} & 0 \\ 0 & 1 \end{bmatrix}$ is an invertible operator. Statement (i) leads to compute of inverse and its inverse is $1-TT^\dagger+TT^*((TS)^*)^\dagger(TS)^\dagger$.

(v) Being self adjoint of SS' and equality (8) imply that $SS' = \begin{bmatrix} 1 & 0 \\ 0 & S_3S_2'+S_4S_4' \end{bmatrix}$. Hence matrix operators yield

$$\begin{aligned}TSS^*SS'T^* &= \begin{bmatrix} T_1(S_1S_1^*+S_2S_2^*) & T_1(S_1S_3^*+S_2S_4^*) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & S_3S_2'+S_4S_4' \end{bmatrix}\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= TSS^*T^*.\end{aligned}$$

Since S has a closed range, then $TSS^*S(S'-S^\dagger)T^* = 0$.

(vi) Condition $SS^*T^\dagger T = T^\dagger TSS^*T^\dagger T$ leads to

$$\begin{aligned}(1-T^\dagger T)SS^*T^\dagger T &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} S_1S_1^*+S_2S_2^* & S_1S_3^*+S_2S_4^* \\ S_3S_1^*+S_4S_2^* & S_3S_3^*+S_4S_4^* \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ S_3S_1^*+S_4S_2^* & 0 \end{bmatrix} = 0.\end{aligned}$$

On the other hand, (6) implies that $(TS)^\dagger = \begin{bmatrix} S_1^*T_1^*D^{-1} & 0 \\ S_2^*T_1^*D^{-1} & 0 \end{bmatrix}$. Therefore $S(TS)^\dagger = \begin{bmatrix} (S_1S_1^*+S_2S_2^*)T_1^*D^{-1} & 0 \\ (S_3S_1^*+S_4S_2^*)T_1^*D^{-1} & 0 \end{bmatrix}$. Then $S(TS)^\dagger = \begin{bmatrix} ET_1^*D^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$. That is $S(TS)^\dagger = T^\dagger$.

(vii) A straightforward computation shows that

$$\begin{aligned}T^\dagger TSS^*T^* &= \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix}\begin{bmatrix} S_1^* & S_3^* \\ S_2^* & S_4^* \end{bmatrix}\begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (S_1S_1^*+S_2S_2^*)T_1^* & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} ET_1^* & 0 \\ 0 & 0 \end{bmatrix},\end{aligned}$$

where $(S_1S_1^* + S_2S_2^*)T_1^* = ET_1^*$ is invertible. On the other

$$\begin{aligned}(T^\dagger TSS^*T^*)^\dagger &= \begin{bmatrix} (T_1^*)^{-1}E^{-1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (T_1^*)^{-1}(T_1^{-1}D(T_1^*)^{-1})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} D^{-1}T_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= (TS(TS)^*)^\dagger T.\end{aligned}$$

Thus $(T^\dagger TSS^*T^*)^\dagger = (TS(TS)^*)^\dagger T$. □

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