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A NOTE ON THE STRUCTURE OF NORMAL HAMILTONIAN MATRICES

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Abstract. The structures of the blocks of a normal Hamiltonian matrix are studied. In this note it is obtained that all four blocks of a normal Hamiltonian matrix $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$ can be expressed as linear combinations of four other matrices.

1. Introduction and preliminaries

Hamiltonian matrices have been a topic of extensive research since they have many applications in engineering and physics. In the context of linear algebra, one of their most important applications is the fact that they are linearizations of gyroscopic systems that can be represented by self–adjoint quadratic matrix polynomials. For more insight on these topics, see [2], [3], [4] and the references therein.

We denote by $\mathbb{C}^{n\times n}$ and $\mathbb{R}^{n\times n}$ the set of complex and real $n\times n$ matrices, respectively. A complex $2n\times 2n$ matrix $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, with $A,B,C,D \in \mathbb{C}^{n\times n}$ is called Hamiltonian if the matrix JH is hermitian, where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. It follows that JH is hermitian if and only if $D = -A^*$ and $B^* = B$ and $C^* = C$. Therefore, the considered Hamiltonian matrix has the general form

$$H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}, \ A,B,C \in \mathbb{C}^{n\times n},\ B^* = B,\ C^* = C.$$

In the remainder we will need the following notation and definitions:

- $\mathbb{I}$ is the set of imaginary numbers and $\mathbb{I}^{n\times n}$ the set of $n\times n$ matrices with imaginary entries.
- $\sigma(A)$ is the set of eigenvalues of a square matrix A.
- $tr(A)$ is the trace of a square matrix A.
- $\|A\|_F$ is the Frobenious norm of a matrix A, $\|A\|_F = \sqrt{tr(A^*A)}$.

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- I_n is the $n \times n$ identity matrix.
- $Re(A) \in \mathbb{R}^{n\times n}$ and $Im(A) \in \mathbb{R}^{n\times n}$ are the real and imaginary parts of a complex matrix A respectively, so that $A = Re(A) + iIm(A)$.
- Let G be a normal matrix such that $G^2 = -I_n$. Then a matrix A is called G–Hamiltonian (resp., G–Skew–Hamiltonian) when $(AG)^* = AG$ (resp., $(AG)^* = -AG$).
- $\overline{A}$ is the complex conjugate of a complex matrix A.

The purpose of this note is to take advantage of the symmetries that a normal Hamiltonian matrix $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$ provides in order to examine the structures of the blocks A,B,C and investigate how these strucrures are related. More precisely, it is proved that the matrices A,B,C of a normal Hamiltonian are linear combinations of four other matrices that satisfy a strong relation. The analysis here is much based on Theorem 1 in the work of Gigola, Lebtahi and Thome [1]. There, the authors give a unitary equivalence result for normal G–Hamiltonian matrices. This definition of G–Hamiltonian matrix is a generalization of Hamiltonian matrices, since J satisfies the conditions of G. A similar theorem for the case of normal G–skew–Hamiltonian matrices can be found in [5]. For clarity, we state them here as items a and b of the following thorem, respectively.

THEOREM 1. *If U is a unitary matrix such that $G = U \begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix} U^*$, then*

a. $A \in \mathbb{C}^{2n\times 2n}$ is a normal G–Hamiltonian matrix if and only if

$$A = U \begin{bmatrix} A_1 & W \\ W^* & A_2 \end{bmatrix} U^*,$$

where $A_1^* = -A_1, A_2^* = -A_2$ and $A_1W = WA_2$.

b. $A \in \mathbb{C}^{2n\times 2n}$ is a normal G–skew–Hamiltonian matrix if and only if

$$A = U \begin{bmatrix} A_1 & W \\ -W^* & A_2 \end{bmatrix} U^*,$$

where $A_1^* = A_1, A_2^* = A_2$ and $A_1W = WA_2$.

The structure of this paper is as follows: In Section 2 we prove the main results for complex normal Hamiltonian matrices. In Section 3 we move to the real setting and exploit the results of Section 2 to explore the structures of the blocks of real normal Hamiltonian matrices. These results can be used to construct normal Hamiltonian matrices, which is not a trivial affair if we exclude the hermitian or skew hermitian cases. This is illustrated through an example. Finally, a last section is included to express similar results for normal–skew–Hamiltonian matrices.

2. Main results

At the beginning we prove that a $2n\times 2n$ unitary matrix that diagonalizes the normal matrix J has a very specific form. Note that $\sigma(J)=\{i,-i\}$.

PROPOSITION 1. *A matrix* $U=\begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}\in\mathbb{C}^{2n\times 2n}$, $U_i\in\mathbb{C}^{n\times n}$, $i=1,2,3,4$ *is unitary such that* $JU=U\begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix}$ *if and only if* $U_3=iU_1$, $U_4=-iU_2$ *and* $U_1U_1^*=U_2U_2^*=\frac{1}{2}I_n$.

Proof. Let U be a unitry matrix such that

$$JU=U\begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}=\begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix}\begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix}$$

or

$$\begin{bmatrix} U_3 & U_4 \\ -U_1 & -U_2 \end{bmatrix}=\begin{bmatrix} iU_1 & -iU_2 \\ iU_3 & -iU_4 \end{bmatrix}$$

The last matrix equality yields $U_3=iU_1$, $U_4=-iU_2$. Moreover,

$$UU^*=I_{2n}$$

or

$$\begin{bmatrix} U_1 & U_2 \\ iU_1 & -iU_2 \end{bmatrix}\begin{bmatrix} U_1^* & -iU_1^* \\ U_2^* & iU_2^* \end{bmatrix}=\begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}$$

or

$$\begin{bmatrix} U_1U_1^*+U_2U_2^* & -iU_1U_1^*+iU_2U_2^* \\ iU_1U_1^*-iU_2U_2^* & U_1U_1^*+U_2U_2^* \end{bmatrix}=\begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}.$$

By this equality we have $U_1U_1^*=U_2U_2^*=\frac{1}{2}I_n$.

Conversely, let $U=\begin{bmatrix} U_1 & U_2 \\ iU_1 & -iU_2 \end{bmatrix}$ with $U_1U_1^*=U_2U_2^*=\frac{1}{2}I_n$. Then U is unitary since

$$UU^*=\begin{bmatrix} U_1 & U_2 \\ iU_1 & -iU_2 \end{bmatrix}\begin{bmatrix} U_1^* & -iU_1^* \\ U_2^* & iU_2^* \end{bmatrix}=\begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}.$$

Moreover, it holds that

$$\begin{aligned} U^*JU &= \begin{bmatrix} U_1^* & -iU_1^* \\ U_2^* & iU_2^* \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ iU_1 & -iU_2 \end{bmatrix} \\ &= \begin{bmatrix} iU_1^* & U_1^* \\ -iU_2^* & U_2^* \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ iU_1 & -iU_2 \end{bmatrix} \\ &= \begin{bmatrix} 2iU_1^*U_1 & 0 \\ 0 & -2iU_2^*U_2 \end{bmatrix} \\ &= \begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix}. \quad \square \end{aligned}$$

Before we prove the main result, we present a lemma that will be useful for its proof.

LEMMA 1. *The matrix* $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$, $A,B,C \in \mathbb{C}^{n\times n}$, $B^* = B$, $C^* = C$ *is a complex normal Hamiltonian matrix if and only if* $AC - BA$ *is skew–hermitian and* $AA^* - A^*A = C^2 - B^2$.

Proof. For the normality of H, it is required that

$$\begin{aligned} &HH^* = H^*H \\ \Longleftrightarrow &\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} \begin{bmatrix} A^* & C \\ B & -A \end{bmatrix} = \begin{bmatrix} A^* & C \\ B & -A \end{bmatrix} \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} \\ \Longleftrightarrow &\begin{bmatrix} AA^* + B^2 & AC - BA \\ CA^* - A^*B & C^2 + A^*A \end{bmatrix} = \begin{bmatrix} A^*A + C^2 & A^*B - CA^* \\ BA - AC & B^2 + AA^* \end{bmatrix} \end{aligned}$$

or equivalently,

$$AA^* + B^2 = A^*A + C^2 \tag{1}$$

and

$$AC - BA = A^*B - CA^*, \tag{2}$$

and the proof is complete. $\square$

COROLLARY 1. *Let the matrix* $H = \begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$, $A,B,C \in \mathbb{C}^{n\times n}$, $B^* = B$, $C^* = C$ *be a complex normal Hamiltonian matrix. Then* $tr(B^2) = tr(C^2)$, *that is* $\|B\|_F = \|C\|_F$.

THEOREM 2. *The matrix* $\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$, $A,B,C \in \mathbb{C}^{n\times n}$, $B^* = B$, $C^* = C$ *is a complex normal Hamiltonian matrix if and only if there are skew hermitian* $K_1, K_2 \in \mathbb{C}^{n\times n}$ *and a matrix* $Z \in \mathbb{C}^{n\times n}$ *satisfying* $K_1Z = ZK_2$, *such that*

$$\begin{aligned} A &= K_1 + K_2 + Z + Z^*, \\ B &= -i(K_1 - K_2) + i(Z - Z^*), \\ C &= i(K_1 - K_2) + i(Z - Z^*). \end{aligned}$$

Proof. According to Theorem 1 and Proposition 1, there is a unitary matrix $U = \begin{bmatrix} U_1 & U_2 \\ iU_1 & -iU_2 \end{bmatrix}$ with $U_1U_1^* = U_2U_2^* = \frac{1}{2}I_n$, skew–hermitian matrices $A_1, A_2 \in \mathbb{C}^{n\times n}$ and a matrix $W \in \mathbb{C}^{n\times n}$ satisfying $A_1W = WA_2$, such that

$$\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} = U \begin{bmatrix} A_1 & W \\ W^* & A_2 \end{bmatrix} U^*$$

or

$$\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \\ iU_1 & -iU_2 \end{bmatrix} \begin{bmatrix} A_1 & W \\ W^* & A_2 \end{bmatrix} \begin{bmatrix} U_1^* & -iU_1^* \\ U_2^* & iU_2^* \end{bmatrix}$$

or

$$\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} = \begin{bmatrix} U_1A_1 + U_2W^* & U_1W + U_2A_2 \\ iU_1A_1 - iU_2W^* & iU_1W - iU_2A_2 \end{bmatrix} \begin{bmatrix} U_1^* & -iU_1^* \\ U_2^* & iU_2^* \end{bmatrix}$$

Performing the last matrix multiplication yields

$$\begin{aligned} A &= U_1A_1U_1^* + U_2W^*U_1^* + U_1WU_2^* + U_2A_2U_2^*, \\ B &= -iU_1A_1U_1^* - iU_2W^*U_1^* + iU_1WU_2^* + iU_2A_2U_2^*, \\ C &= iU_1A_1U_1^* - iU_2W^*U_1^* + iU_1WU_2^* - iU_2A_2U_2^*. \end{aligned}$$

Setting $K_1 = U_1A_1U_1^*$ and $K_2 = U_2A_2U_2^*$ which are skew–hermitian, and $Z = U_1WU_2^*$, we have the desired forms of A, B and C. Finally, keeping in mind that $U_1U_1^* = U_2U_2^* = \frac{1}{2}I_n$, we have

$$\begin{aligned} K_1Z &= U_1A_1U_1^*U_1WU_2^* \\ &= \frac{1}{2}U_1A_1WU_2^* \\ &= \frac{1}{2}U_1WA_2U_2^* \\ &= U_1WU_2^*U_2A_2U_2^* \\ &= ZK_2. \end{aligned}$$

For the converse, assume that there are skew hermitian $K_1, K_2 \in \mathbb{C}^{n\times n}$ and a matrix $Z \in \mathbb{C}^{n\times n}$ satisfying $K_1Z = ZK_2$ such that

$$\begin{aligned} A &= K_1 + K_2 + Z + Z^*, \\ B &= -i(K_1 - K_2) + i(Z - Z^*), \\ C &= i(K_1 - K_2) + i(Z - Z^*). \end{aligned}$$

We will use Lemma 1 to show that the Hamiltonian matrix $\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix}$ is normal. It suffices to show that $AC - BA$ is skew–hermitian and that $AA^* - A^*A = C^2 - B^2$.

Performing the necessary operations, we obtain

$$AA^* = -(K_1 + K_2)^2 + (Z^* + Z)^2 + [K_1Z^* - ZK_1 + K_2Z - Z^*K_2],$$

$$A^*A = -(K_1+K_2)^2 + (Z^*+Z)^2 - [K_1Z^* - ZK_1 + K_2Z - Z^*K_2],$$
$$B^2 = -(K_1-K_2)^2 - (Z-Z^*)^2 - [K_1Z^* - ZK_1 + K_2Z - Z^*K_2],$$

and

$$C^2 = -(K_1-K_2)^2 - (Z-Z^*)^2 + [K_1Z^* - ZK_1 + K_2Z - Z^*K_2].$$

Clearly, $AA^* - A^*A = C^2 - B^2$. Finally,

$$AC - BA = 2i[K_1^2 - K_2^2] + 2i[Z^*Z - ZZ^*],$$

which is skew–hermitian, and the proof is complete. □

REMARK 1. Note that, instead of using any unitary transformation to apply Theorem 2, we can use the matrix $Q = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & I_n \\ iI_n & -iI_n \end{bmatrix}$ which is unitary and

$$Q^*JQ = \frac{1}{2}\begin{bmatrix} I_n & -iI_n \\ I_n & iI_n \end{bmatrix}\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}\begin{bmatrix} I_n & I_n \\ iI_n & -iI_n \end{bmatrix}$$

or

$$Q^*JQ = \begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix},$$

satisfying the conditions required. By doing that we have the submatrices A, B and C of the Hamiltonian expressed directly as linear combinations of matrices A_1, A_2, W, W^* of Theorem 1, and not of K_1, K_2, Z, Z^* which are transformations of them. Verily,

$$H = Q\begin{bmatrix} A_1 & W \\ W^* & A_2 \end{bmatrix}Q^*$$

or

$$\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} = \begin{bmatrix} I_n & I_n \\ iI_n & -iI_n \end{bmatrix}\begin{bmatrix} A_1 & W \\ W^* & A_2 \end{bmatrix}\begin{bmatrix} I_n & -iI_n \\ I_n & iI_n \end{bmatrix}$$

or

$$\begin{bmatrix} A & B \\ C & -A^* \end{bmatrix} = \begin{bmatrix} A_1+A_2+W+W^* & -iA_1+iA_2+iW-iW^* \\ iA_1-iA_2-iW^*+iW & A_1+A_2-W-W^* \end{bmatrix},$$

so that

$$A = A_1 + A_2 + W + W^*,$$
$$A^* = -A_1 - A_2 + W + W^*$$

and

$$B = -i(A_1 - A_2) + i(W - W^*),$$
$$C = i(A_1 - A_2) + i(W - W^*).$$

3. Real normal Hamiltonian matrices

Here we leave the general complex setting, and focus on real normal Hamiltonian matrices.

PROPOSITION 2. *The real Hamiltonian matrix* $H = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}$ *is normal if and only if there is a skew–hermitian matrix* $K_1 \in \mathbb{C}^{n\times n}$ *and a complex symmetric matrix* Z *with* $K_1 Z = Z\overline{K_1}$ *such that*

$$A = 2Re(K_1) + 2Re(Z), \;\; B = 2Im(K_1) - 2Im(Z),$$

and

$$C = -2Im(K_1) - 2Im(Z).$$

Proof. From Theorem 2 we have that there are skew hermitian $K_1, K_2 \in \mathbb{C}^{n\times n}$ and a matrix $Z \in \mathbb{C}^{n\times n}$ satisfying $K_1 Z = ZK_2$ such that

$$\begin{aligned} A &= K_1 + K_2 + Z + Z^*, \\ B &= -i(K_1 - K_2) + i(Z - Z^*), \\ C &= i(K_1 - K_2) + i(Z - Z^*), \\ A^T = A^* &= -K_1 - K_2 + Z + Z^*. \end{aligned}$$

Since A, B, C are all real $n \times n$ matrices, we have:

$$B + C \in \mathbb{R}^{n\times n} \Rightarrow i(Z - Z^*) \in \mathbb{R}^{n\times n} \Rightarrow (Z - Z^*) \in \mathbb{I}^{n\times n},$$

so, if $Z = [z_{ij}]$, $i, j = 1, \cdots, n$, then $z_{ij} - \overline{z_{ji}} \in \mathbb{I}$, and hence,

$$Re(z_{ij}) = Re(z_{ji}). \tag{3}$$

Moreover,

$$A + A^T \in \mathbb{R}^{n\times n} \Rightarrow Z + Z^* \in \mathbb{R}^{n\times n},$$

so, $z_{ij} + \overline{z_{ji}} \in \mathbb{R}$, and consequently,

$$Im(z_{ij}) = Im(z_{ji}). \tag{4}$$

Equations (3) and (4) imply that $z_{ij} = z_{ji}$, making Z a complex symmetric matrix.

Now,

$$C - B \in \mathbb{R}^{n\times n} \Rightarrow i(K_1 - K_2) \in \mathbb{R}^{n\times n} \Rightarrow (K_1 - K_2) \in \mathbb{I}^{n\times n},$$

which yields $Re(K_1) = Re(K_2)$. Similarly,

$$A - A^T \in \mathbb{R}^{n\times n} \Rightarrow K_1 + K_2 \in \mathbb{R}^{n\times n},$$

so, $Im(K_1) = -Im(K_2)$. Therefore, we conclude $K_1 = \overline{K_2}$ and $K_1 Z = Z\overline{K_1}$. The forms of A, B, C follow from the facts that $K_1 = \overline{K_2}$ and Z is complex symmetric making $Z^* = \overline{Z}$. □

The next corollary gives a form similar to that of Theorem 1 for the case of real normal Hamiltonian matrices.

COROLLARY 2. *Let* $H = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}$, $A,B,C \in \mathbb{R}^{n\times n}$ *be a real normal Hamiltonian matrix and* $U_1 \in \mathbb{C}^{n\times n}$ *such that* $U_1U_1^* = \frac{1}{2}I_n$. *If* $U = \begin{bmatrix} U_1 & \overline{U_1} \\ iU_1 & -i\overline{U_1} \end{bmatrix}$. *Then*

a. *U is unitary and* $U^*JU = \begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix}$, *and*

b. *there are matrices* A_1, W, *where* A_1 *is skew–hermitian and* $A_1W = W\overline{A_1}$, *such that* $H = U \begin{bmatrix} A_1 & W \\ W^* & \overline{A_1} \end{bmatrix} U^*$.

Proof.

a. Let $U_1 \in \mathbb{C}^{n\times n}$ so that $U_1U_1^* = \frac{1}{2}I_n$. Then, if $U = \begin{bmatrix} U_1 & \overline{U_1} \\ iU_1 & -i\overline{U_1} \end{bmatrix}$, we have

$$\begin{aligned} UU^* &= \begin{bmatrix} U_1 & \overline{U_1} \\ iU_1 & -i\overline{U_1} \end{bmatrix} \begin{bmatrix} U_1^* & -iU_1^* \\ U_1^T & iU_1^T \end{bmatrix} \\ &= \begin{bmatrix} U_1U_1^* + \overline{U_1}U_1^T & 0 \\ 0 & U_1U_1^* + \overline{U_1}U_1^T \end{bmatrix} \\ &= \begin{bmatrix} U_1U_1^* + (U_1U_1^*)^T & 0 \\ 0 & U_1U_1^* + (U_1U_1^*)^T \end{bmatrix} \\ &= \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}. \end{aligned}$$

Moreover,

$$\begin{aligned} U^*JU &= \begin{bmatrix} U_1^* & -iU_1^* \\ U_1^T & iU_1^T \end{bmatrix} \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} U_1 & \overline{U_1} \\ iU_1 & -i\overline{U_1} \end{bmatrix} \\ &= \begin{bmatrix} iI_n & 0 \\ 0 & -iI_n \end{bmatrix}. \end{aligned}$$

b. According to Theorem 1, we have that there are skew–hermitian matrices A_1, A_2 and a matrix W satisfying $A_1W = WA_2$ such that

$$\begin{aligned} \begin{bmatrix} A_1 & W \\ W^* & A_2 \end{bmatrix} &= U^*HU \\ &= \begin{bmatrix} U_1^* & -iU_1^* \\ U_1^T & iU_1^T \end{bmatrix} \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix} \begin{bmatrix} U_1 & \overline{U_1} \\ iU_1 & -i\overline{U_1} \end{bmatrix}. \end{aligned}$$

Performing the necessary operations we obtain

$$A_1 = U_1^*AU_1 - i(U_1^*CU_1 - U_1^*BU_1) - U_1^*A^TU_1$$

and

$$A_2 = U_1^TA\overline{U_1} + i(U_1^TC\overline{U_1} - U_1^TB\overline{U_1}) - U_1^TA^T\overline{U_1}.$$

Evidently, $\overline{A_1} = A_2$. □

We include a last proposition to investigate the strong relation of Z to A_1. This is useful when we want to apply Proposition 2 to construct a real normal Hamiltonian matrix, a procedure that is not that trivial, unless we are referring to symmetric or skew–symmetric matrices. It is a sylvester equation type result that relates the choice on the entries of Z to the spectrum of the skew–hermitian matrix A_1.

PROPOSITION 3. *Let $K_1 \in \mathbb{C}^{n\times n}$ be a skew–hermitian matrix, and $Z \in \mathbb{C}^{n\times n}$ be a symmetric matrix such that $K_1 Z = Z\overline{K_1}$. Let also $R \in \mathbb{C}^{n\times n}$ be a unitary matrix that diagonalizes K_1, so that, $R^* K_1 R = diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Then $Z = RSR^T$, where S is complex symmetric and $s_{ij} = s_{ji} = 0$, if $\lambda_i + \lambda_j \neq 0$, and $s_{ii} = 0$, if $\lambda_i \neq 0$.*

Proof. $R^* K_1 R = diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, and since the eigenvalues of K_1 are imaginary, we have $R^T \overline{K_1 R} = -diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$. Also, note that $(R^T)^{-1} = \overline{R}$, so

$$K_1 Z = Z\overline{K_1}$$

and

$$R^* K_1 R R^* Z \overline{R} = R^* Z (R^T)^{-1} R^T \overline{K_1 R}.$$

Setting $S = R^* Z \overline{R}$, which is a complex symmetric matrix since $S^T = (R^* Z \overline{R})^T = R^* Z^T \overline{R}$ $= S$, we have $diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\} S = -S diag\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$.

Equating the diagonal entries of the left hand side product and the right hand side product, we obtain $2\lambda_i s_{ii} = 0$ which yields $s_{ii} = 0$ when $\lambda_i \neq 0$, and equating the off diagonal entries, we have $s_{ij}(\lambda_i + \lambda_j) = 0$ by which we have $s_{ij} = s_{ji} = 0$ when $(\lambda_i + \lambda_j) \neq 0$. □

EXAMPLE. Let's illustrate the use of Propositions 2 and 3 in constructing a real normal Hamiltonian matrix.

Let

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1+i}{2} & -\frac{1}{2} & 0 \\ -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{5i}{2\sqrt{15}} & \frac{3+i}{2\sqrt{15}} & \frac{4+3i}{2\sqrt{15}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

be unitary and $K_1 = R diag\{-5i, 5i, -i, 0\} R^*$ be a skew–hermitian matrix.

$$K_1 = \begin{bmatrix} i & \frac{5+3i}{\sqrt{3}} & \frac{-8+6i}{\sqrt{15}} & 0 \\ \frac{-5+3i}{\sqrt{3}} & \frac{-i}{3} & \frac{-9+13i}{\sqrt{45}} & 0 \\ \frac{8+6i}{\sqrt{15}} & \frac{9+13i}{\sqrt{45}} & \frac{-5i}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $Z = RSR^T$, and the entries of S are determined by the eigenvalues of K_1 according to Proposition 3, $\lambda_1 = -5i$, $\lambda_2 = 5i$, $\lambda_3 = -i$, $\lambda_4 = 0$, so $s_{11} = s_{22} = s_{33} = 0$, $s_{44} \in \mathbb{C}$, $s_{12} = s_{21} \in \mathbb{C}$ and all other entries are equal to zero. Setting,

$$S = \begin{bmatrix} 0 & 1-i & 0 & 0 \\ 1-i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}+i \end{bmatrix},$$

we have

$$Z = RSR^T = \begin{bmatrix} 1 & \frac{1-i}{2\sqrt{3}} & \frac{1+2i}{\sqrt{15}} & 0 \\ \frac{1-i}{2\sqrt{3}} & \frac{2-2i}{3} & \frac{-7+i}{2\sqrt{45}} & 0 \\ \frac{1+2i}{\sqrt{15}} & \frac{-7+i}{2\sqrt{45}} & \frac{1+2i}{3} & 0 \\ 0 & 0 & 0 & \sqrt{2}+i \end{bmatrix}.$$

Now, we are ready to construct the blocks A,B,C of the normal Hamiltonian matrix. In particular,

$$A = 2Re(K_1) + 2Re(Z) = \begin{bmatrix} 2 & \frac{11}{\sqrt{3}} & \frac{-14}{\sqrt{15}} & 0 \\ \frac{-9}{\sqrt{3}} & \frac{4}{3} & \frac{-25}{\sqrt{45}} & 0 \\ \frac{18}{\sqrt{15}} & \frac{11}{\sqrt{45}} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 2\sqrt{2} \end{bmatrix},$$

$$B = 2Im(K_1) - 2Im(Z) = \begin{bmatrix} 2 & \frac{7}{\sqrt{3}} & \frac{8}{\sqrt{15}} & 0 \\ \frac{7}{\sqrt{3}} & \frac{2}{3} & \frac{25}{\sqrt{45}} & 0 \\ \frac{8}{\sqrt{15}} & \frac{25}{\sqrt{45}} & \frac{-14}{3} & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

and

$$C = -2Im(K_1) - 2Im(Z) = \begin{bmatrix} -2 & \frac{-5}{\sqrt{3}} & \frac{-16}{\sqrt{15}} & 0 \\ \frac{-5}{\sqrt{3}} & 2 & \frac{-27}{\sqrt{45}} & 0 \\ \frac{-16}{\sqrt{15}} & \frac{-27}{\sqrt{45}} & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

It is a matter of simple computations to show that $H = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}$ is normal.

4. Skew–Hamiltonian matrices

Using the same techniques, similar results can be proved for skew-Hamiltonian matrices. This is done with the use of Theorem 1b.

THEOREM 3. *a. The matrix* $\begin{bmatrix} E & F \\ K & E^* \end{bmatrix}$, $E,F,K \in \mathbb{C}^{n\times n}$, $F^* = -F$, $K^* = -K$ *is a complex normal skew–Hamiltonian matrix if and only if there are hermitian* $M_1, M_2 \in \mathbb{C}^{n\times n}$ *and a matrix* $D \in \mathbb{C}^{n\times n}$ *satisfying* $M_1D = DM_2$ *such that*

$$\begin{aligned} E &= M_1 + M_2 + D - D^*, \\ F &= -i(M_1 - M_2) + i(D + D^*), \\ K &= i(M_1 - M_2) + i(D + D^*). \end{aligned}$$

b. The matrix $\begin{bmatrix} E & F \\ K & E^T \end{bmatrix}$, $E,F,K \in \mathbb{R}^{n\times n}$, $F^* = -F$, $K^* = -K$ *is a real normal skew–Hamiltonian matrix if and only if there is a Hermitian matrix* $M_1 \in \mathbb{C}^{n\times n}$ *and a skew–symmetric complex matrix* $D \in \mathbb{C}^{n\times n}$ *satisfying* $M_1D = D\overline{M_1}$ *such that*

$$\begin{aligned} E &= 2Re(M_1) + 2Re(D), \\ F &= -2Im(M_1) - 2Im(D), \\ K &= 2Im(M_1) - 2Im(D). \end{aligned}$$

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