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SHARP OPERATOR MEAN INEQUALITIES OF THE NUMERICAL RADII

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Abstract. We present several sharp upper bounds and some extension for product operators. Among other inequalities, it is shown that if $0 < mI \leqslant B^* f^2(|X|)B$, $A^* g^2(|X^*|)A \leqslant MI$, f,g are non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$, $(t \geqslant 0)$, then for all non-negative operator monotone decreasing function h on $[0,\infty)$, we obtain that

$$\left\|h\left(B^* f^2(|X|)B\right)\sigma h\left(A^* g^2(|X^*|)A\right)\right\| \leqslant \frac{mk}{M} h\left(|\langle (A^*XB)x,x\rangle|\right),$$

As an application of the above inequality, it is shown that

$$\omega\left(A^*XB\right) \leqslant \frac{mk}{M}\left\|B^* f^2(|X|)B!A^* g^2(|X^*|)A\right\|,$$

where, $k = \dfrac{(M+m)^2}{4mM}$ and σ is an operator mean s.t., $! \leqslant \sigma \leqslant \triangledown$.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. An operator $A \in \mathscr{B}(\mathscr{H})$ is called positive if $\langle Ax,x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. We write $A \geqslant 0$ if A is positive.

A continuous real-valued function f defined on interval J is said to be operator monotone increasing (decreasing) if for every two positive operators A and B with spectral in J, the inequality $A \leqslant B$ implies $f(A) \leqslant f(B)$ $(f(A) \geqslant f(B))$, respectively. As an example, it is well known that the power function x^r on $(0,\infty)$ is operator monotone increasing if $r \in [0,1]$ and operator monotone decreasing if and only if $r \in [-1,0]$.

If $f : J \to \mathbb{R}$ is a convex function and A is a self-adjoint operator with spectrum in J, then

$$f\left(\langle Ax,x\rangle\right) \leqslant \langle f(A)x,x\rangle. \tag{1.1}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$, and the reverse inequality holds if f is concave (see [10]).

The spectral radius and the numerical radius of $A \in \mathscr{B}(\mathscr{H})$ are defined by $r(A) = \sup\{|\lambda| \ : \ \lambda \in sp(A)\}$ and

$$\omega(A) = \sup\{|\langle Ax,x\rangle| \ : \ x \in H, \ \|x\| = 1\},$$

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respectively. It is well-known that $r(A) \leqslant \omega(A)$ and $\omega(.)$ defines a norm on $\mathscr{B}(\mathscr{H})$, which is equivalent to the usual operator norm $\|.\|$.

In fact, for any $A \in \mathscr{B}(\mathscr{H})$,

$$\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|. \tag{1.2}$$

Kittaneh [9] has shown that for $A \in \mathscr{B}(\mathscr{H})$,

$$\omega^2(A) \leqslant \frac{1}{2}\||A|^2 + |A^*|^2\|, \tag{1.3}$$

which is a refinement of right hand side of inequality (1.2).

Dragomir [5] proved that for any $A, B \in \mathscr{B}(\mathscr{H})$ and for all $p \geqslant 1$,

$$\omega^p(B^*A) \leqslant \frac{1}{2}\|(A^*A)^p + (B^*B)^p\|. \tag{1.4}$$

In [12], it has been shown that if $A, B \in \mathscr{B}(\mathscr{H})$ and $p \geqslant 1$, then

$$\omega^p(B^*A) \leqslant \frac{1}{4}\|(AA^*)^p + (BB^*)^p\| + \frac{1}{2}\omega^p(AB^*), \tag{1.5}$$

which is generalization of inequality (1.4) and in particular cases is sharper than this inequality. Shebrawi et al. [11] generalized inequalities (1.3) and (1.4), as follows:

If $A, B, X \in \mathscr{B}(\mathscr{H})$ and $p \geqslant 1$, we have

$$\omega^p(A^*XB) \leqslant \frac{1}{2}\|(A^*|X^*|A)^p + (B^*|X|B)^p\|. \tag{1.6}$$

In this paper, we first derive a new lower bound for inner-product of products A^*XB involving operator monotone decreasing function, and, so we give refinement of the inequalities (1.4) and (1.6). We prove a numerical radius, which is similar to (1.5) in some example is sharper than (1.5).

In particular, we extend inequality (1.5) and also find some example which show that is a refinement of (1.6). In the next, we present numerical radius inequalities for products of operators, which one of the applications of our results is a generalization of (1.3).

2. Main results

We first recall that for positive invertible operators $A, B \in \mathscr{B}(\mathscr{H})$, the weighted operator arithmetic and harmonic means are defined, by

$$A \bigtriangledown_\nu B = (1-\nu)A + \nu B$$

and

$$A!_\nu B = \left((1-\nu)A^{-1} + \nu B^{-1})\right)^{-1}.$$

It is well-known that if σ_ν is an operator mean, then

$$A!_\nu B \leqslant A\sigma_\nu B \leqslant A \bigtriangledown_\nu B.$$

To prove our numerical radius inequalities, we need several lemmas.

LEMMA 2.1. [7] *If* $A \in B(\mathscr{H})$ *and* f,g *are non-negative continuous functions on* $[0,\infty)$ *satisfying* $f(t)g(t) = t$, $(t \geqslant 0)$, *then for each* $x,y \in \mathscr{H}$

$$|\langle Ax,y\rangle| \leqslant \|f(|A|)x\|\|g(|A^*|)y\|.$$

LEMMA 2.2. [6] *Let* $0 < mI \leqslant A,B \leqslant MI$, $0 \leqslant \nu \leqslant 1$, $!_\nu \leqslant \tau_\nu, \sigma_\nu \leqslant \nabla_\nu$ *and* Φ *be a positive unital linear map. If* h *is an operator monotone decreasing function on* $(0,\infty)$, *then*

$$h\big(\Phi(A)\big)\sigma_\nu h\big(\Phi(B)\big) \leqslant kh\big(\Phi(A\tau_\nu B)\big)$$

where, $k = \dfrac{(M+m)^2}{4mM}$ *stands for the known Kantorovich constant.*

LEMMA 2.3. *Let* $A \in \mathscr{B}(\mathscr{H})$ *be a strictly positive operator. Then for all non-negative decreasing continuous function* h *on* $[0,\infty)$, *we have*

$$\|h(A^{-1})\| \leqslant h(\|A\|^{-1}).$$

Proof. From $A \leqslant \|A\|I$, it follows that $\|A\|^{-1}I \leqslant A^{-1}$. That is $sp(A^{-1}) \subseteq (\|A\|^{-1},\infty)$. So $sp(h(A^{-1})) = h(sp(A^{-1})) \subseteq h(\|A\|^{-1},\infty)$. Since h is decreasing, we have $h(A^{-1}) \leqslant h(\|A\|^{-1})I$ and therefore $\|h(A^{-1})\| \leqslant h(\|A\|^{-1})$. $\square$

THEOREM 2.4. *Let* $A,B,X \in \mathscr{B}(\mathscr{H})$ *and* f,g *are non-negative continuous functions on* $[0,\infty)$ *in which,* $f(t)g(t) = t$, $(t \geqslant 0)$.

If $0 < mI \leqslant B^*f^2(|X|)B$, $A^*g^2(|X^*|)A \leqslant MI$, $h : [0,\infty) \to [0,\infty)$ *is an operator monotone decreasing function and* σ *is an arbitrary mean between* ∇ *and* $!$, *then for any unit vextor* $x \in \mathscr{H}$,

$$\left\|h\big(B^*f^2(|X|)B\big)\sigma h\big(A^*g^2(|X^*|)A\big)\right\| \leqslant \frac{mk}{M}h\left(|\langle(A^*XB)x,x\rangle|\right), \tag{2.1}$$

where, $k = \dfrac{(M+m)^2}{4mM}$.

In particular,

$$\left\|h\big(B^*f^2(|X|)B\big)\sigma h\big(A^*g^2(|X^*|)A\big)\right\| \leqslant h\left(|\langle(A^*XB)x,x\rangle|\right). \tag{2.2}$$

Proof. Let $x \in \mathscr{H}$ be a unit vector. Now applying Lemma 2.1, AM-GM inequality and since every operator monotone decreasing function is operator convex [2], we have

$$\begin{aligned}
\frac{m}{M}h\left(|\langle A^*XBx,x\rangle|\right) &= \frac{m}{M}h\left(|\langle XBx,Ax\rangle|\right)\\
&\geqslant \frac{m}{M}h\left(\sqrt{\langle B^*f^2(|X|)Bx,x\rangle\,\langle A^*g^2(|X^*|)Ax,x\rangle}\right)\\
&\geqslant \frac{m}{M}h\left(\left\langle\left(\frac{B^*f^2(|X|)B + A^*g^2(|X^*|)A}{2}\right)x,x\right\rangle\right)
\end{aligned}$$

$$\geqslant h\left(\frac{m}{M}\left\langle\left(\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}\right)x, x\right\rangle\right)$$
$$\geqslant h\left(\frac{m}{M}\left\|\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}\right\|\right)$$

By hypothesis and operator convexity of $t \mapsto t^{-1}$, we obtain,

$$\left\|\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}\right\| \leqslant M$$

and

$$\left\|\left(\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}\right)^{-1}\right\| \leqslant \frac{1}{m}$$

Therefore

$$\left\|\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}\right\| \leqslant \frac{M}{m}\left\|\left(\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}\right)^{-1}\right\|^{-1} \tag{2.3}$$

By using inequality (2.3) and Lemma 2.3, we have

$$\begin{aligned} & h\left(\frac{m}{M}\left\|\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}\right\|\right) \\ & \geqslant h\left(\left\|\left(\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}\right)^{-1}\right\|^{-1}\right) \\ & \geqslant \left\|h\left(\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}\right)\right\| \\ & \geqslant \frac{1}{k}\left\|h\big(B^* f^2(|X|)B\big)\sigma h\big(A^* g^2(|X^*|)A\big)\right\| \end{aligned}$$

where, in the last inequality, we used Lemma 2.2 for $\nu = \frac{1}{2}$. Hence inequality (2.1) is proved. Now by inequality (2.1) and the fact that $\frac{mk}{M} \leqslant 1$, we obtain inequality (2.2). $\square$

REMARK 2.5. In the assumptions of Theorem 2.4, we can replace $0 < mI \leqslant B^* f^2(|X|)B,\ A^* g^2(|X^*|)A \leqslant MI$ with $0 < mI \leqslant \frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2} \leqslant MI$.

So, if we assume that $\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}$ is invertible, we can conclude (2.2).

Similarly, if $\frac{B^* f^2(|X|)B + A^* g^2(|X^*|)A}{2}$ is not invertible, we can prove that

$$\left\|h\big(B^* f^2(|X|)B + \varepsilon I\big)\sigma h\big(A^* g^2(|X^*|)A + \varepsilon I\big)\right\| \leqslant h(|\langle (A^* XB)x, x\rangle|)$$

and taking limit of $\varepsilon \to 0$, we can conclude (2.2) without the assumption $0 < mI \leqslant B^* f^2(|X|)B$, $A^* g^2(|X^*|)A \leqslant MI$.

REMARK 2.6. Under the assumptions of Theorem 2.4, if $!_\nu \leqslant \sigma_\nu \leqslant \nabla_\nu$ and

$$0 < mI \leqslant (B^* f^2(|X|)B)^{\frac{1}{1-\nu}}, (A^* g^2(|X^*|)A)^{\frac{1}{\nu}} \leqslant MI,$$

then by applying (1.1) for the concave function t^ν $(0 < \nu < 1)$ and AM-GM inequality, respectively, we can write

$$\begin{aligned}\frac{m}{M} h\left(|\langle A^* X B x, x\rangle|^2\right) &\geqslant \frac{m}{M} h\left(\langle B^* f^2(|X|)Bx, x\rangle \langle A^* g^2(|X^*|)Ax, x\rangle\right)\\ &\geqslant \frac{m}{M} h\left(\left\langle (B^* f^2(|X|)B)^{\frac{1}{1-\nu}} x, x\right\rangle^{1-\nu} \left\langle (A^* g^2(|X^*|)A)^{\frac{1}{\nu}} x, x\right\rangle^{\nu}\right)\\ &\geqslant \frac{m}{M} h\left(\left\langle \left[(1-\nu)(B^* f^2(|X|)B)^{\frac{1}{1-\nu}} + \nu (A^* g^2(|X^*|)A)^{\frac{1}{\nu}}\right] x, x\right\rangle\right)\end{aligned}$$

Therefore, by similar argument to the proof of Theorem 2.4, we obtain

$$\left\| h\big((B^* f^2(|X|)B)^{\frac{1}{1-\nu}}\big) \sigma_\nu h\big((A^* g^2(|X^*|)A)^{\frac{1}{\nu}}\big) \right\| \leqslant \frac{mk}{M} h\left(|\langle (A^* X B)x, x\rangle|^2\right) \tag{2.4}$$

LEMMA 2.7. [1] *If A, B are positive operators and f is a non-negative non-decreasing convex function on $[0, \infty)$, then*

$$\|f\big((1-\nu)A + \nu B\big)\| \leqslant \|(1-\nu) f(A) + \nu f(B)\|,$$

for all $0 < \nu < 1$.

Applying Theorem 2.4 to the decreasing convex function $h(t) = t^{-1}$ and $\sigma = \nabla (:= \nabla_{\frac{1}{2}})$, we reach the following corollary:

COROLLARY 2.8. *Let $A, B, X \in \mathscr{B}(\mathscr{H})$ and f, g are non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, $(t \geqslant 0)$. If $0 < mI \leqslant B^* f^2(|X|)B$, $A^* g^2(|X^*|)A \leqslant MI$, then*

$$\omega\big(A^* X B\big) \leqslant \frac{mk}{M} \left\| B^* f^2(|X|)B ! A^* g^2(|X^*|)A \right\|. \tag{2.5}$$

Furthermore, for increasing convex function $h' : [0, \infty) \to [0, \infty)$ s.t. $h'(0) = 0$, we have

$$h'\left(\omega\big(A^* X B\big)\right) \leqslant \frac{mk}{2M} \left\| h'\left(B^* f^2(|X|)B\right) + h'\left(A^* g^2(|X^*|)A\right) \right\|. \tag{2.6}$$

In particular, for all $p \geqslant 1$

$$\omega^p\big(A^* X B\big) \leqslant \frac{mk}{2M} \left\| \big(B^* f^2(|X|)B\big)^p + \big(A^* g^2(|X^*|)A\big)^p \right\|. \tag{2.7}$$

Proof. Let $x \in \mathscr{H}$ be a unit vector. Put $h(t) = t^{-1}$ and $\sigma = \nabla$ in (2.1). Then we have

$$\left\| \frac{\big(B^* f^2(|X|)B\big)^{-1} + \big(A^* g^2(|X^*|)A\big)^{-1}}{2} \right\| \leqslant \frac{mk}{M} \left(|\langle (A^* X B)x, x\rangle|\right)^{-1}$$

Therefore

$$
\begin{aligned}
|\langle (A^*XB)x,x\rangle| &\leqslant \frac{mk}{M}\left\|\frac{\left(B^*f^2(|X|)B\right)^{-1}+\left(A^*g^2(|X^*|)A\right)^{-1}}{2}\right\|^{-1}\\
&\leqslant \frac{mk}{M}\left\|\left(\frac{\left(B^*f^2(|X|)B\right)^{-1}+\left(A^*g^2(|X^*|)A\right)^{-1}}{2}\right)^{-1}\right\|.
\end{aligned}
$$

Taking the supremum over $x\in\mathscr{H}$ with $\|x\|=1$ in the above inequality, we obtain (2.5).

Let us prove (2.6). By an inequality (2.5) and Lemma 2.7, we get

$$
\begin{aligned}
h'\left(\omega\left(A^*XB\right)\right) &\leqslant h'\left(\frac{mk}{2M}\left\|B^*f^2(|X|)B+A^*g^2(|X^*|)A\right\|\right)\\
&\leqslant \frac{mk}{M}h'\left(\left\|\frac{B^*f^2(|X|)B+A^*g^2(|X^*|)A}{2}\right\|\right)\\
&\leqslant \frac{mk}{M}\left\|h'\left(\frac{B^*f^2(|X|)B+A^*g^2(|X^*|)A}{2}\right)\right\|\\
&\leqslant \frac{mk}{2M}\left\|h'\left(B^*f^2(|X|)B\right)+h'\left(A^*g^2(|X^*|)A\right)\right\|
\end{aligned}
$$

The third inequality in the above inequalities follows from (1.1) (in fact, a similar argument to the proof of Lemma 2.3, leads to equality). The last inequality obtains from Lemma 2.7.

By taking $h'(t)=t^p(p>1)$, we reach inequality (2.7). □

By taking $f(t)=g(t)=t^{\frac{1}{2}}$ in an inequality (2.5) we get a refinement of inequality (1.6) for $p=1$, and if we put $f(t)=g(t)=t^{\frac{1}{2}}$ in (2.7), we present a refinement of inequality (1.6).

Applying inequality (2.4) to the decreasing convex function $h(t)=t^{-1}$, one can reach the similar results as Corollary 2.8 (we omit the detail).

The following lemma will be useful in the proof of the next result.

LEMMA 2.9. [3] *Let* $A_1,A_2,B_1,B_2\in\mathscr{B}(\mathscr{H})$. *Then*

$$
\begin{aligned}
r(A_1B_1+A_2B_2) &\leqslant \frac{1}{2}(\omega(B_1A_1)+\omega(B_2A_2))\\
&\quad+\frac{1}{2}\sqrt{(\omega(B_1A_1)-\omega(B_2A_2))^2+4\|B_1A_2\|\|B_2A_1\|}.
\end{aligned}
$$

In the next theorem, we give an inequality similar to (1.5).

THEOREM 2.10. *Let* $A,B\in\mathscr{B}(\mathscr{H})$. *Then for all non-negative non-decreasing convex function* h *on* $[0,\infty)$, *we have*

$$
h\left(\omega(A^*B)\right)\leqslant\frac{1}{2}h(\|A\|\|B\|)+\frac{1}{2}h\left(\omega(BA^*)\right). \tag{2.8}
$$

Proof. Let $\theta \in \mathbb{R}$. Letting $A_1 = e^{i\theta}A^*$, $B_1 = B$, $A_2 = B^*$ and $B_2 = e^{-i\theta}A$ in Lemma 2.9 we can write

$$\begin{aligned}\|Re(e^{i\theta}(A^*B))\| &= r(Re(e^{i\theta}(A^*B)) \\ &\leqslant \frac{1}{4}(\omega(BA^*)+\omega(AB^*)) \\ &\quad + \frac{1}{4}\sqrt{(\omega(BA^*)-\omega(AB^*))^2+4\|AA^*\|\|BB^*\|} \\ &= \frac{1}{2}\omega(BA^*)+\frac{1}{2}\|A\|\|B\|\end{aligned}$$

Hence, by Lemma 2.14 (a) and convexity of h, we get (2.8). □

EXAMPLE 2.11. Letting $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$. Since $\frac{1}{4}\|AA^*+BB^*\| = 7.5432$ and $\frac{1}{2}\|A\|\|B\| = 6.1962$, we can say that inequality (2.8), in this example, is a refinement of (1.5).

COROLLARY 2.12. *Let $A,B \in \mathscr{B}(\mathscr{H})$. Then for all $p \geqslant 1$ we have*

$$\omega^p(A^*B) \leqslant \frac{1}{2}\|A\|^p\|B\|^p + \frac{1}{2}\omega^p(BA^*).$$

COROLLARY 2.13. *Let $A \in \mathscr{B}(\mathscr{H})$, $A = U|A|$ be the polar decomposition of A, and f, g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$ $(t \geqslant 0)$ and let $\tilde{A}_{f,g} = f(|A|)Ug(|A|)$ be generalize the Aluthge transform of A. Then for all $p \geqslant 1$,*

$$\omega^p(A) \leqslant \frac{1}{2}\|f(|A|)\|^p\|g(|A|)\|^p + \frac{1}{2}\omega^p(\tilde{A}_{f,g}).$$

Next, we need the following two lemmas. The first lemma in part (a), which contains a very useful formula of numerical radius, can be found in [13]. Part (b) is well-known (see [4]) and two lemma concerning norm inequalities was given in [8].

LEMMA 2.14. *Let A be an operator in $\mathscr{B}(\mathscr{H})$. Then*

(a) $\omega(A) = \sup_{\theta\in\mathbb{R}}\|\mathrm{Re}(e^{i\theta}A)\| = \frac{1}{2}\sup_{\theta\in\mathbb{R}}\|A+e^{i\theta}A^*\|.$

(b) $\omega\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \max(\omega(A),\omega(B)).$

LEMMA 2.15. *If A_1,A_2,B_1,B_2,X and Y are operators in $\mathscr{B}(\mathscr{H})$. Then*

$$2\|A_1XA_2^*+B_1YB_2^*\| \leqslant \left\|\begin{bmatrix} A_1^*A_1X+XA_2^*A_2 & A_1^*B_1Y+XA_2^*B_2 \\ B_1^*A_1X+YB_2^*A_2 & B_1^*B_1Y+YB_2^*B_2 \end{bmatrix}\right\| \tag{2.9}$$

THEOREM 2.16. *Let $A, B, X \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega(A^*XB) \leqslant \frac{1}{4}\|AA^*X + XBB^*\| + \frac{1}{2}\omega\left(\begin{bmatrix} XBA^* & 0 \\ 0 & BA^*X \end{bmatrix}\right) \tag{2.10}$$

Proof. Applying the first inequality in Lemma 2.14 (a) and by letting $A_1 = B_2 = e^{i\theta}A^*$, $A_2 = B_1 = B^*$ and $Y = X^*$ in inequality (2.9), we have

$$\begin{aligned}
\omega(A^*XB) &= \sup_{\theta\in\mathbb{R}} \left\|\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\mathrm{A}^*\mathrm{XB})\right\| \\
&= \frac{1}{2}\sup_{\theta\in\mathbb{R}} \left\|e^{i\theta}A^*XB + e^{-i\theta}B^*X^*A\right\| \\
&\leqslant \frac{1}{4}\sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} AA^*X + XBB^* & e^{-i\theta}AB^*X^* + e^{i\theta}XBA^* \\ e^{i\theta}BA^*X + e^{-i\theta}X^*AB^* & BB^*X^* + X^*AA^* \end{bmatrix}\right\| \\
&\leqslant \frac{1}{4}\sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} AA^*X + XBB^* & 0 \\ 0 & BB^*X^* + X^*AA^* \end{bmatrix}\right\| \\
&\quad + \frac{1}{4}\sup_{\theta\in\mathbb{R}} \left\|\begin{bmatrix} 0 & e^{i\theta}(XBA^* + e^{-2i\theta}AB^*X^* \\ e^{i\theta}(BA^*X + e^{-2i\theta}X^*AB^* & 0 \end{bmatrix}\right\| \\
&= \frac{1}{4}\left\|AA^*X + XBB^*\right\| \\
&\quad + \frac{1}{4}\sup_{\theta\in\mathbb{R}}\left(\max\left\{\left\|XBA^* + e^{-2i\theta}AB^*X^*\right\|, \left\|BA^*X + e^{-2i\theta}X^*AB^*\right\|\right\}\right)
\end{aligned}$$

Using the second equality in Lemma 2.14 (a), (b), respectively, we deduce the desired inequality (2.10). □

REMARK 2.17. By letting $X = I$ in the inequality (2.10), and by using Lemma 2.14 (b), it is easy to see that the inequality (2.10) generalizes inequality (1.5) for $p = 1$.

EXAMPLE 2.18. Taking $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 1 & 5 \end{bmatrix}$ and $X = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. By an easy computation, we find that

$$\frac{1}{2}\|A^*|X^*|A + B^*|X|B\| \approx 59.5407,$$

$$\frac{1}{4}\|AA^*X + XBB^*\| + \frac{1}{2}\omega\left(\begin{bmatrix} XBA^* & 0 \\ 0 & BA^*X \end{bmatrix}\right) \approx 57.7024$$

and $\omega(A^*XB) \approx 42.2677$. This show that the inequality (2.10), in this example, provides an improvement of the inequality (1.6) for $p = 1$.

COROLLARY 2.19. *Let $A \in \mathscr{B}(\mathscr{H})$, $A = U|A|$ be the polar decomposition of A, and f, g be two non-negative continuous functions on $[0, \infty)$ such that $f(x)g(x) = x$ $(x \geqslant 0)$ and let $\tilde{A}_{f,g} = f(|A|)Ug(|A|)$ be generalize the Aluthge transform of A.. Then for all non-negative and increasing convex function h on $[0, \infty)$, we have*

$$h\big(\omega(A)\big) \leqslant \frac{1}{4}\left\|h\big(f^2(|A|)\big) + h\big(g^2(|A|)\big)\right\| + \frac{1}{2}h\big(\omega(\tilde{A}_{f,g})\big). \tag{2.11}$$

Proof. Since

$$\omega(A) = \omega(Ug(|A|)f(|A|)) = \omega(Ug(|A|)UU^*f(|A|)).$$

If we take $A^* = Ug(|A|)$, $X = U$ and $B = U^*f(|A|)$ in (2.10), we get

$$\omega(A) \leqslant \frac{1}{4}\left\|\left(f^2(|A|)+g^2(|A|)\right)U\right\| + \frac{1}{2}\omega(\tilde{A}_{f,g}).$$

By the fact that $\|U\| = 1$ and convexity of h, we obtain (2.11). □

THEOREM 2.20. *Let $A,B,X \in \mathscr{B}(\mathscr{H})$. Then*

$$\omega(A^*XB + B^*XA) \leqslant \left(\frac{1}{2}\left(\|A\|^2+\|B\|^2\right) + \|AB^*\|\right)\omega(X).$$

Proof. By using the first equality in Lemma 2.14 (a) and the fact that $\mathrm{Re}(e^{i\theta}(A^*XB + B^*XA)) = A^*\mathrm{Re}(e^{i\theta}X)B + B^*\mathrm{Re}(e^{i\theta}X)A$ and putting $A_1 = B_2 = A^*$, $X = Y = \mathrm{Re}(e^{i\theta}X)$ and $A_2 = B_1 = B^*$ in inequality (2.9), we get

$$\begin{aligned}
&\sup_{\theta\in\mathbb{R}}\left\|\mathrm{Re}(e^{i\theta}(A^*XB+B^*XA))\right\| \\
&\leqslant \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} AA^*\mathrm{Re}(e^{i\theta}X)+\mathrm{Re}(e^{i\theta}X)BB^* & AB^*\mathrm{Re}(e^{i\theta}X)+\mathrm{Re}(e^{i\theta}X)BA^* \\ BA^*\mathrm{Re}(e^{i\theta}X)+\mathrm{Re}(e^{i\theta}X)AB^* & BB^*\mathrm{Re}(e^{i\theta}X)+\mathrm{Re}(e^{i\theta}X)AA^* \end{bmatrix}\right\| \\
&\leqslant \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} AA^*\mathrm{Re}(e^{i\theta}X)+\mathrm{Re}(e^{i\theta}X)BB^* & 0 \\ 0 & BB^*\mathrm{Re}(e^{i\theta}X)+\mathrm{Re}(e^{i\theta}X)AA^* \end{bmatrix}\right\| \\
&\quad + \frac{1}{2}\sup_{\theta\in\mathbb{R}}\left\|\begin{bmatrix} 0 & AB^*\mathrm{Re}(e^{i\theta}X)+\mathrm{Re}(e^{i\theta}X)BA^* \\ BA^*\mathrm{Re}(e^{i\theta}X)+\mathrm{Re}(e^{i\theta}X)AB^* & 0 \end{bmatrix}\right\|
\end{aligned}$$

Using the first equality in Lemma 2.14 (a), we obtain

$$\omega(A^*XB + B^*XA) \leqslant \frac{1}{2}(\|A\|^2+\|B\|^2)\omega(X) + \|AB^*\|\omega(X).$$

This completes the proof. □

The following lemma is due to Kittaneh [7]

LEMMA 2.21. *Let $A,B \in \mathscr{B}(\mathscr{H})$ such that $|A|B = B^*|A|$. If f and g are non-negative continuous function on $[0,\infty)$ satisfying $f(t)g(t) = t$ $(t \geqslant 0)$, then for any vectors $x,y \in \mathscr{H}$*

$$|\langle ABx,y\rangle| \leqslant r(B)\|f(|A|)x\|\|g(|A^*|)y\|.$$

THEOREM 2.22. *Let $A,B,X \in \mathscr{B}(\mathscr{H})$ satisfying $|A^*|X = X^*|A^*|$ and f,g be two non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$ $(t \geqslant 0)$. If h is a nonnegative increasing convex function on $[0,\infty)$, then*

$$h\left(\omega^2(A^*XB)\right) \leqslant \left\|(1-\nu)h\left(r^2(X)(B^*f^2(|A^*|)B)^{\frac{1}{1-\nu}}\right) + \nu h\left(r^2(X)g^{\frac{2}{\nu}}(|A|)\right)\right\|$$

for all $0 < \nu < 1$. *Moreover, in special case for* $r(X) \leqslant 1$ *and* $h(0) = 0$, *we have*

$$h\big(\omega^2(A^*XB)\big) \leqslant r^2(X)\big\|(1-\nu)h\big((B^*f^2(|A^*|)B)^{\frac{1}{1-\nu}}\big) + \nu h\big(g^{\frac{2}{\nu}}(|A|)\big)\big\|$$

Proof. Setting $y = x$ in Lemma 2.21 and using (1.1) for the concave function t^ν, respectively, we get

$$\begin{aligned}
|\langle A^*XBx, x\rangle|^2 &\leqslant r^2(X)\|f(|A^*|)Bx\|^2\|g(|A|)x\|^2 \\
&= r^2(X)\langle B^*f^2(|A^*|)Bx, x\rangle\langle g^2(|A|)x, x\rangle \\
&= r^2(X)\left\langle \left((B^*f^2(|A|)B)^{\frac{1}{1-\nu}}\right)^{1-\nu} x, x\right\rangle \left\langle \left((g^2(|A|))^{\frac{1}{\nu}}\right)^{\nu} x, x\right\rangle \\
&\leqslant r^2(X)\left\langle (B^*f^2(|A^*|)B)^{\frac{1}{1-\nu}} x, x\right\rangle^{1-\nu} \left\langle (g^2(|A|))^{\frac{1}{\nu}} x, x\right\rangle^{\nu} \\
&\leqslant r^2(X)\left\langle (1-\nu)\left(B^*f^2(|A^*|)B\right)^{\frac{1}{1-\nu}} + \nu g^{\frac{2}{\nu}}(|A|)x, x\right\rangle.
\end{aligned}$$

Hence by taking the supremum over $x \in \mathscr{H}$, we get

$$\omega^2(A^*XB) \leqslant r^2(X)\big\|(1-\nu)\left(B^*f^2(|A^*|)B\right)^{\frac{1}{1-\nu}} + \nu g^{\frac{2}{\nu}}(|A|)\big\|.$$

Since h is an increasing convex function, we have

$$\begin{aligned}
h\big(\omega^2(A^*XB)\big) &\leqslant h\left(r^2(X)\big\|(1-\nu)\left(B^*f^2(|A^*|)B\right)^{\frac{1}{1-\nu}} + \nu g^{\frac{2}{\nu}}(|A|)\big\|\right) \\
&= \left\|h\left(r^2(X)(1-\nu)\left(B^*f^2(|A^*|)B\right)^{\frac{1}{1-\nu}} + \nu g^{\frac{2}{\nu}}(|A|)\right)\right\| \\
&\leqslant \big\|(1-\nu)h\big(r^2(X)(B^*f^2(|A^*|)B)^{\frac{1}{1-\nu}}\big) + \nu h\big(r^2(X)g^{\frac{2}{\nu}}(|A|)\big)\big\|
\end{aligned}$$

where, in the last inequality we used Lemma 2.7. □

Now we present some applications of Theorem 2.22.
Letting $f(t) = t^{1-\nu}$ and $g(t) = t^\nu$ for $0 < \nu < 1$ in Theorem 2.22 we get

COROLLARY 2.23. *Let* $A, B, X \in \mathscr{B}(\mathscr{H})$ *satisfying* $|A^*|X = X^*|A^*|$. *If* h *is a nonnegative increasing convex function on* $[0, \infty)$, *then for all* $0 < \nu < 1$

$$h\big(\omega^2(A^*XB)\big) \leqslant \big\|(1-\nu)h\big(r^2(X)(B^*|A^*|^2B)\big) + \nu h\big(r^2(X)|A|^2\big)\big\|.$$

Inparticullar, for $r(X) \leqslant 1$ *and* $h(0) = 0$

$$h\big(\omega^2(A^*XB)\big) \leqslant r^2(X)\big\|(1-\nu)h(B^*|A^*|^2B) + \nu h(|A|^2)\big\|.$$

By the convexity $h(t) = t^p$ for $p \geqslant 1$ we have

COROLLARY 2.24. *Let* $A, B, X \in \mathscr{B}(\mathscr{H})$, *then for all* $0 < \nu < 1$ *and* $p \geqslant 1$

$$\omega^{2p}(A^*XB) \leqslant r^{2p}(X)\big\|(1-\nu)(B^*|A^*|^2B)^p + \nu|A|^{2p}\big\|.$$

In addition, by using Theorem 2.22 and corollaries 2.23, 2.24 for $X = B = I$, we obtain several generalization of inequality 1.3.

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