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CLOSEDNESS OF RANGES OF UNBOUNDED UPPER TRIANGULAR OPERATOR MATRICES

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Abstract. This paper deals with the closed range property of operator matrices. The necessary and sufficient condition is given for an unbounded upper triangular partial operator matrix to have a closed range completion. In particular, the bounded case is its direct consequence.

1. Introduction

Partial operator matrices are operator matrices the entries of which are specified only on a subset of its positions, while a completion of a partial operator matrix is the operator matrix resulting from filling in its unspecified entries. The operator matrix completion problem was shown to be very useful in various pure and applied mathematical fields, e.g., in operator theory, numerical analysis, optimal control theory, systems theory and engineering sciences (see [2] and references therein). In this problem, one is concerned with conditions under which a partial operator matrix has completions with some given properties. Recently, many results were given dealing with invertible or closed range completion of operator matrices [1, 3, 4, 10, 11, 13]. These results, in fact, are concerned with bounded operator matrices. Because the entries of operator matrices often appear as unbounded operators in infinite dimensional systems, it is expected to study the completion problem of unbounded cases.

Let $\mathscr{L}(X_1,X_2)$ be the collection of all (linear) operators between Hilbert spaces X_1 and X_2. For $T\in\mathscr{L}(X_1,X_2)$, T^* denotes its adjoint operator; the domain, range and kernel of T are, respectively, represented by $\mathscr{D}(T)$, $\mathscr{R}(T)$ and $\mathscr{N}(T)$; write $n(T)=\dim\mathscr{N}(T)$ and $d(T)=\dim\mathscr{R}(T)^{\perp}$.

In [1], the closedness of the range $\mathscr{R}(M_C)$ of the bounded partial operator matrix $M_C=\left(\begin{smallmatrix}A & C\\ 0 & B\end{smallmatrix}\right)$ was investigated by the method of decomposing spaces. It is shown that for the given bounded operators A and B, there exists a bounded operator C such that $M_C=\left(\begin{smallmatrix}A & C\\ 0 & B\end{smallmatrix}\right)$ has a closed range if and only if

$$\begin{cases} n(B)=\infty, & \text{if } \mathscr{R}(A) \text{ is not closed and } \mathscr{R}(B) \text{ is closed;}\\ d(A)=\infty, & \text{if } \mathscr{R}(A) \text{ is closed and } \mathscr{R}(B) \text{ is not closed;}\\ n(B)=d(A)=\infty, & \text{if none of } \mathscr{R}(A) \text{ and } \mathscr{R}(B) \text{ is closed.}\end{cases} \tag{1.1}$$

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Here one has three cases to consider to address the description (1.1), which are based on the discussions for the closedness of $\mathscr{R}(A)$ and $\mathscr{R}(B)$.

In the present paper we consider the closed range completion of unbounded operator matrices. In this case, the domain of an unbounded operator does not necessarily be split into an orthogonal sum under some given orthogonal decomposition of its domain space, so it can not be represented as a row operator form; also, for unbounded operators T and S, ST^{-1} and $T^{-1}S$ are not bounded any more. Based on discussions for the dimension $d(A)$ of $\mathscr{R}(A)^{\perp}$, the preceding setbacks can be effectively avoided, and necessary and sufficient conditions are given for a partial unbounded upper triangular operator matrix to have a closed range completion: Let A be a densely defined closed operator and B be a closed operator. If $d(A) < \infty$, then there exists a closable operator C such that $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \mathscr{D}(A) \oplus \mathscr{D}(B) \subset X_1 \oplus X_2 \to X_3 \oplus X_4$ is a closed operator with closed range if and only if

(i) $\mathscr{R}(B)$ is closed,

(ii) $\mathscr{R}(A)$ is closed or $n(B) = \infty$;

while if $d(A) = \infty$, and if B is further densely defined, then there exists a closable operator C such that $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \mathscr{D}(A) \oplus \mathscr{D}(B) \subset X_1 \oplus X_2 \to X_3 \oplus X_4$ is a closed operator with closed range if and only if $\mathscr{R}(A)$ is closed or $n(B) = \infty$.

In the case when A is closed and B is a densely defined closed operator, or when A is an arbitrary linear operator and B is a bounded operator, we investigate the closed range properties of M_C based on the dimension $n(B)$ of $\mathscr{N}(B)$.

2. Auxiliary propositions

In this section, we present some basic lemmas and auxiliary propositions, which are necessary to prove the main results of this paper.

In what follows, we always assume $A \in \mathscr{L}(X_1, X_3)$, $B \in \mathscr{L}(X_2, X_4)$ and $C \in \mathscr{L}(X_2, X_3)$, where X_1, X_2, X_3 and X_4 are all complex infinite dimensional separable Hilbert spaces. For a subspace $\mathscr{G}$ of a Hilbert space, $P_{\mathscr{G}}$ represents the orthogonal projection onto $\mathscr{G}$ along $\mathscr{G}^{\perp}$ (if $\mathscr{G}$ is closed) and $T|_{\mathscr{G}}$ stands for the restriction of T to $\mathscr{G}$.

Let T and S be operators with the same domain space X_1 such that $\mathscr{D}(T) \subset \mathscr{D}(S)$ and

$$\|Su\| \leqslant a\|u\| + b\|Tu\|, \ u \in \mathscr{D}(T),$$

where a, b are nonnegative constants. Then we say that S is relatively bounded with respect to T or simply T-bounded (see [8]).

LEMMA 2.1. *Let $T : \mathscr{D}(T) \subset X_1 \to X_2$ be a closed operator and let $S : \mathscr{D}(S) \subset X_1 \to X_2$ be T-bounded and* $\dim \mathscr{R}(S) < \infty$. *Then, $\mathscr{R}(T+S)$ is closed if and only if $\mathscr{R}(T)$ is closed.*

Proof. Since S is T-bounded, the desired result can be reduced to the special case when T and S are bounded. Indeed, setting

$$|||u||| = \|u\| + \|Tu\|, \ u \in \mathscr{D}(T),$$

we easily see that $\mathscr{D}(T)$ becomes a Banach space $\hat{X}_1$ if $|||\cdot|||$ is chosen as the norm. Then T and S can be regarded as bounded operators from $\hat{X}_1$ into X_2 (see [8, Remark IV.1.4]). By Lemma 2.1 in [1], the desired result follows immediately. $\square$

REMARK 2.2. If T is closed and S is closable, the inclusion $\mathscr{D}(T)\subset\mathscr{D}(S)$ implies that S is T-bounded (see [8, Remark IV.1.5]), and hence, by Lemma 2.1, $\mathscr{R}(T+S)$ is closed if and only if $\mathscr{R}(T)$ is closed.

LEMMA 2.3. *Let* $M=\left(\begin{smallmatrix}A & C\\ 0 & B\end{smallmatrix}\right):\mathscr{D}(A)\oplus(\mathscr{D}(B)\cap\mathscr{D}(C))\subset X_1\oplus X_2\to X_3\oplus X_4$ *with* $\mathscr{R}(M)$ *closed. If* $\overline{\mathscr{R}(A)}=X_3$, *then* $\mathscr{R}(B|_{\mathscr{D}(B)\cap\mathscr{D}(C)})$ *is closed.*

Proof. Write $B_1=B|_{\mathscr{D}(B)\cap\mathscr{D}(C)}$ and $C_1=C|_{\mathscr{D}(B)\cap\mathscr{D}(C)}$. Let $\{v_n\}_{n=1}^{\infty}\subset\mathscr{R}(B_1)$ be a sequence with $v_n\to v\in X_4$ $(n\to\infty)$. To prove the closedness of $\mathscr{R}(B_1)$, it suffices to verify $\left(\begin{smallmatrix}0\\ v\end{smallmatrix}\right)\in\mathscr{R}(M)$. In fact, if $\left(\begin{smallmatrix}0\\ v\end{smallmatrix}\right)\in\mathscr{R}(M)$, then there exists a vector $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)\in\mathscr{D}(M)$ such that $M\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)=\left(\begin{smallmatrix}0\\ v\end{smallmatrix}\right)$, i.e.

$$\begin{cases} Ax+C_1y=0,\\ \qquad B_1y=v,\end{cases}$$

so $v\in\mathscr{R}(B_1)$.

For $v_n\in\mathscr{R}(B_1)$, there exists a vector $y_n\in\mathscr{D}(B)\cap\mathscr{D}(C)$ such that $B_1y_n=v_n$.

Since $\overline{\mathscr{R}(A)}=X_3$, for $-C_1y_n\in X_3$, there exists an element, say $x_n\in\mathscr{D}(A)$, such that $|Ax_n+C_1y_n|<\frac{1}{n}$ for each $n\in\mathbb{N}$. Thus, $M\left(\begin{smallmatrix}x_n\\ y_n\end{smallmatrix}\right)\to\left(\begin{smallmatrix}0\\ v\end{smallmatrix}\right)$ $(n\to\infty)$. Therefore, $\left(\begin{smallmatrix}0\\ v\end{smallmatrix}\right)\in\mathscr{R}(M)$ follows from the fact that $\mathscr{R}(M)$ is closed. $\square$

An operator between Hilbert spaces admits column representation under every orthogonal decomposition of its range space. Using this property, we may give the following two results.

PROPOSITION 2.4. *Let* $M=\left(\begin{smallmatrix}A & C\\ 0 & B\end{smallmatrix}\right):\mathscr{D}(A)\oplus(\mathscr{D}(B)\cap\mathscr{D}(C))\subset X_1\oplus X_2\to X_3\oplus X_4$ *be a linear operator, where* $B|_{\mathscr{D}(B)\cap\mathscr{D}(C)}$ *is closed and* C *is closable. If* $\mathscr{R}(M)$ *is closed and* $d(A)<\infty$, *then* $\mathscr{R}(B|_{\mathscr{D}(B)\cap\mathscr{D}(C)})$ *is closed.*

Proof. As an operator from $X_1\oplus X_2$ to $\overline{\mathscr{R}(A)}\oplus\mathscr{R}(A)^{\perp}\oplus X_4$, M has the following block representation

$$M=\begin{pmatrix}A & C\\ 0 & B\end{pmatrix}=\begin{pmatrix}A_1 & C_1\\ 0 & C_2\\ 0 & B\end{pmatrix},$$

where $A_1=P_{\overline{\mathscr{R}(A)}}A$, $C_1=P_{\overline{\mathscr{R}(A)}}C$ and $C_2=P_{\mathscr{R}(A)^{\perp}}C$. Since $\mathscr{R}(A_1)=\mathscr{R}(A)$, $\mathscr{R}(A_1)$ is clearly dense in $\overline{\mathscr{R}(A)}$. According to Lemma 2.3, the closedness of $\mathscr{R}(M)$ implies that $\mathscr{R}\begin{pmatrix}C_2\\ B|_{\mathscr{D}(B)\cap\mathscr{D}(C)}\end{pmatrix}$ is closed. Here C_2 is a $B|_{\mathscr{D}(B)\cap\mathscr{D}(C)}$-bounded operator with $\dim\mathscr{R}(C_2)<\infty$. By Lemma 2.1, we see that $\mathscr{R}(B|_{\mathscr{D}(B)\cap\mathscr{D}(C)})$ is closed. $\square$

PROPOSITION 2.5. *Let $M = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) : \mathscr{D}(A) \oplus \mathscr{D}(B) \subset X_1 \oplus X_2 \to X_3 \oplus X_4$ be a densely defined closed operator, where B is closed and C is B-bounded such that C^* is A^*-bounded, with both relative bounds smaller than one. If $\mathscr{R}(M)$ is closed and $n(B) < \infty$, then $\mathscr{R}(A)$ is closed.*

Proof. Write $T = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & C \\ 0 & 0 \end{smallmatrix}\right)$. Since the upper triangular operator M is closed, A is clearly closed, and hence T is closed. From the assumptions of relative boundedness, it follows that S is T-bounded and S^* is T^*-bounded, with both relative bounds smaller than one. Thus, $M^* = T^* + S^* = \left(\begin{smallmatrix} A^* & 0 \\ C^* & B^* \end{smallmatrix}\right)$ by Corollary 1 in [6].

According to the closed range theorem, the closedness of $\mathscr{R}(M)$ implies that of $\mathscr{R}(M^*)$. Also, as an operator from $X_3 \oplus X_4$ to $X_1 \oplus \mathscr{R}(B^*)^\perp \oplus \overline{\mathscr{R}(B^*)}$, M^* admits the following block representation

$$M^* = \begin{pmatrix} A^* & 0 \\ C^* & B^* \end{pmatrix} = \begin{pmatrix} A^* & 0 \\ C_1^* & 0 \\ C_2^* & B_1^* \end{pmatrix},$$

where $C_1^* = P_{\mathscr{R}(B^*)^\perp} C^*$, $C_2^* = P_{\overline{\mathscr{R}(B^*)}} C^*$ and $B_1^* = P_{\overline{\mathscr{R}(B^*)}} B^*$. Note that $\mathscr{R}(B_1^*)$ is dense in $\overline{\mathscr{R}(B^*)}$ and $n(B) = \dim \mathscr{R}(B^*)^\perp$. Similar to the proof of Proposition 2.4, we see that $\mathscr{R}(A^*)$ is closed, and hence $\mathscr{R}(A)$ is closed. $\square$

Clearly, we have the result of Proposition 2.5 without any artificial assumptions for bounded operator matrix. In fact, we claim that this still holds true for the unbounded case. In order to remove such assumptions in Proposition 2.5, however, we require the following well known lemma:

LEMMA 2.6. *Let $T : \mathscr{D}(T) \subset X_1 \to X_2$ and $S : \mathscr{D}(S) \subset X_3 \to X_2$ be linear operators. If $\mathscr{R}(T) \subset \mathscr{R}(S)$, then there exists a linear operator $G : \mathscr{D}(T) \to X_3$ such that $T = SG$. In addition, if T is bounded on X_1 and S is closed, then G is bounded on X_1.*

As is stated previously, the domain of an unbounded operator can not be decomposed arbitrarily. But if $n(B|_{\mathscr{D}(B)\cap\mathscr{D}(C)}) < \infty$, then $\mathscr{N}(B|_{\mathscr{D}(B)\cap\mathscr{D}(C)})$ is a closed subspace of $\mathscr{D}(B) \cap \mathscr{D}(C)$, which may provide a useful decomposition method.

PROPOSITION 2.7. *Let $M = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) : \mathscr{D}(A) \oplus (\mathscr{D}(B) \cap \mathscr{D}(C)) \subset X_1 \oplus X_2 \to X_3 \oplus X_4$ be a closed operator. If $\mathscr{R}(M)$ is closed and $n(B) < \infty$, then $\mathscr{R}(A)$ is closed.*

Proof. Write $B_1 = B|_{\mathscr{D}(B)\cap\mathscr{D}(C)}$, then $n(B_1) < \infty$ from $n(B) < \infty$, and hence $\mathscr{N}(B_1)$ is closed. As an operator from $X_1 \oplus \mathscr{N}(B_1) \oplus \mathscr{N}(B_1)^\perp$ to $\overline{\mathscr{R}(A)} \oplus \mathscr{R}(A)^\perp \oplus X_4$, M can be written as

$$M = \begin{pmatrix} A_1 & C_1 & C_2 \\ 0 & C_3 & C_4 \\ 0 & 0 & B_{11} \end{pmatrix},$$

where $A_1 = P_{\overline{\mathscr{R}(A)}} A$, $C_1 = P_{\overline{\mathscr{R}(A)}} C|_{\mathscr{N}(B_1)}$, $C_2 = P_{\overline{\mathscr{R}(A)}} C|_{\mathscr{N}(B_1)^\perp \cap (\mathscr{D}(B)\cap\mathscr{D}(C))}$, $C_3 = P_{\mathscr{R}(A)^\perp} C|_{\mathscr{N}(B_1)}$, $C_4 = P_{\mathscr{R}(A)^\perp} C|_{\mathscr{N}(B)^\perp \cap \mathscr{D}(B)\cap\mathscr{D}(C)}$ and $B_{11} = B|_{\mathscr{N}(B_1)^\perp \cap \mathscr{D}(B)\cap\mathscr{D}(C)}$. From

$n(B_1) < \infty$, we know that $\begin{pmatrix} C_1 \\ C_3 \end{pmatrix}$ is of finite rank, and hence

$$M_1 = \begin{pmatrix} A_1 & 0 & C_2 \\ 0 & 0 & C_4 \\ 0 & 0 & B_{11} \end{pmatrix}$$

is a closed operator and $\mathscr{R}(M_1)$ is closed. Set

$$Q = \begin{pmatrix} I_{\overline{\mathscr{R}(A)}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \overline{\mathscr{R}(A)} \\ \mathscr{R}(A)^{\perp} \\ X_4 \end{pmatrix} \to \begin{pmatrix} \overline{\mathscr{R}(A)} \\ \mathscr{R}(A)^{\perp} \\ X_4 \end{pmatrix}.$$

Then, it follows from $\mathscr{R}(A) \subset \mathscr{R}(M_1)$ that $\mathscr{R}(Q) \subset \mathscr{R}(M_1)$.

By Lemma 2.6, then there exists a bounded G such that

$$Q = M_1 G.$$

Because G is a bounded operator defined on the whole space, it can be written as the following block operator matrix,

$$G = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix} : \begin{pmatrix} \overline{\mathscr{R}(A)} \\ \mathscr{R}(A)^{\perp} \\ X_4 \end{pmatrix} \to \begin{pmatrix} X_1 \\ \mathscr{N}(B_1) \\ \mathscr{N}(B_1)^{\perp} \end{pmatrix}.$$

Then

$$\begin{aligned} Q &= \begin{pmatrix} A_1 & 0 & C_2 \\ 0 & 0 & C_4 \\ 0 & 0 & B_{11} \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix} \\ &= \begin{pmatrix} A_1G_{11} + C_2G_{31} & A_1G_{12} + C_2G_{32} & A_1G_{13} + C_2G_{33} \\ C_4G_{31} & C_4G_{32} & C_4G_{33} \\ B_{11}G_{31} & B_{11}G_{32} & B_{11}G_{33} \end{pmatrix}. \end{aligned}$$

From the above equation, we see that

$$\begin{aligned} A_1G_{11} + C_2G_{31} &= I_{\overline{\mathscr{R}(A)}}, \\ B_{11}G_{31} &= 0. \end{aligned}$$

Thus, $G_{31} = 0$ since B_{11} is injective, and hence $A_1G_{11} = I_{\overline{\mathscr{R}(A)}}$. Note that A_1 is a closed operator and G_{11} is bounded. Therefore, A_1 is right invertible, i.e., $\overline{\mathscr{R}(A)} = \mathscr{R}(A_1) = \mathscr{R}(A)$. This proves that $\mathscr{R}(A)$ is closed. $\square$

In Proposition 2.7, the operator matrix M is required to be closed. The following lemma is devoted to the study for more general cases, which follows from Kato's Lemma ([9, Lemma 331]) by considering the quotient $(\mathscr{N}(S) + \mathscr{R}(T))/\mathscr{N}(S)$.

LEMMA 2.8. *Assume that $T : \mathscr{D}(T) \subset X_1 \to X_2$ is a linear operator, and $S : \mathscr{D}(S) \subset X_2 \to X_3$ is a closed operator with closed range. Then, $\mathscr{R}(ST)$ is closed if $\mathscr{N}(S) + \mathscr{R}(T)$ is closed. Here, in fact, X_1, X_2 and X_3 could be any Banach spaces.*

REMARK 2.9. In particular, when S is bounded with closed range, the similar argument of (1.2) in [5, Theorem 1] holds, i.e., $\mathscr{R}(ST)$ is closed if and only if $\mathscr{N}(S)+\mathscr{R}(T)$ is closed.

PROPOSITION 2.10. *Let $M=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}:\mathscr{D}(A)\oplus\mathscr{D}(B)\subset X_1\oplus X_2\to X_3\oplus X_4$ be a linear operator, and let B be a closed operator with closed range and $n(B)<\infty$. If $\mathscr{R}(\begin{pmatrix} A & C \\ 0 & I \end{pmatrix})$ is closed, then $\mathscr{R}(M)$ is closed. In addition, if B and C are further bounded operators on Y, then the closedness of $\mathscr{R}(A)$ implies that of $\mathscr{R}(M)$.*

Proof. Evidently, the factorization formula

$$M=\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \tag{2.1}$$

holds. Write $S=\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}$, $T_1=\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}$ and $T_2=\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$. From the assumptions, it follows that S is closed with closed range and $n(S)<\infty$. Since $\mathscr{R}(\begin{pmatrix} A & C \\ 0 & I \end{pmatrix})$ is closed, $\mathscr{N}(S)+\mathscr{R}(T_1T_2)$ is closed. Thus, $\mathscr{R}(M)=\mathscr{R}(ST_1T_2)$ is closed by Lemma 2.8.

If B and C are bounded operators on X_2, then T_1 is a bounded operator with a bounded inverse defined on the whole space. Therefore, $\mathscr{R}(T_1T_2)$ is closed if and only if $\mathscr{R}(T_2)$ is closed, which is equivalent to the closedness of $\mathscr{R}(A)$. □

COROLLARY 2.11. *Let $M=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}:\mathscr{D}(A)\oplus X_2\subset X_1\oplus X_2\to X_3\oplus X_4$ be a linear operator with B and C bounded. If $n(B)<\infty$ and $\mathscr{R}(M)$ is closed, then $\mathscr{R}(A)$ is closed.*

Proof. Make the factorization as in (2.1). When B is bounded, S is clearly bounded. From Remark 2.9, it follows that $\mathscr{N}(S)+\mathscr{R}(T_1T_2)$ is closed. Since C is bounded and $n(B)<\infty$, we deduce that $\mathscr{R}(T_2)$ is closed, and hence $\mathscr{R}(A)$ is closed. □

3. Main results

In the following, we analyze the closed range properties of partial triangular operator matrices in the cases $d(A)<\infty$ and $d(A)=\infty$, respectively.

THEOREM 3.1. *Let A be a densely defined closed operator, and let B be a closed operator. If $d(A)<\infty$, then there exists a closable operator C such that $M_C=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}:\mathscr{D}(A)\oplus\mathscr{D}(B)\subset X_1\oplus X_2\to X_3\oplus X_4$ is a closed operator with closed range if and only if*

(i) $\mathscr{R}(B)$ is closed; and

(ii) $\mathscr{R}(A)$ is closed or $n(B)=\infty$.

Proof. Assume that there exists a desired closable operator C such that $M_C=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a closed operator defined on $\mathscr{D}(A)\oplus\mathscr{D}(B)$ with closed range. Then, the claim (i) follows from Proposition 2.4. If $n(B)<\infty$, $\mathscr{R}(A)$ is closed by Proposition 2.7, and hence (ii) holds.

Conversely, if $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are both closed, then M_C is obviously a closed operator with closed range when we take $C=0$.

If $\mathscr{R}(A)$ is not closed and $\mathscr{R}(B)$ is closed with $n(B)=\infty$, then we know $\dim\overline{\mathscr{R}(A)}=\infty$, and hence we may let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be orthogonal bases of $\mathscr{N}(B)$ and $\overline{\mathscr{R}(A)}$, respectively. Define the unitary operator C_1 by

$$C_1 f_i = g_i, \quad i=1,2,3,\ldots.$$

Then, taking $C=\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} : \mathscr{N}(B)\oplus\mathscr{N}(B)^{\perp}\to\overline{\mathscr{R}(A)}\oplus\mathscr{R}(A)^{\perp}$, we immediately obtain the desired operator M_C. In fact, $\begin{pmatrix} A_1 & C_1 \end{pmatrix} : X_1\oplus\mathscr{N}(B)\to\overline{\mathscr{R}(A)}$ is a densely defined closed operator and $\begin{pmatrix} A_1 & C_1 \end{pmatrix}^* = \begin{pmatrix} A_1^* \\ C_1^* \end{pmatrix}$, where $A_1 = P_{\overline{\mathscr{R}(A)}}A$. Thus, $\begin{pmatrix} A_1 & C_1 \end{pmatrix}\begin{pmatrix} A_1^* \\ C_1^* \end{pmatrix} = A_1A_1^* + I_{\overline{\mathscr{R}(A)}} : \overline{\mathscr{R}(A)}\to\overline{\mathscr{R}(A)}$ is a densely defined closed operator with bounded inverse ([7, Proposition 2.14]), which implies that $\mathscr{R}(\begin{pmatrix} A_1 & C_1 \end{pmatrix}\begin{pmatrix} A_1 & C_1 \end{pmatrix}^*)$ is closed in $\overline{\mathscr{R}(A)}$. By [7, Proposition 2.11], we see that $\mathscr{R}(\begin{pmatrix} A_1 & C_1 \end{pmatrix})$ is closed. This together with the closedness of $\mathscr{R}(B)$ deduces that M_C is a closed operator with closed range. □

The following is a simple illustrating example of the result above.

EXAMPLE 3.2. Denote by $L^2[0,+\infty)$ the Hilbert space of square Lebesgue integrable complex-valued functions on $[0,+\infty)$, and by $\mathscr{A}$ the space of complex-valued functions on $[0,+\infty)$ that are absolutely continuous on every compact subinterval of $[0,+\infty)$. Let $X_i = L^2[0,+\infty), i=1,2,3,4$. Consider the operators $B=0$ and A in X_1 defined by

$$\mathscr{D}(A)=\{y\in X_1\cap\mathscr{A} : y'\in X_1, y(0)=0\},$$

$Ay=y'-y$. Clearly, $d(A)<\infty$, $n(B)=\infty$ and $\mathscr{R}(B)$ is closed. Then, by Theorem 3.1 and its proof, we can easily find the desired operator C such that $M_C=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is a closed operator with closed range. Since $B=0$, this example in fact reduces to the completion problem of row operators.

THEOREM 3.3. *Let A and B be densely defined closed operators. If $d(A)=\infty$, then there exists a closable operator C such that $M_C=\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \mathscr{D}(A)\oplus\mathscr{D}(B)\subset X_1\oplus X_2\to X_3\oplus X_4$ is a closed operator with closed range if and only if $\mathscr{R}(A)$ is closed or $n(B)=\infty$.*

Proof. The proof of necessity is the same as that in Theorem 3.1. Now we prove the sufficiency. If $\mathscr{R}(B)$ is closed, then the proof is similar to that in Theorem 3.1.

If $\mathscr{R}(A)$ is closed with $d(A)=\infty$ and $\mathscr{R}(B)$ is not closed, we can take $C=\begin{pmatrix} 0 & 0 \\ 0 & C_4 \end{pmatrix} : \mathscr{N}(B)\oplus\mathscr{N}(B)^{\perp}\to\mathscr{R}(A)\oplus\mathscr{R}(A)^{\perp}$, where the unitary operator C_4 is defined by

$$C_4 f_i = g_i, \quad i=1,2,3,\ldots.$$

Note that the non-closedness of $\mathscr{R}(B)$ implies that $\dim\mathscr{R}(B)=\infty$, and hence $\dim\mathscr{N}(B)^{\perp}=\infty$, and $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are orthogonal bases of $\mathscr{N}(B)^{\perp}$ and $\mathscr{R}(A)^{\perp}$, respectively. In order to prove the closedness of $\mathscr{R}(M_C)$, it suffices to prove that $\mathscr{R}(\begin{pmatrix} C_4 \\ B_1 \end{pmatrix})$

is closed, where $B_1 = B|_{\mathscr{N}(B)^\perp \cap \mathscr{D}(B)}$. Since $\begin{pmatrix} C_4 \\ B_1 \end{pmatrix}$ is a densely defined closed operator and C_4 is bounded, we have $\begin{pmatrix} C_4 \\ B_1 \end{pmatrix}^* = \begin{pmatrix} C_4^* & B_1^* \end{pmatrix}$. Thus, $\mathscr{R}(\begin{pmatrix} C_4^* & B_1^* \end{pmatrix}\begin{pmatrix} C_4 \\ B_1 \end{pmatrix})$ is closed, since $\begin{pmatrix} C_4^* & B_1^* \end{pmatrix}\begin{pmatrix} C_4 \\ B_1 \end{pmatrix} = I_{\mathscr{N}(B)^\perp} + B_1^* B_1$ is a boundedly invertible closed operator. Therefore, $\mathscr{R}\begin{pmatrix} C_4^* & B_1^* \end{pmatrix}$ is closed, and hence $\mathscr{R}\begin{pmatrix} C_4 \\ B_1 \end{pmatrix}$ is closed.

If none of $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are closed and $n(B) = \infty$, we take $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_4 \end{pmatrix}$: $\mathscr{N}(B) \oplus \mathscr{N}(B)^\perp \to \overline{\mathscr{R}(A)} \oplus \mathscr{R}(A)^\perp$. Here C_1 and C_4 are unitary operators defined as follows:

$$C_1 f_i^{(1)} = g_i^{(1)}, \ \ i = 1,2,3,\dots,$$

$$C_4 f_i^{(2)} = g_i^{(2)}, \ \ i = 1,2,3,\dots.$$

Note that the non-closedness of $\mathscr{R}(A)$ implies that $\dim \overline{\mathscr{R}(A)} = \infty$, and $\{f_i^{(1)}\}_{i=1}^\infty$, $\{f_i^{(2)}\}_{i=1}^\infty$, $\{g_i^{(1)}\}_{i=1}^\infty$ and $\{g_i^{(2)}\}_{i=1}^\infty$ are orthogonal bases of $\mathscr{N}(B)$, $\mathscr{N}(B)^\perp$, $\overline{\mathscr{R}(A)}$ and $\mathscr{R}(A)^\perp$, respectively. Thus, $\mathscr{R}(M_C) = \mathscr{R}(\begin{pmatrix} A_1 & C_1 \end{pmatrix}) \oplus \mathscr{R}(\begin{pmatrix} C_4 \\ B_1 \end{pmatrix})$, $\mathscr{R}(M_C M_C^*) = \mathscr{R}(\begin{pmatrix} A_1 & C_1 \end{pmatrix}\begin{pmatrix} A_1 & C_1 \end{pmatrix}^*) \oplus \mathscr{R}(\begin{pmatrix} C_4 \\ B_1 \end{pmatrix}\begin{pmatrix} C_4 \\ B_1 \end{pmatrix}^*)$, and the closedness of $\mathscr{R}(M_C)$ is equivalent to the closedness of $\mathscr{R}(\begin{pmatrix} A_1 & C_1 \end{pmatrix})$ and $\mathscr{R}(\begin{pmatrix} C_4 \\ B_1 \end{pmatrix})$, where $A_1 = P_{\overline{\mathscr{R}(A)}} A$ and B_1 is defined as in last paragraph. Finally, we can easily obtain our result by [7, Proposition 2.11]. □

Based on the discussions in the cases $n(B) < \infty$ and $n(B) = \infty$, we may similarly have the following two theorems. Note that they can not be proved by employing the adjoint operation to the original operator matrix, since the operator matrices involved are unbounded (even not necessarily densely defined).

THEOREM 3.4. *Let A be a closed operator, and let B be a densely defined closed operator. If $n(B) < \infty$, then there exists a closable operator C such that $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$: $\mathscr{D}(A) \oplus \mathscr{D}(B) \subset X_1 \oplus X_2 \to X_3 \oplus X_4$ is a closed operator with closed range if and only if*

(i) $\mathscr{R}(A)$ is closed; and

(ii) $\mathscr{R}(B)$ is closed or $d(A) = \infty$.

Proof. The necessity follows from Propositions 2.4 and 2.7. Conversely, if $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are both closed, then taking $C = 0$ will demonstrate that M_C is a closed operator with closed range. If $\mathscr{R}(A)$ is closed with $d(A) = \infty$ and $\mathscr{R}(B)$ is not closed, we can take $C = \begin{pmatrix} 0 & 0 \\ 0 & C_4 \end{pmatrix} : \mathscr{N}(B) \oplus \mathscr{N}(B)^\perp \to \mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$, where the unitary operator C_4 is defined by

$$C_4 f_i = g_i, \ \ i = 1,2,3,\dots.$$

Note that the non-closeness of $\mathscr{R}(B)$ implies that $\dim \mathscr{N}(B)^\perp = \infty$, and $\{f_i\}_{i=1}^\infty$ and $\{g_i\}_{i=1}^\infty$ are orthonormal bases of $\mathscr{N}(B)^\perp$ and $\mathscr{R}(A)^\perp$, respectively. In order to prove the closeness of $\mathscr{R}(M_C)$, it suffices to prove that $\mathscr{R}(\begin{pmatrix} C_4 \\ B_1 \end{pmatrix})$ is closed, where $B_1 =$

$B|_{\mathscr{D}(B)\cap\mathscr{N}(B)^{\perp}}$. Since $\begin{pmatrix} C_4 \\ B_1 \end{pmatrix} : \mathscr{D}(B)\cap\mathscr{N}(B)^{\perp} \to \mathscr{R}(A)^{\perp}\oplus W$ is a densely defined closed operator and C_4 is bounded, we have $\begin{pmatrix} C_4 \\ B_1 \end{pmatrix}^* = \begin{pmatrix} C_4^* & B_1^* \end{pmatrix}$. Thus, $\mathscr{R}(\begin{pmatrix} C_4^* & B_1^* \end{pmatrix}\begin{pmatrix} C_4 \\ B_1 \end{pmatrix})$ is closed, since $\begin{pmatrix} C_4^* & B_1^* \end{pmatrix}\begin{pmatrix} C_4 \\ B_1 \end{pmatrix} = I_{\mathscr{N}(B)^{\perp}} + B_1^*B_1 : \mathscr{N}(B)^{\perp} \to \mathscr{N}(B)^{\perp}$ is a boundedly invertible closed operator. Therefore, $\mathscr{R}\begin{pmatrix} C_4^* & B_1^* \end{pmatrix}$ is closed, and hence $\mathscr{R}\begin{pmatrix} C_4 \\ B_1 \end{pmatrix}$ is closed. □

THEOREM 3.5. *Let A and B be densely defined closed operators. If $n(B) = \infty$, then there exists a closable operator C such that $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \mathscr{D}(A)\oplus\mathscr{D}(B) \subset X_1 \oplus X_2 \to X_3 \oplus X_4$ is a linear operator with closed range if and only if $\mathscr{R}(B)$ is closed, or $\mathscr{R}(B)$ is not closed and $d(A) = \infty$.*

Proof. The proof of the necessity is the same as in Theorem 3.4. Conversely, the case of A with closed range is similar to that in Theorem 3.4, and the case of A with non-closed range is similar to that in Theorem 3.3. □

By Corollary 2.11, for a general linear operator (not necessarily densely defined closed) A, we actually have the following theorem.

THEOREM 3.6. *Let A be a linear operator, and let B be a bounded operator. If $n(B) < \infty$, then there exists a bounded operator C such that $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \mathscr{D}(A)\oplus\mathscr{D}(B) \subset X_1 \oplus X_2 \to X_3 \oplus X_4$ is a linear operator with closed range if and only if*

(i) $\mathscr{R}(A)$ is closed; and

(ii) $\mathscr{R}(B)$ is closed or $d(A) = \infty$.

Proof. The proof of the necessity holds by Corollary 2.11. The rest of the proof is analogous to that in Theorem 3.3. □

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