Operators and Matrices
Volume 15, Number 2 (2021), 463–470 doi:10.7153/oam-2021-15-32

EIGENVECTORS AND SPECTRA OF SOME WEIGHTED COMPOSITION OPERATORS ON L^p SPACES

GAJATH GUNATILLAKE

(*Communicated by G. Misra*)

Abstract. Let φ be a self map of $[0,1]$, and $\mathscr{W}$ be a map on $[0,1]$. If f belongs to the L^p space of $[0,1]$, then the operator $C_{\mathscr{W},\varphi}$ defined by $C_{\mathscr{W},\varphi}(f) = \mathscr{W}\cdot f\circ\varphi$, is a weighted composition operator. The spectrum of such an operator when φ is a monotonic contraction map and $\mathscr{W}$ is a Lipschitz continuous function is computed in this work.

1. Introduction

Let $1 \leqslant p < \infty$, and φ be a self map of $[0,1]$. Assume that f is in the L^p space of $[0,1]$. The operator that takes f to $f\circ\varphi$ is a composition operator and is denoted by C_φ. For more details on composition operators on L^p spaces see Chapter 2 of [3].

Now let $\mathscr{W}$ be a function on $[0,1]$. The operator that takes f to $\mathscr{W}\cdot f\circ\varphi$ is a weighted composition operator and is denoted by $C_{\mathscr{W},\varphi}$. See [1] and [2] for weighted composition operators on different spaces.

In this work we take φ to be a strictly monotonic contraction map which maps the interval $[0,1]$ into itself that induces a bounded composition operator on L^p. We take $\mathscr{W}$ to be a Lipschitz continuous function on $[0,1]$. We first construct some eigenvectors of $C_{\mathscr{W},\varphi}$. Next we estimate its spectral radius and this allows us to compute the spectrum of $C_{\mathscr{W},\varphi}$.

2. Preliminaries

Let $1 \leqslant p < \infty$. Denote the interval $[0,1]$ by I and the Lebesgue measure by m. A measurable function ϕ from I into I is said to be non-singular, if $m(\phi^{-1}(S)) = 0$, whenever $m(S) = 0$ for measurable S.

Assume that φ is a Lebesgue measurable non-singular self map of I. Suppose that there is a positive constant K such that for all Lebesgue measurable subsets E of I,

$$m(\varphi^{-1}(E)) \leqslant Km(E). \tag{1}$$

Then, a bounded linear operator C_φ on L^p can be defined by

$$C_\varphi(f) = f\circ\varphi$$

Mathematics subject classification (2020): Primary 47B37; Secondary 47B33.
Keywords and phrases: Weighted composition operator, L^p spaces.

Paper OaM-15-32

see Chapter 2 of [3] for a proof. Now let $\mathscr{W}$ be a Lipschitz continuous function on I. Denote the operator that takes f to $\mathscr{W} \cdot f \circ \varphi$ by $C_{\mathscr{W},\varphi}$, i.e

$$C_{\mathscr{W},\varphi}(f) = \mathscr{W} \cdot f \circ \varphi$$

Since we have taken C_φ as a bounded operator and $\mathscr{W}$ a continuous function, it easily follows that $C_{\mathscr{W},\varphi}$ is a bounded operator. However, in general it is possible to have bounded weighted composition operators induced by weight functions that are not continuous.

Let n be an integer greater than 1. We use φ^n to denote φ composed with itself n times. Moreover, we take $\varphi^0(x) = x$. Also we use C_φ^n to denote the operator C_φ composed with itself n times. It is easy to see that $C_\varphi^n(f) = f \circ \varphi^n$. Therefore,

$$C_\varphi^n = C_{\varphi^n}$$

Similarly, it is easy to see that $C_{\mathscr{W},\varphi}^n(f) = (\mathscr{W})(\mathscr{W} \circ \varphi) \cdots (\mathscr{W} \circ \varphi^{n-1}) \cdot f \circ \varphi^n$ hence

$$C_{\mathscr{W},\varphi}^n = C_{(\mathscr{W})(\mathscr{W} \circ \varphi)(\mathscr{W} \circ \varphi^2) \cdots (\mathscr{W} \circ \varphi^{n-1}),\varphi^n}$$

We refer to strictly increasing functions as increasing functions and strictly decreasing functions as decreasing functions.

Suppose that φ is a strictly monotonic contraction map whose Lipschitz constant is β. Thus, for any x, y in I,

$$|\varphi(x) - \varphi(y)| \leqslant \beta \cdot |x - y|$$

where $0 < \beta < 1$. If C_φ is bounded, and φ is differentiable at x_0, then from (1) it follows that $|\varphi'(x_0)| \geqslant \frac{1}{K}$. Notice that φ' exists almost everywhere. It is well known that φ has a unique fixed point ζ and if $x \neq \zeta$, the sequence $\{\varphi^n(x)\}$ converges to ζ. Here φ is a strictly monotonic map, hence if $x \neq \zeta$, then $\varphi^n(x) \neq \zeta$, for any n. Whenever the one sided derivatives of φ at 0 and 1 exist, we denote them by $\varphi'(0)$ and $\varphi'(1)$.

3. Spectra

We begin our work by constructing some eigenvectors for C_φ.

LEMMA 3.1. *Let φ be an increasing contraction map that takes I into itself. Let the unique fixed point of φ be ζ. Suppose that C_φ is bounded on L^p and $\varphi'(\zeta)$ exist. Then the point spectrum of C_φ contains the open disk of radius $(\varphi'(\zeta))^{-\frac{1}{p}}$ centered at the origin.*

Proof. We will first consider the case $0 \leqslant \zeta < 1$.

Since φ has only one fixed point, $\varphi(1) < 1$. Moreover, φ is an increasing function, thus $\varphi^n(1) < \varphi^{n-1}(1)$, for all $n \geqslant 1$. The function φ is a contraction map, thus the sequence $\{\varphi^n(1)\}$ converges to ζ. For a positive integer n, let $A_n = (\varphi^n(1), \varphi^{n-1}(1)]$. If $0 < |\lambda| < (\varphi'(\zeta))^{-\frac{1}{p}}$, then define

$$f(x) = \begin{cases} \lambda^{n-1}, & \text{if } x \in A_n \\ 0, & \text{if } x \leqslant \zeta \end{cases}$$

Clearly $\int_I |f(x)|^p dx = \sum_{n=1}^{\infty} m(A_n)|\lambda^{n-1}|^p$. It can be very easily seen that $\dfrac{m(A_n)}{m(A_{n-1})} = \dfrac{\varphi^{n-1}(1) - \varphi^n(1)}{\varphi^{n-2}(1) - \varphi^{n-1}(1)}$. Now, since $\lim_{n\to\infty} \varphi^n(1) = \zeta$, it follows that $\lim\limits_{n\to\infty} \dfrac{m(A_n)}{m(A_{n-1})} = \varphi'(\zeta)$. Therefore, if $0 < |\lambda| < (\varphi'(\zeta))^{-\frac{1}{p}}$, then

$$\lim_{n\to\infty} \frac{m(A_n)|\lambda^{n-1}|^p}{m(A_{n-1})|\lambda^{n-2}|^p} < 1.$$

Hence $f \in L^p$.

To prove that f is an eigenvector, first, let $x \in [0, \zeta]$. Then $f(x) = 0$. Since φ is increasing, if $0 \leqslant x \leqslant \zeta$, then $0 \leqslant \varphi(x) \leqslant \zeta$, hence $f(\varphi(x)) = 0$.

Next let $x \in (\zeta, 1]$. Then $x \in A_n$, for some n and hence $f(x) = \lambda^{n-1}$. Since φ is increasing it easily follows that $A_{n+1} = \varphi(A_n)$. Therefore, $\varphi(x) \in A_{n+1}$, hence $f(\varphi(x)) = \lambda^n$.

Therefore, for all x in I we get that

$$f(\varphi(x)) = \lambda f(x)$$

This proves that λ is an eigenvalue when $0 \leqslant \zeta < 1$.

Now assume that $\zeta = 1$.

Then $\{\varphi^n(0)\}$ is an increasing sequence that converges to 1. For a positive integer n, let $B_n = [\varphi^{n-1}(0), \varphi^n(0))$. If $0 < |\lambda| < (\varphi'(1))^{-\frac{1}{p}}$, define

$$g(x) = \begin{cases} \lambda^{n-1}, & \text{if } x \in B_n \\ 0, & \text{if } x = 1 \end{cases}$$

Using arguments very similar to the ones used when $\zeta < 1$, it can be proved that g is an eigenvector for eigenvalue λ.

Clearly $m(\varphi(I)) < 1$, and the characteristic function of $I \setminus \varphi(I)$ is in the kernel of C_φ. Therefore 0 is in the point spectrum as well. $\square$

Let T be a bounded linear operator on L^p. Below, the spectral radius of T is denoted by $r(T)$ and the supremum of the set $\{\|T(f)\|_p : \|f\|_p = 1\}$ is denoted by $\|T\|$. Moreover, $\sigma(T)$ denotes the spectrum of T.

LEMMA 3.2. *Let φ be an increasing contraction map that takes I into itself. Let ζ be the unique fixed point of φ. Assume that φ' exists and is continuous at ζ. If C_φ is bounded on L^p, then the spectral radius of C_φ is not larger than $(\varphi'(\zeta))^{-\frac{1}{p}}$.*

Proof. First assume that $0 < \zeta < 1$.

Let $f \in L^p$. Then $\|C_\varphi^n(f)\|_p^p = \int_I |f(\varphi^n(x))|^p dx$. By a change of variables it can be easily seen that

$$\|C_\varphi^n(f)\|_p^p = \int_{A_n} |f(y)|^p \frac{1}{(\varphi^n)'((\varphi^n)^{-1}(y))} dy \tag{2}$$

where $A_n = [\varphi^n(0), \varphi^n(1)]$ and $(\varphi^n)^{-1}$ is defined on A_n. Let $z = (\varphi^n)^{-1}(y)$. By applying the chain rule repeatedly we get $(\varphi^n)'(z) = \prod_{j=1}^n \varphi'(\varphi^{n-j}(z))$.

Let $0 < \varepsilon < 1$. Clearly $\varphi^n(I) = [\varphi^n(0), \varphi^n(1)]$. Notice that both sequences $\{\varphi^n(0)\}$ and $\{\varphi^n(1)\}$ tend to ζ. Hence, $\varphi'(\zeta)\cdot(1-\varepsilon) < \varphi'(\varphi^{n+N}(z))$ for $n > N$, for some N and all $z \in I$. Thus $(\varphi'(\zeta)\cdot(1-\varepsilon))^{n-N}\omega^N < \prod_{j=1}^n \varphi'(\varphi^{n-j}(z))$, where ω is the infimum of φ' on I. Therefore, from (2) it follows that

$$\|C_\varphi^n(f)\|_p^p \leqslant \frac{1}{(\varphi'(\zeta)\cdot(1-\varepsilon))^{n-N}\omega^N} \int_I |f(y)|^p dy$$

Hence $\|C_\varphi^n\| \leqslant \dfrac{1}{((\varphi'(\zeta)\cdot(1-\varepsilon))^{n-N}\omega^N)^{\frac{1}{p}}}$. Now it easily follows that

$$\|C_\varphi^n\|^{\frac{1}{n}} \leqslant \frac{1}{((\varphi'(\zeta)\cdot(1-\varepsilon))^{1-\frac{N}{n}}\omega^{\frac{N}{n}})^{\frac{1}{p}}}.$$

Thus

$$\lim_{n\to\infty} \|C_\varphi^n\|^{\frac{1}{n}} \leqslant \frac{1}{(\varphi'(\zeta)\cdot(1-\varepsilon))^{\frac{1}{p}}}$$

Since the inequality above is true for all ε in (0,1), it easily follows that $r(C_\varphi) \leqslant (\varphi'(\zeta))^{-\frac{1}{p}}$.

If $\zeta = 0$, then $\{\varphi^n(1)\}$ tends to 0 and $\varphi^n(I) = [0, \varphi^n(1)]$. If $\zeta = 1$, then $\{\varphi^n(0)\}$ tends to 1 and $\varphi^n(I) = [\varphi^n(0), 1]$. Thus, a proof similar to the one used for $0 < \zeta < 1$, yields the desired result when $\zeta = 0$ or $\zeta = 1$. □

Using the result above, next we estimate the spectral radius of $C_{\mathscr{W},\varphi}$.

LEMMA 3.3. *Let φ be an increasing contraction map that takes I into itself and $\mathscr{W}$ be a Lipschitz continuous map on I. Let ζ be the unique fixed point of φ. Suppose that φ' exists and continuous at ζ. If C_φ is bounded on L^p, then the spectral radius of $C_{\mathscr{W},\varphi}$ is not larger than $|\mathscr{W}(\zeta)|\cdot(\varphi'(\zeta))^{-\frac{1}{p}}$.*

Proof. First assume that $0 < \zeta < 1$.

Let $f \in L^p$. Then $\|C_{\mathscr{W},\varphi}^n(f)\|_p^p = \int_I |\prod_{j=1}^n \mathscr{W}(\varphi^j(x))\cdot f(\varphi^n(x))|^p dx$.

Let $0 < \varepsilon$. Clearly $\varphi^n(I) = [\varphi^n(0), \varphi^n(1)]$. Notice that both sequences $\{\varphi^n(0)\}$ and $\{\varphi^n(1)\}$ tend to ζ. Thus $|\mathscr{W}(\varphi^j(x))| \leqslant |\mathscr{W}(\zeta)| + \varepsilon$ for $j > N$, for some N. Therefore,

$$\|C_{\mathscr{W},\varphi}^n(f)\|_p^p \leqslant (|\mathscr{W}(\zeta)| + \varepsilon)^{p(n-N)} \int_I |C_{\mathscr{W},\varphi}^N C_\varphi^{n-N}(f)(x)|^p dx$$

Thus, $\|C^n_{\mathscr{W},\varphi}(f)\|_p^p \leqslant (|\mathscr{W}(\zeta)|+\varepsilon)^{p(n-N)}\|C^N_{\mathscr{W},\varphi}\|^p \cdot \|C_\varphi^{n-N}\|^p \cdot \|f\|_p^p$ and now it follows that

$$\|C^n_{\mathscr{W},\varphi}\| \leqslant (|\mathscr{W}(\zeta)|+\varepsilon)^{(n-N)}\|C^N_{\mathscr{W},\varphi}\| \cdot \|C_\varphi^{n-N}\|$$

Therefore

$$\|C^n_{\mathscr{W},\varphi}\|^{\frac{1}{n}} \leqslant (|\mathscr{W}(\zeta)|+\varepsilon)|^{(1-\frac{N}{n})}\|C^N_{\mathscr{W},\varphi}\|^{\frac{1}{n}} \cdot (\|C_\varphi^{n-N}\|^{\frac{1}{(n-N)}})^{1-\frac{N}{n}}$$

By letting n tend to infinity we get

$$r(C_{\mathscr{W},\varphi}) \leqslant \frac{|\mathscr{W}(\zeta)|+\varepsilon}{(\varphi'(\zeta))^{\frac{1}{p}}}$$

Since ε is arbitrary, it easily follows that $r(C_{\mathscr{W},\varphi}) \leqslant |\mathscr{W}(\zeta)| \cdot (\varphi'(\zeta))^{-\frac{1}{p}}$.

If $\zeta = 0$, then $\{\varphi^n(1)\}$ tends to 0 and $\varphi^n(I) = [0, \varphi^n(1)]$. If $\zeta = 1$, then $\{\varphi^n(0)\}$ tends to 1 and $\varphi^n(I) = [\varphi^n(0), 1]$. Thus, a proof similar to the one used for $0 < \zeta < 1$, yields the desired result when $\zeta = 0$ or $\zeta = 1$. □

In order to create eigenvectors for weighted composition operators we first investigate some infinite products.

LEMMA 3.4. *Let φ be a contraction map that takes I into itself. Assume that ζ is the unique fixed point of φ. If Φ is a Lipschitz continuous function on I and* $\Phi(\zeta) = 1$, *then the infinite product*

$$\left(\prod_{n=0}^{\infty} \Phi \circ \varphi^n\right)$$

converges to a bounded function that is non-zero on a neighborhood of ζ.

Proof. Let $\tilde{\gamma}$ be the Lipschitz constant of Φ and β be the Lipschitz constant of φ. Then, $|\Phi(\varphi^n(x)) - \Phi(\zeta)| \leqslant \tilde{\gamma}|\varphi^n(x) - \zeta|$. Thus for all x in I,

$$|\Phi(\varphi^n(x)) - 1| \leqslant \tilde{\gamma}\beta^n$$

Hence the infinite product $\prod_{n=0}^{\infty} \Phi(\varphi^n(x))$ converges to a function $W(x)$. Moreover,

$$|\prod_{n=0}^{\infty} \Phi(\varphi^n(x))| \leqslant e^{\frac{\tilde{\gamma}}{(1-\beta)}}$$

for all $x \in I$. See page 162 of [6] for convergence and the upper bound of the infinite product.

There is a neighborhood U of positive measure that contains ζ such that $\Phi(x) \neq 0$, for $x \in U$. Clearly $\varphi^n(U) \subseteq U$, thus $\Phi(\varphi^n(x)) \neq 0$, when $x \in U$ and all $n \in \mathbb{N}$. Thus $W(x) \neq 0$, for all $x \in U$; see page 163 of [6]. □

The infinite product above allows us to construct eigenvectors of weighted composition operators.

LEMMA 3.5. *Let φ be a contraction map that takes I into itself. Assume that ζ is the unique fixed point of φ. Further assume that Φ is a Lipschitz continuous function on I and $\Phi(\zeta)=1$. Suppose that C_φ is a bounded operator and λ is a non-zero eigenvalue with the eigenvector f. Then λ is also an eigenvalue of $C_{\Phi,\varphi}$ with the eigenvector*

$$\left(\prod_{n=0}^{\infty}\Phi\circ\varphi^n\right)\cdot f$$

Proof. Let $W(x)=\prod_{n=0}^{\infty}\Phi\circ\varphi^n(x)$. Then W is bounded on I. Thus, $W\cdot f$ is in L^p. Now,

$$\begin{aligned}C_{\Phi,\varphi}(W\cdot f)&=\Phi\cdot W\circ\varphi\cdot f\circ\varphi\\&=\Phi\cdot\left(\prod_{n=0}^{\infty}\Phi\circ\varphi^n\right)\circ\varphi\cdot(\lambda f)\\&=\lambda\cdot\Phi\cdot\left(\prod_{n=0}^{\infty}\Phi\circ\varphi^{n+1}\right)\cdot f\end{aligned}$$

Incorporating the term Φ into the infinite product results in

$$C_{\Phi,\varphi}(W\cdot f)=\lambda\cdot\left(\prod_{n=0}^{\infty}\Phi\circ\varphi^n\right)\cdot f$$

Finally we prove that $W\cdot f$ is not the zero function. Let $n\geqslant 1$. Since f is an eigenvector for λ, it follows that $C_\varphi^n(f)=\lambda^n f$. Therefore

$$f\circ\varphi^n=\lambda^n f, \tag{3}$$

almost everywhere. If U is a neighborhood of ζ, then $\varphi^n(I)\subset U$, for all n large enough. If f is zero almost everywhere on U, then it follows from equation above that f is zero almost everywhere on I. Since f is an eigenvector this is impossible, hence f cannot be zero a.e on any neighborhood of ζ.

Since W is non-zero on some neighborhood of ζ now it follows that $W\cdot f$ is a non-zero element in L^p. Thus λ is an eigenvalue of $C_{\Phi,\varphi}$ with the eigenvector $W\cdot f$. $\square$

Notice that results in Lemma 3.4 and Lemma 3.5 were obtained without assuming that φ is monotonic. It suffices that φ is a contraction map.

If $r>0$, we denote the open disc of radius r centered at the origin by $B(r)$.

LEMMA 3.6. *Let φ be an increasing contraction map that takes I into itself and $\mathscr{W}$ be a Lipschitz continuous map on I. Let ζ be the unique fixed point of φ. Suppose that φ' exists and continuous at ζ. Assume that C_φ is bounded on L^p and $\mathscr{W}(\zeta)\neq 0$. If λ is in $B(|\mathscr{W}(\zeta)|\cdot(\varphi'(\zeta))^{-\frac{1}{p}})\setminus\{0\}$, then λ is an eigenvalue of $C_{\mathscr{W},\varphi}$.*

Proof. Let $\Phi(x) = \dfrac{\mathscr{W}(x)}{\mathscr{W}(\zeta)}$ for $x \in I$. Now consider the weighted composition operator $C_{\Phi,\varphi}$. Let $\lambda \in B((\varphi'(\zeta))^{-\frac{1}{p}}) \setminus \{0\}$. Then λ is an eigenvalue of C_φ, and hence it is an eigenvalue of $C_{\Phi,\varphi}$; see Lemma 3.5 and Lemma 3.1. Since $\mathscr{W}(\zeta)C_{\Phi,\varphi} = C_{\mathscr{W},\varphi}$, it is easy to see that $\mathscr{W}(\zeta)\lambda$ is an eigenvalue of $C_{\mathscr{W},\varphi}$. This is the desired result. $\square$

Next we compute the spectrum when φ is increasing.

THEOREM 3.7. *Let φ be an increasing contraction map that takes I into itself and $\mathscr{W}$ be a Lipschitz continuous map on I. Let ζ be the unique fixed point of φ. Suppose that φ' exists and continuous at ζ. If C_φ is bounded on L^p, then the spectrum of $C_{\mathscr{W},\varphi}$ is the closed disk of radius $|\mathscr{W}(\zeta)| \cdot (\varphi'(\zeta))^{-\frac{1}{p}}$ centered at the origin.*

Proof. If $\mathscr{W}(\zeta) = 0$, then $r(C_{\mathscr{W},\varphi}) = 0$, and hence the spectrum is $\{0\}$.

Now assume that $\mathscr{W}(\zeta) \neq 0$. From Lemma 3.6 it follows that $\sigma(C_{\mathscr{W},\varphi})$ contains $B(|\mathscr{W}(\zeta)| \cdot (\varphi'(\zeta))^{-\frac{1}{p}}) \setminus \{0\}$, and from Lemma 3.3 it follows that $\sigma(C_{\mathscr{W},\varphi})$ is contained in the closure of $B(|\mathscr{W}(\zeta)| \cdot (\varphi'(\zeta))^{-\frac{1}{p}})$. Since $\sigma(C_{\mathscr{W},\varphi})$ is a closed set, the desired result follows. $\square$

Below, we denote the unit circle centered at origin by $\mathbb{T}$ and the point spectrum of an operator T by $\sigma_p(T)$.

Next, we compute the spectrum when φ is decreasing.

THEOREM 3.8. *Let φ be a decreasing contraction map that takes I into itself and $\mathscr{W}$ be a Lipschitz continuous map on I. Let ζ be the unique fixed point of φ. Suppose that φ' exists and continuous at ζ. If C_φ is bounded on L^p, then the spectrum of $C_{\mathscr{W},\varphi}$ is the closed disk of radius $|\mathscr{W}(\zeta)| \cdot (\varphi'(\zeta))^{-\frac{1}{p}}$ centered at the origin.*

Proof. First assume that $\mathscr{W}(\zeta) \neq 0$. If φ is decreasing, then φ^2 is increasing. Thus, from Lemma 3.1 it follows that $\sigma_p(C_{\varphi^2})$ contains $B(((\varphi^2)'(\zeta))^{-\frac{1}{p}})$. Recall that $C_\varphi^2 = C_{\varphi^2}$. Thus, $(\sigma_p(C_\varphi))^2 = \sigma_p(C_{\varphi^2})$; see [5, p.266]. If $\lambda \in \sigma_p(C_\varphi) \setminus \mathbb{T}$, and $0 \leqslant \theta < 2\pi$, then $\lambda e^{i\theta}$ also belongs to $\sigma_p(C_\varphi)$; see [4]. Since $(\varphi^2)'(\zeta) = (\varphi'(\zeta))^2$ it follows that $\sigma_p(C_\varphi)$ contains $B((\varphi'(\zeta))^{-\frac{1}{p}}) \setminus \mathbb{T}$.

If λ is a non-zero eigenvalue of C_φ, then λ is also an eigenvalue of $C_{\Phi,\varphi}$ where $\Phi(x) = \dfrac{\mathscr{W}(x)}{\mathscr{W}(\zeta)}$, therefore, $\mathscr{W}(\zeta)\lambda \in \sigma_p(C_{\mathscr{W},\varphi})$; see Lemma 3.5. Thus it follows that $\sigma_p(C_{\mathscr{W},\varphi})$ contains $B(|\mathscr{W}(\zeta)| \cdot (\varphi'(\zeta))^{-\frac{1}{p}}) \setminus (\{0\} \cup \mathbb{T})$.

It is not difficult to see that $\mathscr{W} \cdot \mathscr{W} \circ \varphi$ is Lipschitz continuous on I. Thus from Theorem 3.7 it follows that $\sigma(C_{\mathscr{W}\cdot\mathscr{W}\circ\varphi,\varphi^2})$ is the closure

of $B(|\mathscr{W}(\zeta)|^2((\varphi^2)'(\zeta))^{-\frac{1}{p}})$.

Recall that $C_{\mathscr{W},\varphi}^2 = C_{\mathscr{W}\cdot\mathscr{W}\circ\varphi,\varphi^2}$. Thus, $(\sigma(C_{\mathscr{W},\varphi}))^2 = \sigma(C_{\mathscr{W}\cdot\mathscr{W}\circ\varphi,\varphi^2})$. Therefore, $r(C_{\mathscr{W},\varphi}) = (|\mathscr{W}(\zeta)|^2((\varphi^2)'(\zeta))^{-\frac{1}{p}})^{\frac{1}{2}}$. Since $\sigma(C_{\mathscr{W},\varphi})$ is a closed set, the desired result follows.

If $\mathscr{W}(\zeta)=0$, then $r(C_{\mathscr{W},\varphi})=0$ and hence $\sigma(C_{\mathscr{W},\varphi})=\{0\}$. $\square$

We close this paper with the following example.

Let $\varphi(x)=kx$, where $0<k<1$ and $\mathscr{W}(x)=e^x$. The spectrum of $C_{\mathscr{W},\varphi}$ on L^p is the closed disk of radius $k^{-\frac{1}{p}}$; see Theorem 3.7.

Now, $\left(\prod_{n=0}^{\infty}\mathscr{W}\circ\varphi^n\right)(x)=e^{x(1+k+k^2+\cdots)}$, which further simplifies to $e^{\frac{1}{1-k}x}$. Now let f_λ be an eigenvector for C_φ as described in Lemma 3.1. Then $e^{\frac{1}{1-k}x}f_\lambda(x)$ is an eigenvector for $C_{\mathscr{W},\varphi}$.

REFERENCES

[1] P. BUDZYŃSKI, Z. JABŁOŃSKI, I. B. JUNG AND J. STOCHEL, *Unbounded Weighted Composition Operators in L^2 spaces*, Springer, 2018.

[2] J. W. CARLSON, *The Spectra and Commutants of some Weighted Composition Operators*, Trans. Amer. Math. Soc., 317 (1990), 631–654.

[3] R. K. SINGH AND J. S. MANHAS, *Composition Operators on Function Spaces*, Elsevier, 1993.

[4] W. C. RIDGE, *Spectrum of a Composition Operator*, Proc. Amer. Math. Soc., 37 (1973), 121–127.

[5] W. RUDIN, *Functional Analysis*, TATA McGraw-Hill, 2006.

[6] W. A. VEECH, *A Second Course in Complex Analysis*, W. A. Benjamin. Inc, New York, 1967.

(Received September 18, 2019)

Gajath Gunatillake
Department of Mathematics and Statistics
American University of Sharjah
P. O. BOX 26666, Sharjah, UAE
e-mail: mgunatillake@aus.edu; gajathg@yahoo.com