Operators and Matrices
Volume 15, Number 2 (2021), 435–450
doi:10.7153/oam-2021-15-30

NEW RESULTS ON α–SPECTRAL RADIUS OF GRAPHS

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(*Communicated by R. A. Brualdi*)

Abstract. For $0 \leqslant \alpha < 1$, Nikiforov proposed to study the spectral properties of the family of matrices $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$ of a graph G, where $D(G)$ is the degree diagonal matrix and $A(G)$ is the adjacency matrix. The α-spectral radius of G is the largest eigenvalue of $A_\alpha(G)$. For $0 \leqslant \alpha < 1$, we give a lower bound for the α-spectral radius, and bounds for the maximum and minimum entries of the α-Perron vector, and we determine the unique graph with maximum α-spectral radius among graphs with given number of odd vertices.

1. Introduction

We consider simple graphs. Let G be a graph on n vertex set $V(G) = \{v_1, \dots, v_n\}$ and edge set $E(G)$. For $v \in V(G)$, let $\delta_G(v)$ (or δ_v) and $N_G(v)$ be the degree of v and the set of neighbors of v in G, respectively. We say G is r-regular if the degree of each vertex is r. The adjacency matrix $A(G)$ of G is an $n \times n$ matrix (a_{ij}), where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Let $D(G)$ be the diagonal matrix of the degrees of G. The signless Laplacian matrix of G is known as $Q(G) = D(G) + A(G)$. The spectral properties of the adjacency matrix and the signless Laplacian matrix of a graph have been investigated for a long time, see, e.g., [3, 4, 7, 11, 12, 13]. For any real $\alpha \in [0,1)$, Nikiforov [8] proposed to study the spectral properties of the family of matrices $A_\alpha(G)$ defined by the convex linear combination:

$$A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G).$$

It is easily seen that $A(G) = A_0(G)$ and $Q(G) = 2A_{\frac{1}{2}}(G)$.

For any real $\alpha \in [0,1)$, $A_\alpha(G)$ is a symmetric nonnegative matrix, and thus its eigenvalues are all real. We call the largest eigenvalue of $A_\alpha(G)$ the α-spectral radius of G, denoted by $\rho_\alpha(G)$. If G is connected, then for $0 \leqslant \alpha < 1$, $A_\alpha(G)$ is irreducible, we have by the Perron-Frobenius theorem that $\rho_\alpha(G)$ is simple and positive, and there is a unique positive unit eigenvector corresponding to $\rho_\alpha(G)$, which is called the α-Perron vector of G, see [8].

Let K_n be the complete graph on n vertices. Nikiforov [8] showed that the r-partite Turán graph is the unique graph with maximum α-spectral radius for $0 < \alpha <$

Mathematics subject classification (2020): 05C50, 15A18.
Keywords and phrases: α-spectral radius, α-Perron vector, graph, odd vertices, maximum degree.
This research was supported by the National Natural Science Foundation of China (Nos. 11801410 and 11671156).
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$1-\frac{1}{r}$ among K_{r+1}-free graphs on n vertices with $r \geqslant 2$. Nikiforov and Rojo [9] determined the unique graph with maximum α-spectral radius among connected graphs on n vertices with diameter (at least) k. Guo and Zhou [5] gave upper bounds for α-spectral radius for unicyclic graphs with given maximum degree connected irregular graphs with given maximum degree and some other graph parameters, and graphs with given domination number, respectively. They also determined the unique tree with maximum α-spectral radius among trees with given diameter.

A vertex in a graph is said to be odd (even, respectively) if its degree is odd (even, respectively). It is well known that the number of odd vertices in a graph is always even.

In this paper, for $0 \leqslant \alpha < 1$, we obtain a lower bound for the α-spectral radius, and bounds for the maximum and minimum entries of the α-Perron vector, and determine the unique graph with maximum α-spectral radius among graphs with given number of odd vertices.

2. Preliminaries

Let G be a graph with $V(G)=\{v_1,\dots,v_n\}$. A column vector $x=(x_{v_1},\dots,x_{v_n})^\top \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ that maps vertex v_i to x_{v_i}, i.e., $x(v_i)=x_{v_i}$ for $i=1,\dots,n$. Then

$$x^\top A_\alpha(G)x = \alpha \sum_{u\in V(G)} \delta_G(u)x_u^2 + 2(1-\alpha)\sum_{uv\in E(G)} x_u x_v.$$

Moreover, λ is an eigenvalue of $A_\alpha(G)$ if and only if $x \neq 0$ and for each $u \in V(G)$, we have the following eigenequation:

$$\lambda x_u = \alpha\delta_G(u)x_u + (1-\alpha)\sum_{v\in N_G(u)} x_v.$$

LEMMA 1. [8] *Let G be a connected graph. If H is an induced subgraph of G, then for $0 \leqslant \alpha < 1$, $\rho_\alpha(H) < \rho_\alpha(G)$.*

LEMMA 2. [8] *Let G be a connected graph with η being an automorphism of G. Let $0 \leqslant \alpha < 1$ and x be the α-Perron vector of G. Then for $u,v \in V(G)$, $\eta(u)=v$ implies that $x_u = x_v$.*

For a vertex subset $W \subseteq V(G)$, let $G[W]$ be the subgraph of G induced by W.

For an edge subset S of G, $G-S$ denotes the graph obtained from G by deleting the edges in S. For an edge subset S' of the complement of G, $G+S'$ denotes the graph obtained from G by adding the edges in S'. If $S=\{e\}$ and $S'=\{e'\}$, then we simplify $G-\{e\}$ as $G-e$ and $G+\{e'\}$ as $G+e'$.

LEMMA 3. [5] *Let G be a connected graph with $u,v \in V(G)$. Suppose that $v_1,\dots,v_s \in N_G(v)\setminus(N_G(u)\cup\{u\})$ with $1 \leqslant s \leqslant \delta_G(v)$. Let $G'=G-\{vv_i : i=1,\dots,s\}+\{uv_i : i=1,\dots,s\}$. Let $0 \leqslant \alpha < 1$ and x be the α-Perron vector of G. If $x_u \geqslant x_v$, then $\rho_\alpha(G') > \rho_\alpha(G)$.*

For two vertex disjoint graphs G and H, the join of G and H, written as $G \vee H$, is the graph obtained by joining each vertex of $V(G)$ to each vertex of $V(H)$.

Denote by $J_{s\times t}$ the $s\times t$ all ones matrix, and as usual, I_n denotes the unit matrix of order n.

3. Lower bounds for the α-spectral radius

Let G be a graph on n vertices with m edges. For $0 \leqslant \alpha < 1$,

$$\rho_\alpha(G) \geqslant \frac{2m}{n}$$

with equality if and only if G is regular. From [8, Proposition 18] and [1], if G is irregular graph, then $\rho_\alpha(G) \geqslant \rho_0(G)$ and $\rho_0(G) - \frac{2m}{n} \geqslant \frac{1}{n(\Delta+2)}$, so

$$\rho_\alpha(G) - \frac{2m}{n} \geqslant \frac{1}{n(\Delta+2)}, \tag{1}$$

where Δ is the maximum degree of G. Here we give lower bounds for the difference between $\rho_\alpha(G)$ and $\frac{2m}{n}$ by using the techniques in [1].

THEOREM 1. *Let G be a connected graph on n vertices with m edges. Suppose that $\emptyset \neq S \subseteq V(G)$ and $s = |S|$. Then for $0 \leqslant \alpha < 1$,*

$$\rho_\alpha(G) - \frac{2m}{n} \geqslant \frac{\alpha}{cn}\sum_{u\in S}\left(\frac{s\delta_u^3}{\sum_{w\in S}\delta_w^2} - \delta_u\right) + \frac{2(1-\alpha)}{cn}\left(\sqrt{s\sum_{u\in S}\delta_u^2} - \sum_{u\in S}\delta_u\right),$$

where $c = 1$ if S is an independent set, and $c = 2$ otherwise.

Proof. Write $D = D(G)$ and $A = A(G)$. Let $s = |S|$. If $s = 1$, the result is trivial. Suppose in the following that $s \geqslant 2$.

Suppose first that S is an independent set. Let x be a positive vector such that $x_u = \frac{a_u}{\sqrt{n}}$ for $u \in S$ and $x_u = \frac{1}{\sqrt{n}}$ for $u \in V(G)\setminus S$ with $\sum_{u\in S} a_u^2 = s$, where the value of a_u for $u \in S$ will be determined later. It is easily seen that x is unit. Then

$$\begin{aligned}
\rho_\alpha(G) - \frac{2m}{n} &\geqslant x^\top(\alpha D + (1-\alpha)A)x - \frac{2m}{n}\\
&= \alpha\sum_{u\in V(G)}\delta_u x_u^2 + (1-\alpha)\sum_{vw\in E(G)} 2x_v x_w - \frac{\alpha+1-\alpha}{n}\sum_{u\in V(G)}\delta_u\\
&= \alpha\sum_{u\in V(G)}\left(\delta_u x_u^2 - \frac{\delta_u}{n}\right) + (1-\alpha)\left(\sum_{vw\in E(G)} 2x_v x_w - \sum_{u\in V(G)}\frac{\delta_u}{n}\right)\\
&= \frac{\alpha}{n}\left(\sum_{u\in S}\delta_u(a_u^2-1)\right) + \frac{2(1-\alpha)}{n}\left(\sum_{u\in S}a_u\delta_u - \sum_{u\in S}\delta_u\right).
\end{aligned} \tag{2}$$

Let $S=\{u_1,\dots,u_s\}$. Choose $a_{u_1},\dots,a_{u_s}$ such that $\frac{a_{u_1}}{\delta_{u_1}}=\cdots=\frac{a_{u_s}}{\delta_{u_s}}$. Then

$$\left(\sum_{u\in S} a_u\delta_u\right)^2=\left(\sum_{u\in S} a_u^2\right)\left(\sum_{u\in S}\delta_u^2\right)=s\sum_{u\in S}\delta_u^2.$$

As $\sum_{u\in S} a_u^2=s$, we have $a_{u_i}^2=\frac{s\delta_{u_i}^2}{\sum_{u\in S}\delta_u^2}$ for $1\leqslant i\leqslant s$. Then from (2), we have

$$\rho_\alpha(G)-\frac{2m}{n}\geqslant\frac{\alpha}{n}\sum_{u\in S}\left(\frac{s\delta_u^3}{\sum_{w\in S}\delta_w^2}-\delta_u\right)+\frac{2(1-\alpha)}{n}\left(\sqrt{s\sum_{u\in S}\delta_u^2}-\sum_{u\in S}\delta_u\right),$$

as desired.

Now suppose that S is not an independent set in G. Let G' be the bipartite graph with vertex set $V(G)\times\{1,2\}$ such that for $(u,i),(v,j)\in V(G)\times\{1,2\}$, (u,i) is adjacent to (v,j) in G' if and only if u is adjacent to v in G and $i\neq j$. Then

$$A_\alpha(G')=\begin{pmatrix}\alpha D & (1-\alpha)A\\ (1-\alpha)A & \alpha D\end{pmatrix},$$

and thus

$$\begin{aligned}\det(tI_{2n}-A_\alpha(G'))&=\det\begin{pmatrix}tI_n-\alpha D & -(1-\alpha)A\\ -(1-\alpha)A & tI_n-\alpha D\end{pmatrix}\\ &=\det\left((tI_n-\alpha D)^2-(1-\alpha)^2A^2\right)\\ &=\det((tI_n-(\alpha D+(1-\alpha)A))(tI_n-(\alpha D-(1-\alpha)A)))\\ &=\det(tI_n-A_\alpha(G))\cdot\det(tI_n-(\alpha D-(1-\alpha)A)).\end{aligned}$$

It follows that the eigenvalues of $A_\alpha(G')$ are just the union of the eigenvalues of $A_\alpha(G)$ and the eigenvalues of $\alpha D-(1-\alpha)A$. By [6, Corollary 2.1, p. 38], for any eigenvalue λ of $\alpha D-(1-\alpha)A$, $|\lambda|\leqslant\rho_\alpha(G)$. Thus $\rho_\alpha(G')=\rho_\alpha(G)$. Note that $S\times\{1\}$ is an independent set in G'. Applying (2) to G', we have

$$\begin{aligned}\rho_\alpha(G)-\frac{2m}{n}&=\rho_\alpha(G')-\frac{4m}{2n}\\ &\geqslant\frac{\alpha}{2n}\sum_{u\in S}\left(\frac{s\delta_u^3}{\sum_{w\in S}\delta_w^2}-\delta_u\right)+\frac{1-\alpha}{n}\left(\sqrt{s\sum_{u\in S}\delta_u^2}-\sum_{u\in S}\delta_u\right),\end{aligned}$$

as desired. $\square$

COROLLARY 1. *Let G be an irregular graph on n vertices with m edges, maximum degree Δ and minimum degree δ. For $0\leqslant\alpha<1$,*

$$\rho_\alpha(G)-\frac{2m}{n}\geqslant\frac{\alpha(\Delta-\delta)^2(\Delta+\delta)}{2n(\Delta^2+\delta^2)}+\frac{(1-\alpha)(\Delta-\delta)^2}{n\left(\sqrt{2(\Delta^2+\delta^2)}+\Delta+\delta\right)}. \tag{3}$$

Proof. Let u,v be two vertices such that $\delta_G(u)=\Delta$ and $\delta_G(v)=\delta$. Let $S=\{u,v\}$. By Theorem 1, the result follows. $\square$

Now we compare (1) and (3). For the star $K_{1,n-1}$ on n vertices with $n\geqslant 7$, the bound in (1) is $\frac{1}{n(n+1)}$, while the bound in (3) is $\frac{\alpha(n-2)^2n}{2n(n^2-2n+2)}+\frac{(1-\alpha)(n-2)^2}{n\left(\sqrt{2(n^2-2n+2)}+n\right)}\geqslant \frac{(n-2)^2}{2n(n^2-2n+2)}>\frac{1}{3n}$. The lower bound in (3) is larger than the one in (1).

For the path P_n on n vertices with $n\geqslant 3$, the bound in (1) is $\frac{1}{3n}$, while the bound in (3) is $\frac{3\alpha}{10n}+\frac{(1-\alpha)}{n(\sqrt{10}+3)}<\frac{3}{10n}$. The lower bound in (1) is larger than the one in (3). Thus the lower bounds in (1) and (3) are incomparable in general.

4. Extreme entries of the α-Perron vector

For a connected graph G and $0\leqslant\alpha<1$, let $x_{\max}$ and $x_{\min}$ be the maximum and minimum entries of the α-Perron vector x of G, respectively. For $\alpha=0$, Cioabă and Gregory [2] proved

$$\sqrt{\frac{\Delta}{\Delta+\rho_0(G)^2}}\geqslant x_{\max}\geqslant\frac{\rho_0(G)}{\sqrt{\sum_{v\in V(G)}\delta_v^2}}$$

and

$$x_{\min}<\frac{(\Delta-\rho_0(G))\sqrt{n}}{n\Delta-2m},$$

and Nikiforov [10] proved

$$x_{\min}\leqslant\sqrt{\frac{\delta}{\rho_0(G)^2+\delta(n-\delta)}},$$

where n is the number of vertices, m is the number of edges, and Δ and δ are the maximum and minimum degrees, respectively. The arguments in [2, 10] lead to the following results. For completeness, we include a proof here.

THEOREM 2. *Let G be a connected graph with maximum degree $\Delta\geqslant 1$. Let $\rho_\alpha=\rho_\alpha(G)$. For $0\leqslant\alpha<1$,*

$$\frac{\rho_\alpha-\alpha\Delta}{(1-\alpha)\sqrt{\sum_{u\in V(G)}\delta_u^2}}\leqslant x_{\max}\leqslant(1-\alpha)\sqrt{\frac{\Delta}{(1-\alpha)^2\Delta+(\rho_\alpha-\alpha\Delta)^2}}.$$

The left equality holds if and only if G is regular, and the right equality holds if and only if G is the join of a vertex u and a regular graph on $n-1$ vertices.

Proof. For $u\in V(G)$, we have

$$(\rho_\alpha-\alpha\Delta)x_u\leqslant(\rho_\alpha-\alpha\delta_u)x_u=(1-\alpha)\sum_{v\in N_G(u)}x_v\leqslant(1-\alpha)\delta_u x_{\max},$$

i.e.,

$$(\rho_\alpha - \alpha\Delta)^2 x_u^2 \leqslant x_{\max}^2 (1-\alpha)^2 \delta_u^2.$$

Note that $\sum_{u\in V(G)} x_u^2 = 1$. Summing the above equation for each vertex $u \in V(G)$, we have

$$(\rho_\alpha - \alpha\Delta)^2 \leqslant x_{\max}^2 (1-\alpha)^2 \sum_{u\in V(G)} \delta_u^2.$$

Thus

$$x_{\max} \geqslant \frac{\rho_\alpha - \alpha\Delta}{(1-\alpha)\sqrt{\sum_{u\in V(G)} \delta_u^2}}$$

with equality if and only if $x_u = x_{\max}$ for each $u \in V(G)$, that is, G is regular.

Suppose that u is a vertex in $V(G)$ such that $x_u = x_{\max}$. By Cauchy-Schwarz inequality, we have

$$(\rho_\alpha - \alpha\delta_u)x_u = (1-\alpha)\sum_{v\in N_G(u)} x_v \leqslant (1-\alpha)\sqrt{\delta_u \sum_{v\in N_G(u)} x_v^2},$$

i.e.,

$$\sum_{v\in N_G(u)} x_v^2 \geqslant \frac{(\rho_\alpha - \alpha\delta_u)^2 x_u^2}{(1-\alpha)^2\delta_u}.$$

Then

$$1 = \sum_{v\in V(G)} x_v^2 \geqslant x_u^2 + \sum_{v\in N_G(u)} x_v^2 \geqslant x_u^2\left(1 + \frac{(\rho_\alpha - \alpha\delta_u)^2}{(1-\alpha)^2\delta_u}\right) \geqslant x_u^2\left(1 + \frac{(\rho_\alpha - \alpha\Delta)^2}{(1-\alpha)^2\Delta}\right),$$

and thus

$$x_{\max} = x_u \leqslant \frac{1}{\sqrt{1 + \frac{(\rho_\alpha - \alpha\Delta)^2}{(1-\alpha)^2\Delta}}} = (1-\alpha)\sqrt{\frac{\Delta}{(1-\alpha)^2\Delta + (\rho_\alpha - \alpha\Delta)^2}}.$$

Suppose that the equality holds. Then all above inequalities are equalities, and thus $V(G) = \{u\} \cup N_G(u)$ and $x_v = \frac{(\rho_\alpha - \alpha\delta_u)x_u}{(1-\alpha)\delta_u}$ for $v \in N_G(u)$. Since for $v_1, v_2 \in N_G(u)$,

$$\rho_\alpha x_{v_1} = \alpha\delta_{v_1} x_{v_1} + (1-\alpha)\left(x_u + (\delta_{v_1} - 1)x_{v_1}\right) \text{ and}$$
$$\rho_\alpha x_{v_2} = \alpha\delta_{v_2} x_{v_2} + (1-\alpha)\left(x_u + (\delta_{v_2} - 1)x_{v_2}\right),$$

we have $\delta_{v_1} = \delta_{v_2}$. Then $G - u$ is regular, and thus G is the join of a vertex u and a regular graph on $n-1$ vertices. Conversely, suppose that G is the join of a vertex u and a regular graph H of degree r on $n-1$ vertices. Let

$$c = \frac{\alpha(n-2) - r + \sqrt{(\alpha(n-2) - r)^2 + 4(1-\alpha)^2(n-1)}}{2(1-\alpha)}.$$

Evidently, note that $c \geqslant 1$ is equivalent to

$$\sqrt{(\alpha(n-2)-r)^2+4(1-\alpha)^2(n-1)} \geqslant 2+r-\alpha n,$$

i.e.,

$$\alpha n-(r+1+\alpha)+(1-\alpha)(n-1)=(n-1)-(r+1) \geqslant 0.$$

Let y be a vector defined on $V(G)$ such that $y_w = c$ if $w = u$ and $y_w = 1$ otherwise. Then

$$\begin{aligned}
A_\alpha(G)y &= \begin{pmatrix} \alpha(n-1) & (1-\alpha)J_{1\times(n-1)} \\ (1-\alpha)J_{(n-1)\times 1} & A_\alpha(H)+\alpha I_{n-1} \end{pmatrix} \begin{pmatrix} c \\ J_{(n-1)\times 1} \end{pmatrix} \\
&= \begin{pmatrix} (\alpha c+1-\alpha)(n-1) \\ ((1-\alpha)c+r+\alpha)J_{(n-1)\times 1} \end{pmatrix} \\
&= ((1-\alpha)c+r+\alpha) \begin{pmatrix} \frac{(\alpha c+1-\alpha)(n-1)}{(1-\alpha)c+r+\alpha} \\ J_{(n-1)\times 1} \end{pmatrix} \\
&= ((1-\alpha)c+r+\alpha) \begin{pmatrix} c \\ J_{(n-1)\times 1} \end{pmatrix}.
\end{aligned}$$

Note that y is a positive vector. By the Perron-Frobenius theorem, $x := \frac{y}{\sqrt{c^2+n-1}}$ is the α-Perron vector of G and $\rho_\alpha = (1-\alpha)c+r+\alpha$. By the expression of c, we have $\frac{(\alpha c+1-\alpha)(n-1)}{\rho_\alpha} = c$, so $c = \frac{(1-\alpha)(n-1)}{\rho_\alpha-\alpha(n-1)}$. Thus

$$x_{\max} = \frac{c}{\sqrt{c^2+n-1}} = \frac{1}{\sqrt{1+\frac{(\rho_\alpha-\alpha(n-1))^2}{(1-\alpha)^2(n-1)}}},$$

as desired. □

THEOREM 3. *Let G be a connected irregular graph on n vertices with m edges and maximum degree Δ. Write $\rho_\alpha = \rho_\alpha(G)$. For $0 \leqslant \alpha < 1$,*

$$x_{\min} < \frac{(\Delta-\rho_\alpha)\sqrt{n}}{n\Delta-2m}.$$

Proof. For $u \in V(G)$,

$$\rho_\alpha x_u = \alpha\delta_u x_u + (1-\alpha)\sum_{v\in N_G(u)} x_v.$$

Summing the above equation for each vertex $u \in V(G)$,

$$\begin{aligned}
\rho_\alpha \sum_{u\in V(G)} x_u &= \alpha \sum_{u\in V(G)} \delta_u x_u + (1-\alpha)\sum_{u\in V(G)}\sum_{v\in N_G(u)} x_v \\
&= \alpha \sum_{u\in V(G)} \delta_u x_u + (1-\alpha)\sum_{u\in V(G)} \delta_u x_u \\
&= \sum_{u\in V(G)} \delta_u x_u.
\end{aligned}$$

Then

$$\begin{aligned}(\Delta-\rho_\alpha)\sqrt{n} &\geqslant (\Delta-\rho_\alpha)\sum_{u\in V(G)} x_u\\ &= \sum_{u\in V(G)}(\Delta-\delta_u)x_u\\ &\geqslant \sum_{u\in V(G)}(\Delta-\delta_u)x_{\min}\\ &= x_{\min}(n\Delta-2m),\end{aligned}$$

where the first inequality follows from Cauchy-Schwarz inequality and the fact that $\sum_{u\in V(G)} x_u^2 = 1$. Thus

$$x_{\min} \leqslant \frac{(\Delta-\rho_\alpha)\sqrt{n}}{n\Delta-2m}.$$

If the equality holds, then the above inequalities are all equalities, and thus from Cauchy-Schwarz inequality, all entries of the α-Perron vector of G are equal, implying that G is regular, which is impossible. Hence

$$x_{\min} < \frac{(\Delta-\rho_\alpha)\sqrt{n}}{n\Delta-2m},$$

as desired. □

From Theorem 3 and (1), we have the following result immediately.

COROLLARY 2. *Let G be a connected irregular graph on n vertices with m edges and maximum degree Δ. For $0 \leqslant \alpha < 1$,*

$$x_{\min} < \frac{\Delta n - 2m - \frac{1}{\Delta+2}}{\sqrt{n}(\Delta n - 2m)}.$$

THEOREM 4. *Let G be a connected graph on $n \geqslant 2$ vertices with minimum degree δ. Write $\rho_\alpha = \rho_\alpha(G)$. For $0 \leqslant \alpha < 1$,*

$$x_{\min} \leqslant (1-\alpha)\sqrt{\frac{\delta}{(\rho_\alpha-\alpha\delta)^2+(1-\alpha)^2(n-\delta)\delta}}$$

with equality if G is a regular graph or the join of $(n-\delta)K_1$ and an r-regular graph on δ vertices, where $n+r > 2\delta$.

Proof. Let u be a vertex in $V(G)$ such that $\delta_G(u) = \delta$. By Cauchy-Schwarz inequality, we have

$$(\rho_\alpha-\alpha\delta)x_{\min} \leqslant (\rho_\alpha-\alpha\delta)x_u \leqslant (1-\alpha)\sqrt{\delta\sum_{v\in N_G(u)} x_v^2}.$$

Since $\sum_{v\in V(G)} x_v^2 = 1$, we have

$$(\rho_\alpha - \alpha\delta)x_{\min} \leqslant (1-\alpha)\sqrt{\delta\left(1 - \sum_{v\in V(G)\setminus N_G(u)} x_v^2\right)} \leqslant (1-\alpha)\sqrt{\delta\left(1-(n-\delta)x_{\min}^2\right)},$$

and thus

$$x_{\min} \leqslant (1-\alpha)\sqrt{\frac{\delta}{(\rho_\alpha-\alpha\delta)^2+(1-\alpha)^2(n-\delta)\delta}}.$$

It is easily seen that the equality holds if G is regular. Suppose that $G = (n-\delta)K_1 \vee H$ with H being an r-regular graph on δ vertices, where $n-\delta+r > \delta$. Let

$$c = \frac{-\alpha(n-2\delta)-r+\sqrt{(\alpha(n-2\delta)+r)^2+4(1-\alpha)^2\delta(n-\delta)}}{2(1-\alpha)(n-\delta)}.$$

Evidently, $c > 0$. It may be easily checked that $c < 1$. Let y be a vector defined on $V(G)$ such that $y_w = c$ if $w \notin V(H)$ and $y_w = 1$ if $w \in V(H)$. Then

$$\begin{aligned}
A_\alpha(G)y &= \begin{pmatrix} \alpha\delta I_{n-\delta} & (1-\alpha)J_{(n-\delta)\times\delta} \\ (1-\alpha)J_{\delta\times(n-\delta)} & A_\alpha(H_\delta)+\alpha(n-\delta)I_\delta \end{pmatrix} y \\
&= \begin{pmatrix} (\alpha\delta c+(1-\alpha)\delta)J_{(n-\delta)\times 1} \\ ((1-\alpha)c(n-\delta)+r+(n-\delta)\alpha)J_{\delta\times 1} \end{pmatrix} \\
&= ((1-\alpha)c(n-\delta)+r+(n-\delta)\alpha)\begin{pmatrix} \frac{\alpha\delta c+(1-\alpha)\delta}{(1-\alpha)c(n-\delta)+r+(n-\delta)\alpha} J_{(n-\delta)\times 1} \\ J_{\delta\times 1} \end{pmatrix} \\
&= ((1-\alpha)c(n-\delta)+r+(n-\delta)\alpha)y
\end{aligned}$$

By Perron-Frobenius theorem, $x := \frac{y}{\sqrt{(n-\delta)c^2+\delta}}$ is the α-Perron vector of G and $\rho_\alpha = (1-\alpha)c(n-\delta)+r+(n-\delta)\alpha$. Since $\frac{\alpha\delta c+(1-\alpha)\delta}{\rho_\alpha} = \frac{\alpha\delta c+(1-\alpha)\delta}{(1-\alpha)c(n-\delta)+r+(n-\delta)\alpha} = c$, we have $c = \frac{(1-\alpha)\delta}{\rho_\alpha-\alpha\delta}$. Thus

$$x_{\min} = \frac{c}{\sqrt{(n-\delta)c^2+\delta}} = (1-\alpha)\sqrt{\frac{\delta}{(\rho_\alpha-\alpha\delta)^2+(1-\alpha)^2(n-\delta)\delta}},$$

as desired. $\square$

Let $G = K_{1,3}$ with vertex set $\{v_1,\ldots,v_4\}$, where $\delta_G(v_1) = 3$ and $\delta_G(v_2) = \delta_G(v_3) = \delta_G(v_4) = 1$. By direct calculation, for $\alpha = \frac{1}{4}$, we have $\rho_{\frac{1}{4}} = \frac{1+\sqrt{7}}{2}$, $x_{\min} = \frac{2\sqrt{7}-1}{\sqrt{12(14-\sqrt{7})}} \approx 0.367654$. Using the notations of Theorem 3, $\Delta = 3$, $n = 4$, $m = 3$, and so $x_{\min} < 1 - \frac{\sqrt{7}+1}{6} \approx 0.39237$. Using the notations of Theorem 4, $\delta = 1$, $n = 4$, and so $x_{\min} \leqslant 3\sqrt{\frac{1}{56+4\sqrt{7}}} = \frac{2\sqrt{7}-1}{\sqrt{12(14-\sqrt{7})}}$. The upper bound in Theorem 4 is smaller than the one in Theorem 3.

Let $G = K_1 \vee 2K_2$. By direct calculation, for $\alpha = \frac{1}{4}$, we have $\rho_{\frac{1}{4}} = \frac{9}{8} + \frac{\sqrt{145}}{8}$, $x_{\min} = \frac{6}{1+\sqrt{145}} \cdot \frac{1}{\sqrt{\left(\frac{6}{1+\sqrt{145}}\right)^2+4}} = \frac{6}{\sqrt{620+8\sqrt{145}}} \approx 0.22418$. Using the notations of Theorem 3, $\Delta = 4$, $n = 5$, $m = 6$, and so $x_{\min} < \frac{\sqrt{5}(23-\sqrt{145})}{64} \approx 0.38287$. Using the notations of Theorem 4, $\delta = 2$, $n = 5$, and so $x_{\min} \leqslant 6\sqrt{\frac{2}{366+10\sqrt{145}}} \approx 0.38474$. The upper bound in Theorem 3 is smaller than the one in Theorem 4. Therefore the upper bounds in Theorems 3 and 4 are incomparable in general.

5. Maximum α-spectral radius of graphs with given number of odd vertices

For an even integer $n \geqslant 2$, let $\widetilde{K}_n$ be the graph obtained from K_n by deleting a perfect matching. For integers n,k and an even integer t with $n \geqslant 4$, $t < n$ and $0 \leqslant k \leqslant \frac{t}{2} - 1$, let $B_{n,t,k} = ((K_1 \cup K_{2k+1}) \vee \widetilde{K}_{t-2k-2}) \vee K_{n-t}$. In particular, $B_{n,t,0} \cong \widetilde{K}_t \vee K_{n-t}$ and $B_{n,t,k} \cong (K_1 \cup K_{2k+1}) \vee K_{n-t}$ if $k = \frac{t}{2} - 1$.

LEMMA 4. *Let n,k,t be integers such that $n \geqslant 4$, $n > t$, $0 \leqslant k \leqslant \frac{t}{2} - 1$ and t is even. For $0 \leqslant \alpha < 1$, if $n - \alpha n \geqslant 1$, then $\rho_\alpha(B_{n,t,k}) \leqslant \rho_\alpha(B_{n,t,0})$ with equality if and only if $k = 0$.*

Proof. Let $G = B_{n,t,k}$. Denote by V_1, V_2, V_3 and V_4 the vertex sets of the graphs K_1, K_{2k+1}, $\widetilde{K}_{t-2k-2}$ and K_{n-t}, respectively, appearing in the definition of $B_{n,t,k}$. Let x be the α-Perron vector of G. By Lemma 2, all entries of x corresponding to vertices in V_i are equal if $|V_i| > 1$ for $i = 2,3,4$. Denote by x_1, x_2, x_3 and x_4 the entry of x corresponding to a vertex in V_1, V_2, V_3 and V_4, respectively. Let $\rho_{\alpha,k} = \rho_\alpha(B_{n,t,k})$ and $\beta = 1 - \alpha$. Then by the eigenequations of G at a vertex in V_1, V_2, V_3 and V_4, we have

$$\begin{aligned}
\big(\rho_{\alpha,k} - \alpha(n-2k-2)\big)x_1 - \beta(t-2k-2)x_3 - \beta(n-t)x_4 &= 0,\\
\big(\rho_{\alpha,k} - \alpha(n-2) - 2\beta k\big)x_2 - \beta(t-2k-2)x_3 - \beta(n-t)x_4 &= 0,\\
-\beta x_1 - \beta(2k+1)x_2 + \big(\rho_{\alpha,k} - \alpha(n-2) - \beta(t-2k-4)\big)x_3 - \beta(n-t)x_4 &= 0,\\
-\beta x_1 - \beta(2k+1)x_2 - \beta(t-2k-2)x_3 + \big(\rho_{\alpha,k} - \alpha(n-1) - \beta(n-t-1)\big)x_4 &= 0.
\end{aligned}$$

We view these equations as a homogeneous linear system in the four variables x_1, x_2, x_3 and x_4. Since it has a nontrivial solution, the determinant of the matrix of coefficients of this homogeneous linear system is zero. By direct calculation, this determinant is equal to $f_{\alpha,k}(\rho_{\alpha,k})$, where

$$\begin{aligned}
&f_{\alpha,k}(\rho)\\
=&\rho^4 + \rho^3\left(-n(3\alpha+1) + 2\alpha k + 2\alpha + 5\right) + \rho^2\big(3\alpha n^2(\alpha+1)\\
&-n(13\alpha + 2\alpha k + 4\alpha^2 k + 4\alpha^2 + 4) + 10\alpha + t + 10\alpha k - \alpha t + 8\big)\\
&+\rho\big(-\alpha^2 n^3(\alpha+3) + \alpha n^2(11\alpha + 4\alpha k + 2\alpha^2 k + 2\alpha^2 + 8)\\
&-2n(10\alpha - k + 6\alpha k + \alpha t + 5\alpha^2 k - \alpha^2 t + 6\alpha^2 + 2)
\end{aligned}$$

$$
\begin{aligned}
&-2(2k-8\alpha-t-12\alpha k-4\alpha k^2+2\alpha^2 k+\alpha^2 t+2k^2+2\alpha^2 k^2-\alpha kt+\alpha^2 kt-2))\\
&+\alpha^3 n^4-\alpha^2 n^3(3\alpha+2\alpha k+4)+\alpha n^2(12\alpha-2k+12\alpha k+\alpha t-\alpha^2 t+2\alpha^2+4)\\
&+2n(2k-6\alpha-6\alpha k-\alpha t+2\alpha k^2-6\alpha^2 k+2\alpha^3 k+\alpha^3 t-4\alpha^2-4\alpha^2 k^2\\
&+2\alpha^3 k^2-\alpha^2 kt+\alpha^3 kt)-2(2k-4\alpha-8\alpha k-2\alpha t+kt-4\alpha k^2+2\alpha^2 k\\
&+2\alpha^2 t+2k^2+2\alpha^2 k^2-4\alpha kt+3\alpha^2 kt).
\end{aligned}
$$

In the above, we assume that $k<\frac{t}{2}-1$. If $k=\frac{t}{2}-1$, then $V_3=\emptyset$. We consider the homogeneous linear system in three variables x_1,x_2 and x_4, whose determinant of the matrix of coefficients is $\frac{1}{\rho_{\alpha,k}-\alpha n+2}\cdot f(\rho_{\alpha,k})$. So, for any $0\leqslant k\leqslant\frac{t}{2}-1$, $\rho_{\alpha,k}$ is the largest root of the equation $f_{\alpha,k}(\rho)=0$. Noting that

$$
\begin{aligned}
&f_{\alpha,0}(\rho)\\
=&(\rho-\alpha n+2\alpha)(\rho-\alpha n+2)(\rho^2-\rho(\alpha n+n-3)+\alpha n^2-\alpha n-2n+t-\alpha t+2),
\end{aligned}
$$

we have $\rho_{\alpha,0}=\frac{n(1+\alpha)-3+\sqrt{(n(1-\alpha)+1)^2-4t(1-\alpha)}}{2}$, which is also true for $t=2$. Observe that

$$
\rho_{\alpha,0}>\frac{n(1+\alpha)-3+\sqrt{(n(1-\alpha)+1)^2-4n(1-\alpha)}}{2}=n-2\geqslant\alpha n-1.
$$

In the following, suppose that $k\geqslant 1$. Then since $n-\alpha n\geqslant 1$, we have

$$
\begin{aligned}
f_{\alpha,k}(\rho_{\alpha,0})=&2k(1-\alpha)^2\left((n-2k-2)\rho_{\alpha,0}-\alpha n^2+2n(\alpha k+\alpha+1)-t-2k-2\right)\\
>&2k(1-\alpha)^2((n-2k-2)(\alpha n-1)-\alpha n^2+2n(\alpha k+\alpha+1)-t-2k-2)\\
=&2k(1-\alpha)^2(n-t)\\
>&0,\\
f^{(2)}_{\alpha,k}(\rho_{\alpha,0})=&6\rho_{\alpha,0}((1-\alpha)(n-2k-2)+2k+1)-6\alpha n^2(1-\alpha)\\
&-4\alpha n(1+k)(2\alpha+1)-10\alpha n+16n+20\alpha(1+k)-10t(1-\alpha)-8\\
\geqslant&6(n-2)(n(1-\alpha)+2\alpha(1+k)-1)-6\alpha n^2(1-\alpha)\\
&-4\alpha n(1+k)(2\alpha+1)-10\alpha n+16n+20\alpha(1+k)\\
&-10(n-1)(1-\alpha)-8\\
=&6(n(1-\alpha)-1)^2+4\alpha(1+k)(2n(1-\alpha)-1)+8-10\alpha\\
=&6(n(1-\alpha)-1)^2+4\alpha(1+k)+8-10\alpha\\
>&0,\\
f^{(3)}_{\alpha,k}(\rho_{\alpha,0})=&24\rho_{\alpha,0}+6(-n(3\alpha+1)+2\alpha+2\alpha k+5)\\
>&24(n-2)+6(-n(3\alpha+1)+2\alpha+2\alpha k+5)\\
=&18(n(1-\alpha)-1)+12\alpha(1+k)\\
>&0,
\end{aligned}
$$

$$f_{\alpha,k}^{(4)}(\rho_{\alpha,0}) = 24.$$

It remains to determine the sign of $f_{\alpha,k}^{(1)}(\rho_{\alpha,0})$. By direct calculation,

$$\begin{aligned}
f_{\alpha,k}^{(1)}(\rho_{\alpha,0}) =& \rho_{\alpha,0}(n^2(1-\alpha)^2+2\alpha(k+1)(n-\alpha n+1)-2t(1-\alpha)-1)\\
&-\alpha n^3(1-\alpha)^2-2n^2(1-\alpha)(\alpha+\alpha^2k+\alpha^2-1)\\
&-n(t-4\alpha t+3\alpha^2t-2k+6\alpha^2-5\alpha+4\alpha^2k)+4\alpha-2\\
&-4k^2(1-\alpha)^2-4k(\alpha^2-3\alpha+1)-t(1+4\alpha k+3\alpha-4\alpha^2-4\alpha^2k)\\
\geqslant& (n-2)(n^2(1-\alpha)^2+2\alpha(k+1)(n-\alpha n+1)-2t(1-\alpha)-1)\\
&-\alpha n^3(1-\alpha)^2-2n^2(1-\alpha)(\alpha+\alpha^2k+\alpha^2-1)\\
&-n(t-4\alpha t+3\alpha^2t-2k+6\alpha^2-5\alpha+4\alpha^2k)+4\alpha-2\\
&-4k^2(1-\alpha)^2-4k(\alpha^2-3\alpha+1)-t(1+4\alpha k+3\alpha-4\alpha^2k-4\alpha^2)\\
=& (1-\alpha)(n^2(1-\alpha)(n-\alpha n+2\alpha+2\alpha k)+n(2\alpha+2k-1)\\
&-4k(k+1)(1-\alpha)-t(4\alpha+4\alpha k+3n-3\alpha n-3))\\
\geqslant& (1-\alpha)(n^2(1-\alpha)(n-\alpha n+2\alpha+2\alpha k)+n(2\alpha+2k-1)\\
&-4k(k+1)(1-\alpha)-(n-1)(4\alpha+4\alpha k+3n-3\alpha n-3))\\
=& (1-\alpha)(n^2(1-\alpha)((1-\alpha)(n-2-2k)+2k-1)\\
&+n((5+2k)(1-\alpha)-2\alpha k)-4k(k+1)(1-\alpha)+4\alpha(1+k)-3)\\
\geqslant& (1-\alpha)(n^2(1-\alpha)((1-\alpha)(n-2-2k)+2k-1)\\
&+n((5+2k)(1-\alpha)-2\alpha k)-2k(n-1)(1-\alpha)+4\alpha(1+k)-3)\\
=& (1-\alpha)(n^2(1-\alpha)((1-\alpha)(n-2-2k)+k-1)\\
&+n(5-5\alpha+k(n-\alpha n-2\alpha))+2k-3+4\alpha+2k\alpha),
\end{aligned}$$

where the first inequality is obtained by

$$\begin{aligned}
& n^2(1-\alpha)^2+2\alpha(k+1)(n-\alpha n+1)-2t(1-\alpha)-1\\
>& n^2(1-\alpha)^2+2\alpha(k+1)(n-\alpha n+1)-2(n-1)(1-\alpha)-1\\
=& n(1-\alpha)^2(n-2)+2\alpha kn(1-\alpha)+2\alpha k+1\\
>& 0,
\end{aligned}$$

the second inequality is obtained by $n>t$ and

$$4\alpha+4\alpha k+3n-3\alpha n-3 \geqslant 4\alpha+4\alpha k \geqslant 0,$$

and the third inequality is obtained by $n>t\geqslant 2+2k$. If $0\leqslant\alpha\leqslant\frac{1}{2}$, then

$$\begin{aligned}
f_{\alpha,k}^{(1)}(\rho_{\alpha,0}) \geqslant& (1-\alpha)(n^2(1-\alpha)((1-\alpha)(n-2-2k)+k-1)\\
&+n(5-5\alpha+k(n-\alpha n-2\alpha))+2k-3+4\alpha+2k\alpha)\\
>& 0.
\end{aligned}$$

If $\frac{1}{2} < \alpha < 1$, then, as $n - \alpha n \geqslant 1$, we have

$$
\begin{aligned}
f^{(1)}_{\alpha,k}(\rho_{\alpha,0}) \geqslant & (1-\alpha)(n^2(1-\alpha)((1-\alpha)(n-2-2k)+k-1) \\
& + n(5-5\alpha+k(n-\alpha n-2\alpha))+2k-3+4\alpha+2k\alpha) \\
= & (1-\alpha)(2k(\alpha n^2(1-\alpha)-\alpha n+\alpha+1)+n^2(1-\alpha)^2(n-2)-n^2(1-\alpha) \\
& -5\alpha n+5n+4\alpha-3) \\
\geqslant & (1-\alpha)(2(\alpha n^2(1-\alpha)-\alpha n+\alpha+1)+n^2(1-\alpha)(n-2)(1-\alpha) \\
& -n^2(1-\alpha)-5\alpha n+5n+4\alpha-3) \\
= & (1-\alpha)(n^2(1-\alpha)(2\alpha-1)+n^2(1-\alpha)^2(n-2)-7\alpha n+5n+6\alpha-1) \\
\geqslant & (1-\alpha)(n(2\alpha-1-7\alpha+5)+n^2(1-\alpha)^2(n-2)+6\alpha-1) \\
\geqslant & (1-\alpha)(n(-5\alpha+4)+(n-2)+6\alpha-1) \\
= & (1-\alpha)(n(-5\alpha+5)+6\alpha-3) \\
> & 0.
\end{aligned}
$$

Thus, for $i = 0,1,2,3$, we have $f^{(4-i)}_{\alpha,k}(\rho_{\alpha,0}) > 0$, so $f^{(3-i)}_{\alpha,k}(\rho) \geqslant f^{(3-i)}_{\alpha,k}(\rho_{\alpha,0}) > 0$ for $\rho \geqslant \rho_{\alpha,0}$. Particularly, $f_{\alpha,k}(\rho_{\alpha,0}) > 0$ for $\rho \geqslant \rho_{\alpha,0}$, which together with the fact that $f_{\alpha,k}(\rho) > 0$ if $\rho > \rho_{\alpha,k}$, implies that $\rho_{\alpha,k} < \rho_{\alpha,0}$. $\square$

For a positive integer n and an even integer t with $0 \leqslant t \leqslant n$, let $\mathbb{G}(n,t)$ be the set of connected graphs with n vertices and t odd vertices, and let $H_{n,t}$ be the graph obtained from K_n by deleting $\frac{t}{2}$ disjoint edges if n is odd, and the graph obtained from K_n by deleting $\frac{n-t}{2}$ disjoint edges if n is even.

THEOREM 5. *Let $G \in \mathbb{G}(n,t)$, where $0 \leqslant t \leqslant n$, $n \geqslant 4$ and t is even. For $0 \leqslant \alpha < 1$, if $n - \alpha n \geqslant 1$, then*

$$
\rho_\alpha(G) \leqslant \begin{cases} \dfrac{n(1+\alpha)-3+\sqrt{(n(1-\alpha)+1)^2-4t(1-\alpha)}}{2} & \textit{if } n \textit{ is odd} \\[2ex] \dfrac{n(1+\alpha)-3+\sqrt{(n(1-\alpha)+1)^2-4(n-t)(1-\alpha)}}{2} & \textit{if } n \textit{ is even} \end{cases}
$$

with equality if and only if $G \cong H_{n,t}$.

Proof. Let G be a graph that maximizes the α-spectral radius over graphs in $\mathbb{G}(n,t)$.

Suppose that V_o and V_e be the sets of vertices of odd degree and even degree in G, respectively. Obviously, $|V_o| = t$. Let x be the α-Perron vector of G.

CLAIM. Each vertex of V_o is adjacent to each vertex of V_e if $V_o \neq \emptyset$ and $V_e \neq \emptyset$.

Suppose that there are two vertices $u \in V_o$ and $v \in V_e$ such that u is not adjacent to v. Let $G' = G + uv$. Noting that $\delta_{G'}(u) = \delta_G(u) + 1$, $\delta_{G'}(v) = \delta_G(v) + 1$ and $\delta_{G'}(w) = \delta_G(w)$ for $w \in V(G) \setminus \{u,v\}$, we have $G' \in \mathbb{G}(n,t)$. By Lemma 1, we have $\rho_\alpha(G') > \rho_\alpha(G)$, a contradiction. This proves our claim.

Case 1. n is odd.

In this case, $0 \leqslant t \leqslant n-1$. If $t=0$, then it is trivial that $G \cong K_n \cong H_{n,0}$ by Lemma 1. Suppose that $t \geqslant 2$. Then $V_o \neq \emptyset$ and $V_e \neq \emptyset$. By the Claim, $G \cong G[V_o] \vee G[V_e]$. Evidently, $G[V_e]$ is a spanning subgraph of K_{n-t}. Suppose that $G[V_e]$ is not a complete graph. Then it is a proper spanning subgraph of K_{n-t}, so G is a proper subgraph of $G[V_o] \vee K_{n-t}$. Note that $G[V_o] \vee K_{n-t} \in \mathbb{G}(n,t)$. By Lemma 1, we have $\rho_\alpha(G) < \rho_\alpha(G[V_o] \vee K_{n-t})$, a contradiction. This shows that $G[V_e] \cong K_{n-t}$. That is, $G \cong G[V_o] \vee K_{n-t}$. So, each even vertex is of degree $n-1$ and each odd vertex is of degree at most $n-2$. For any odd vertex v of G, $\delta_{G[V_o]}(v)+n-t=\delta_G(v) \leqslant n-2$, so $\delta_{G[V_o]}(v) \leqslant t-2$.

Next, we show that $G[V_o]$ is a $(t-2)$-regular graph. This is trivial if $t=2$. Suppose that it is not true. Then $t \geqslant 4$ and $\delta_{G[V_o]}(u) \leqslant t-4$ for some $u \in V_o$. Assume that x_u is minimum among the vertices in V_o with degree at most $t-4$ in $G[V_o]$.

Let $\mathcal{N}_u$ be the set of vertices except u that are not adjacent to u in G. By the above claim, $\mathcal{N}_u \subseteq V_o$. As $u \in V_o$ and t is even, $|\mathcal{N}_u| = 2k+1 \leqslant t-1$ for some $k \geqslant 1$, so $1 \leqslant k \leqslant \frac{t}{2}-1$. Let $\mathcal{N}_u = \{u_1, \ldots, u_{2k+1}\}$. Suppose that $G[\mathcal{N}_u]$ is not complete, say u_1 is not adjacent to u_{2k+1}. Let $G' = G + uu_1 + uu_{2k+1} + u_1u_{2k+1}$. Obviously, $G' \in \mathbb{G}(n,t)$. By Lemma 1, we have $\rho_\alpha(G') > \rho_\alpha(G)$, a contradiction. Thus $G[\mathcal{N}_u]$ is a complete graph, i.e., $G[\mathcal{N}_u] \cong K_{2k+1}$.

Suppose without loss of generality that $x_{u_1} = \min\{x_{u_i} : 1 \leqslant i \leqslant 2k+1\}$. Suppose that $x_u \geqslant x_{u_1}$. Let $G'' = G - u_1u_{2k+1} + uu_1 + uu_{2k+1}$. It is easily seen that $G'' \in \mathbb{G}(n,t)$. Since

$$
\begin{aligned}
&\rho_\alpha(G'') - \rho_\alpha(G) \\
\geqslant& x^\top (A_\alpha(G'') - A_\alpha(G))x \\
=&\alpha \sum_{v \in V(G)} (\delta_{G''}(v) - \delta_G(v))x_v^2 + 2(1-\alpha)\left(\sum_{vz \in E(G'')} x_v x_z - \sum_{vz \in E(G)} x_v x_z\right) \\
=&2\alpha x_u^2 + 2(1-\alpha)\left(-x_{u_1}x_{u_{2k+1}} + x_u x_{u_1} + x_u x_{u_{2k+1}}\right) \\
=&2\alpha x_u^2 + 2(1-\alpha)\left(x_u x_{u_1} + (x_u - x_{u_1})x_{u_{2k+1}}\right) \\
>&0,
\end{aligned}
$$

we have $\rho_\alpha(G'') > \rho_\alpha(G)$, a contradiction. Thus $x_u < x_{u_1}$.

If $k = \frac{t}{2}-1$, then we have by Lemma 1 that $G[V_o] = K_1 \cup K_{t-1}$, and thus $G \cong B_{n,t,\frac{t}{2}-1}$.

Suppose that $1 \leqslant k \leqslant \frac{t}{2}-2$. Then $N_G(u) \cap V_o \neq \emptyset$. Suppose that $\delta_{G[V_o]}(u_i) \leqslant t-4$ for some i with $1 \leqslant i \leqslant 2k+1$. Then there are at least two vertices, say w_1 and w_2, in $N_G(u) \cap V_o$, that are not adjacent to u_i. Let $G^* = G - uw_1 - uw_2 + u_iw_1 + u_iw_2$. Note that $G^* \in \mathbb{G}(n,t)$. By the choice of u and Lemma 3, we have $\rho_\alpha(G^*) > \rho_\alpha(G)$, a contradiction. Thus each vertex of $\mathcal{N}_u$ is of degree $t-2$ in $G[V_o]$.

Suppose that $w \in N_G(u) \cap V_o$ is of degree less that $t-4$ in $G[V_o]$. Then there are at least three vertices, say v_1, v_2 and v_3, in $N_G(u) \cap V_o$, that are not adjacent to w. Let $G^{**} = G - uv_1 - uv_2 + wv_1 + wv_2$. It is easily seen that $G^{**} \in \mathbb{G}(n,t)$. Recall that $x_u = \min\{x_v : \delta_G(v) \leqslant n-4, v \in V_o\}$. By Lemma 3, we have $\rho_\alpha(G^{**}) > \rho_\alpha(G)$, a contradiction. Thus each vertex of $N_G(u)\cap V_o$ is of degree $t-2$ in $G[V_o]$. That is, $G[N_G(u)\cap V_o]\cong \widetilde{K}_{t-2k-2}$. Thus $G\cong B_{n,t,k}$.

Note that $B_{n,t,0}\in \mathbb{G}(n,t)$. By Lemma 4, we have $\rho_\alpha(G)=\rho_\alpha(B_{n,t,k})<\rho_\alpha(B_{n,t,0})$ for $1\leqslant k\leqslant \frac{t}{2}-1$, also a contradiction. Therefore, $G[V_o]$ is indeed a $(t-2)$-regular graph, and $G\cong H_{n,t}$.

Case 2. n is even.

If $t=n$, then n is even, and by Lemma 1, we have $G\cong K_n\cong H_{n,n}$. If $t=0$, then it is evident that $G\cong H_{n,0}$, which is the only graph on n vertices with no odd vertices that is regular of degree $n-2$. Suppose that $2\leqslant t\leqslant n-2$. By Lemma 1 and similar argument as in Case 1, $G[V_o]\cong K_t$, $G[V_e]$ is an $(n-t-2)$-regular graph, and so by the claim, $G\cong H_{n,t}$.

Combining the above two cases, we have $G\cong H_{n,t}$. The expression for $\rho_\alpha(H_{n,t})$ follows by direct computation as the α-Perron vector of $H_{n,t}$ has at most two different entries or from the proof of Lemma 4 and direct check if $t=0$ and $t=n$ for even n as $H_{n,t}\cong B_{n,t,0}$ for odd n and $H_{n,t}\cong B_{n,n-t,0}$ for even n and $2\leqslant t<n$. $\square$

COROLLARY 3. *Let $G\in\mathbb{G}(n,t)$, where $0\leqslant t\leqslant n$, t is even and $n\geqslant 4$. Then*

$$\rho_0(G)\leqslant\rho_0(H_{n,t}) \text{ and } \rho_{\frac{1}{2}}(G)\leqslant\rho_{\frac{1}{2}}(H_{n,t})$$

with either equality if and only if $G\cong H_{n,t}$.

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(Received August 16, 2019)

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