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PRESERVERS OF THE c–NUMERICAL RADIUS OF OPERATOR JORDAN SEMI–TRIPLE PRODUCTS

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Abstract. Let $\mathscr{H}$ be a complex Hilbert space with $\dim \mathscr{H} \geqslant 3$, let $\mathfrak{B}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$ and let $\mathfrak{B}^s(\mathscr{H})$ be the real Jordan algebra of all self-adjoint operators in $\mathfrak{B}(\mathscr{H})$. Let $\mathfrak{A} = \mathfrak{B}(\mathscr{H})$ or $\mathfrak{B}^s(\mathscr{H})$. We characterize the surjective maps on $\mathfrak{A}$ preserving the c-numerical radius of Jordan semi-triple products of operators. Further, the maps on $\mathfrak{A}$ preserving the c-numerical range of Jordan semi-triple products are characterized according to different cases of c.

1. Introduction

Motivated by the theory and applications, it is always of interest to characterize maps with special properties such as leaving certain functions, subsets or relations invariant, which are called preserving problems. For some given set $\mathscr{A}$ of matrices or operators, there are interesting results showing $\phi : \mathscr{A} \to \mathscr{A}$ will have a nice structure if

$$F(\phi(A) \circ \phi(B)) = F(A \circ B) \qquad (A, B \in \mathscr{A}) \tag{1.0}$$

for some suitable functional F and some product $\circ$ of matrices or operators (see [2, 3, 4, 6, 12]).

Let $\mathfrak{B}(\mathscr{H})$ be the Banach algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity I and $\mathfrak{B}^s(\mathscr{H})$ be the real Jordan algebra of all self-adjoint operators in $\mathfrak{B}(\mathscr{H})$. If $\dim \mathscr{H} = n < \infty$, $\mathfrak{B}(\mathscr{H})$ and $\mathfrak{B}^s(\mathscr{H})$ are regarded as $M_n(\mathbb{C})$ and $M_n^s(\mathbb{C})$, respectively. Here $M_n(\mathbb{C})$ is the set of all complex $n \times n$ matrices and $M_n^s(\mathbb{C})$ is the set of all complex self-adjoint $n \times n$ matrices. In [2], Bendaoud et al. showed the form of ϕ satisfying (1.0), when $\mathscr{A}$ is some set of $\mathfrak{B}(\mathscr{H})$, $\circ$ is either the usual product or Jordan semi-triple product and $F : \mathfrak{B}(\mathscr{H}) \to [0, \infty)$ has the following properties:

(i) $F(\lambda UAU^*) = F(A)$ for complex unit λ, $A \in \mathfrak{B}(\mathscr{H})$ and unitary $U \in \mathfrak{B}(\mathscr{H})$.

(ii) For every rank-one nilpotent $N \in \mathfrak{B}(\mathscr{H})$, the map $t \mapsto F(tN)$ on $[0, \infty)$ is strictly increasing.

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(iii) $F(A) = 0 \Leftrightarrow A = 0$.

As applications, the results were used to characterize the preservers of the numerical radius, the spectral norm, the pseudo spectral radius, etc. The preservers of pseudo spectral radius were also characterized in [1].

In [12], when $\circ$ is the Jordan semi-triple product, $\mathscr{A}$ is $\mathfrak{B}^s(\mathscr{H})$, and $F : \mathfrak{B}(\mathscr{H}) \to [d,\infty)$ with $d \geqslant 0$ satisfies (i), (ii) and (III), where (III) $F(A) = d \Leftrightarrow A = 0$, the form of ϕ satisfying (1.0) was described. The result was used to characterize maps on $\mathfrak{B}^s(\mathscr{H})$ preserving the pseudo spectral radius.

For $c = (c_1,\cdots,c_k)^t \in \mathbb{R}^k \setminus \{0\}$ and $k \leqslant \dim \mathscr{H}$, recall that the c-numerical range and the c-numerical radius of $A \in \mathfrak{B}(\mathscr{H})$ are respectively defined as

$$W_c(A) = \left\{ \sum_{j=1}^{k} c_j \langle Ae_j, e_j \rangle : \{e_1,\ldots,e_k\} \text{ is an orthonormal subset in } \mathscr{H} \right\},$$

$$r_c(A) = \sup\{|\lambda| : \lambda \in W_c(A)\}.$$

Obviously, nothing changes if the components $c_i (i = 1,\cdots,k)$ are reordered in descending order. Thus we always assume $c_1 \geqslant \cdots \geqslant c_k$. If $k = 1$ and $c_1 = 1$, we get the classical numerical range $W(A)$ and the numerical radius $w(A)$ of A. If $(c_1,\ldots,c_k) = (1,\ldots,1)$, $W_c(A)$ and $r_c(A)$ reduce to the k-numerical range and the k-numerical radius of A, respectively (see [5, 9, 10]). Obviously, the c-numerical radius is unitary similarity. Also it has the property "$r_c(A)$ is a norm if and only if $\sum_{i=1}^{k} c_i \neq 0$ and not all c_i's are equal". When $r_c(A)$ is a norm or it reduces to the k-numerical radius, the maps on $\mathfrak{B}(\mathscr{H})$ preserving the c-numerical radius of Jordan semi-triple product can be characterized from the result in [2]. However, when $\sum_{i=1}^{k} c_i = 0$ and not all c_i's are equal, $r_c(A) = 0$ if and only if A is a scalar multiple of the identity. No general result can be applied to characterize the maps preserving the c-numerical radius. Motivated by this, we consider the question of the maps on $\mathfrak{B}(\mathscr{H})$ and $\mathfrak{B}^s(\mathscr{H})$ satisfying (1.0) when $\circ$ is the Jordan semi-triple product and $F : \mathfrak{B}(\mathscr{H}) \to [0,\infty)$ has the following properties:

(P_1) $F(UAU^*) = F(A)$ for any $A \in \mathfrak{B}(\mathscr{H})$ and unitary $U \in \mathfrak{B}(\mathscr{H})$.

(P_2) For every $A \in \mathfrak{B}(\mathscr{H})$, $F(A) = 0$ if and only if $A \in \mathbb{C}I$.

(P_3) There are non-negative real numbers α, β with $\alpha^2 + \beta^2 \neq 0$ such that $F(T) = \alpha\|T\| + \beta|\mathrm{tr}(T)|$ for each rank-one $T \in \mathfrak{B}(\mathscr{H})$.

In this paper, let $\dim \mathscr{H} \geqslant 3$ and let $\mathfrak{A} = \mathfrak{B}(\mathscr{H})$ or $\mathfrak{B}^s(\mathscr{H})$. Let $\mathscr{W}$, $\mathscr{V}$ be subsets of $\mathfrak{A}$ containing all rank-one operators in $\mathfrak{A}$. When $F : \mathfrak{B}(\mathscr{H}) \to [0,\infty)$ satisfies the properties (P_1), (P_2) and (P_3), we give a characterization of surjective maps $\Phi : \mathscr{W} \to \mathscr{V}$ satisfying $F(\Phi(A)\Phi(B)\Phi(A)) = F(ABA)$ for all $A, B \in \mathscr{W}$. Based on the result, the form of surjective maps on $\mathfrak{A}$ preserving the c-numerical radius is obtained. Further, we give the results about the maps on $\mathfrak{A}$ preserving the c-numerical range according to different cases of c.

The paper is organized as follows: in Section 2, firstly we study the general case of the maps preserving some unitary similarity functional satisfying (P_1) – (P_3) on

Jordan semi-triple products of operators in $\mathfrak{B}(\mathscr{H})$. Then as an application, we give the form of surjective maps preserving the c-numerical radius for operators in $\mathfrak{B}(\mathscr{H})$. In Section 3, we obtain results about maps preserving the c-numerical range of Jordan semi-triple products for operators in $\mathfrak{B}(\mathscr{H})$ according to three cases of c. In Section 4, we characterize the maps in $\mathfrak{B}^s(\mathscr{H})$ preserving the c-numerical radius and the c-numerical range of operator Jordan semi-triple products, respectively.

Throughout this paper, denote the set of the complex field, the set of the real field and the unit circle of complex field by $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{T}$, respectively. For $A \in \mathfrak{B}(\mathscr{H})$, we write A^* for its adjoint and A^{tr} for the transpose of A for an arbitrary but fixed orthogonal basis of $\mathscr{H}$. For any $x, f \in \mathscr{H}$, the notation $x \otimes f$ denotes a rank-one operator on $\mathscr{H}$ defined by $z \mapsto \langle z, f\rangle x$ for every $z \in \mathscr{H}$; and every rank-one operator can be written in this form. Let $\mathscr{F}_1(\mathscr{H})$ denote the set of all rank-one operators in $\mathscr{B}(\mathscr{H})$. Fix an arbitrary orthogonal basis $\{e_i\}_{i\in\Gamma}$, any $x \in \mathscr{H}$ can be written to $x = \sum_{i\in\Gamma} \xi_i e_i$ and define the conjugate operator $J : \mathscr{H} \to \mathscr{H}$ by $Jx = \bar{x} = \sum_{i\in\Gamma} \overline{\xi_i} e_i$. The notation $\overline{A}$ denotes the bounded linear operator JAJ in $\mathfrak{B}(\mathscr{H})$. Notice that $\langle \overline{A} e_i, e_j\rangle = \overline{\langle Ae_i, e_j\rangle}$ for all $i, j \in \Gamma$.

2. Preservers of the c-numerical radius for operators in $\mathfrak{B}(\mathscr{H})$

Firstly, we give a general result that provides a characterization of maps preserving some unitary similarity functional satisfying $(\mathrm{P}_1)-(\mathrm{P}_3)$ on Jordan semi-triple product of operators in $\mathfrak{B}(\mathscr{H})$.

THEOREM 1. *Let $\mathscr{H}$ be a complex Hilbert space with $\dim \mathscr{H} \geqslant 3$ and $\mathscr{W}$, $\mathscr{V}$ be subsets of $\mathfrak{B}(\mathscr{H})$ containing $\mathscr{F}_1(\mathscr{H})$. And let $F : \mathfrak{B}(\mathscr{H}) \to [0, +\infty)$ be a functional satisfying (P_1)-(P_3). Suppose $\Phi : \mathscr{W} \to \mathscr{V}$ is a surjective map satisfying*

$$F(\Phi(A)\Phi(B)\Phi(A)) = F(ABA)$$

for all $A, B \in \mathscr{W}$. Then there exist a unitary operator U on $\mathscr{H}$ and a functional $f : \mathscr{W} \to \mathbb{T}$ such that $\Phi(A) = f(A)UA^{\ddagger}U^*$ for all $A \in \mathscr{W}$, where $A^{\ddagger}$ has only one of the forms in $\{A,\ JAJ,\ A^*,\ JA^*J\}$ for all $A \in \mathscr{W}$.

The following two lemmas show properties of the c-numerical radius (the c-numerical range). One may see [10, 13] for more information.

LEMMA 1. *Let $A \in \mathfrak{B}(\mathscr{H})$, then*

1. $W_c(UAU^*) = W_c(A)$ *for any unitary* $U \in \mathfrak{B}(\mathscr{H})$.
2. $W_c(\lambda A) = \lambda W_c(A)$ *for any* $\lambda \in \mathbb{C}$.
3. $W_c(\lambda I + A) = \lambda \sum_{i=1}^{k} c_i + W_c(A)$ *for any* $\lambda \in \mathbb{C}$.
4. *Suppose c_i's are not equal. Then $W_c(A)$ is a singleton if and only if A is a scalar multiple of the identity.*

5. *The c-numerical radius $r_c(A)$ is a norm on $B(\mathscr{H})$ if and only if $\sum_{i=1}^{k} c_i \neq 0$ and not all c_i's are equal.*

LEMMA 2. [13, Proposition 2.4] *Let $\mathscr{H}$ be a complex Hilbert space and not all c_i in $c \in \mathbb{R}^k \setminus \{0\}$ be equal. Suppose $T \in \mathfrak{B}(\mathscr{H})$ is rank-one, then the followings hold:*

1. *$W_c(T)$ is an elliptical disk with foci $\bar{c}_1 \mathrm{tr}(T)$, $\tilde{c}_k \mathrm{tr}(T)$ and minor axis $(\bar{c}_1 - \tilde{c}_k)\sqrt{\| T \|^2 - | \mathrm{tr}(T) |^2}$, or it is a line segment with end points $\bar{c}_1 \mathrm{tr}(T)$ and $\tilde{c}_k \mathrm{tr}(T)$.*

2. *The c-numerical radius of T is $\frac{(\bar{c}_1 - \tilde{c}_k)}{2}\|T\| + \frac{|\bar{c}_1 + \tilde{c}_k|}{2}|tr(T)|$.*

Here $\bar{c}_1 = \begin{cases} c_1, & \text{if } \dim \mathscr{H} = k, \\ \max\{c_1, 0\}, & \text{if } \dim \mathscr{H} > k, \end{cases}$ *and* $\tilde{c}_k = \begin{cases} c_k, & \text{if } \dim \mathscr{H} = k, \\ \min\{c_k, 0\}, & \text{if } \dim \mathscr{H} > k. \end{cases}$

Next we give one of our main results, which is about maps preserving the c-numerical radius of operator Jordan semi-triple products.

THEOREM 2. *Let $k \geqslant 3$ and not all c_i in $c \in \mathbb{R}^k \setminus \{0\}$ be equal. Suppose $\mathscr{W}$ and $\mathscr{V}$ are subsets of $\mathfrak{B}(\mathscr{H})$ containing $\mathscr{F}_1(\mathscr{H})$. Then the surjective map $\Phi : \mathscr{W} \to \mathscr{V}$ satisfies*

$$r_c(ABA) = r_c(\Phi(A)\Phi(B)\Phi(A)) \qquad (A, B \in \mathscr{W}) \tag{2.0}$$

if and only if there are a unitary operator $U \in \mathfrak{B}(\mathscr{H})$ and a functional $f : \mathscr{W} \to \mathbb{T}$ such that $\Phi(A) = f(A) U A^{\ddagger} U^*$ for all $A \in \mathscr{W}$, where $A^{\ddagger}$ is one of the form among A, JAJ, A^* and JA^*J for all $A \in \mathscr{W}$.

Proof. From Lemma 1 and Lemma 2, we know that $r_c(\cdot)$ satisfies (P_1) and (P_3) of $F(\cdot)$. When $\sum_{i=1}^{k} c_i = 0$, $r_c(\cdot)$ also satisfies (P_2), then the conclusion can be deduced by Theorem 1. When $\sum_{i=1}^{k} c_i \neq 0$, it can be deduced from [1, Theorem 2.1 and Theorem 2.2]. $\square$

To prove Theorem 1, we need the following lemmas. The first two are quoted from [7]. They characterize the maps preserving zero Jordan semi-triple product of operators and matrices.

LEMMA 3. *Let $\mathscr{B}(X)$ be the algebra of all bounded linear operators on an infinite dimensional complex Banach space X, and $\mathscr{W}, \mathscr{V} \subseteq \mathscr{B}(X)$ contain all rank-one idempotents. Suppose that $\Phi : \mathscr{W} \to \mathscr{V}$ is a surjective map such that*

$$ABA = 0 \Leftrightarrow \Phi(A)\Phi(B)\Phi(A) = 0 \quad (A, B \in \mathscr{W}). \tag{1}$$

Then there is a functional $l : \mathscr{W} \to \mathbb{C} \setminus \{0\}$ and either there is a bounded invertible linear or a conjugate linear operator U on X such that $\Phi(A) = l(A) U A U^{-1}$ for each *rank-one* $A \in \mathscr{W}$*, or* X *is reflexive and there is a bounded invertible linear or conjugate linear operator* U *from* X^**, the dual of* X*, into* X *such that* $\Phi(A) = l(A)UA^*U^{-1}$ *for all* $A \in \mathscr{W}$*.*

LEMMA 4. *Let* $n \geqslant 3$ *and* $\mathscr{W}, \mathscr{V} \subseteq M_n(\mathbb{C})$ *contain all rank-one matrices. Suppose that* $\Phi : \mathscr{W} \to \mathscr{V}$ *is a surjective map satisfying* (1)*. Then there are a functional* $l : \mathscr{W} \to \mathbb{C} \setminus \{0\}$*, an invertible matrix* $U \in M_n(\mathbb{C})$ *and a field monomorphism* η *on* $\mathbb{C}$ *such that either* $\Phi(A) = l(A)UA^\eta U^{-1}$ *for all* $A \in \mathscr{W}$*, or* $\Phi(A) = l(A)U(A^\eta)^{\mathrm{tr}}U^{-1}$ *for all* $A \in \mathscr{W}$*.*

LEMMA 5. [3, Lemma 2.4] *Let* $T \in \mathfrak{B}(\mathscr{H})$ *be a positive invertible operator. Then* T *is a scalar multiple of the identity if and only if there is a constant* $\alpha > 0$ *such that* $\| Tx \| \| T^{-1}x \| = \alpha$ *for each unit vector* $x \in \mathscr{H}$*.*

LEMMA 6. *Let* $A, B \in \mathfrak{B}(\mathscr{H})$*. If* $ABA \in \mathbb{C}I \setminus \{0\}$ *and* $BAB \in \mathbb{C}I \setminus \{0\}$*, then* $A^3, B^3 \in \mathbb{C}I \setminus \{0\}$*.*

Proof. $ABA \in \mathbb{C}I \setminus \{0\}$ entails that A is injective and surjective. Thus A is a bijection. Assume $ABA = \lambda I$ and $BAB = \mu I$, where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. Then $B = \lambda(A^{-1})^2$. We get $\lambda^2 (A^{-1})^3 = \mu I$, that is $A^3 = \frac{\lambda^2}{\mu} I$. Similarly we get $B^3 = \frac{\mu^2}{\lambda} I$. $\square$

The following lemma is essential to the proof of Theorem 1.

LEMMA 7. *Suppose* Φ *and* F *satisfy the condition in Theorem* 1*. Then* Φ *preserves zero Jordan semi-triple products on both sides, i.e.* Φ *satisfies* (1)*.*

Proof. For $F : \mathfrak{B}(\mathscr{H}) \to [0, \infty)$ satisfies (P_1) and (P_2), firstly we show $A = 0 \Leftrightarrow \Phi(A) = 0$. Assume $\Phi(A) = 0$. For any $B \in \mathscr{W}$, there is $\dot{B} \in \mathscr{V}$ such that $\Phi(B) = \dot{B}$. Taking $B = x \otimes f$ for any $x, f \in \mathscr{H}$, we get $F((x \otimes f)A(x \otimes f)) = F(\dot{B}\Phi(A)\dot{B}) = 0$. Thus $\langle Ax, f \rangle x \otimes f \in \mathbb{C}I$. This entails $\langle Ax, f \rangle = 0$ for any $x, f \in \mathscr{H}$. So $A = 0$.

Next we show for any $A, B \in \mathscr{W}$, $ABA = 0 \Rightarrow \Phi(A)\Phi(B)\Phi(A) = 0$. If not, there must be non-zero operators $A_0, B_0 \in \mathscr{W}$ such that $A_0B_0A_0 = 0$ and $\Phi(A_0)\Phi(B_0)\Phi(A_0) \in \mathbb{C}I \setminus \{0\}$. Then there is a non-zero complex λ_0 such that $\Phi(A_0)\Phi(B_0)\Phi(A_0) = \lambda_0 I$. Obviously $\Phi(A_0)$ is injective. On the other hand, for any $x \in \mathscr{H}$, we have $\Phi(A_0)(\frac{1}{\lambda_0}\Phi(B_0)\Phi(A_0)x) = x$. Then $\Phi(A_0)$ is also surjective. So both $\Phi(A_0)$ and $\Phi(B_0)$ are invertible.

Now we consider the kernel of A_0. If $\ker A_0 = \{0\}$, then $\dim \mathscr{H} = \infty$ and $A_0B_0A_0 = 0$ implies $B_0A_0 = 0$. The range of A_0 is contained in $\ker B_0$. Then the dimension of $\ker B_0$ is infinite. Choose $x \in \ker B_0$ and consider $F(B_0 x \otimes x B_0)$ and $F(x \otimes x B_0 x \otimes x)$, we get $\Phi(B_0)\Phi(x \otimes x)\Phi(B_0), \Phi(x \otimes x)\Phi(B_0)\Phi(x \otimes x) \in \mathbb{C}I \setminus \{0\}$. With Lemma 6, we get $\Phi(x \otimes x)^3 \in \mathbb{C}I \setminus \{0\}$. However, for F satisfies (P_3), $F((x \otimes x)^3) \neq 0$, which is a contradiction. Thus "$\ker A_0 = \{0\}$" is impossible.

If $\ker A_0 \neq \{0\}$, choose a non-zero element $y \in \ker A_0$. By considering $F(A_0 y \otimes y A_0)$ and $F(y \otimes y A_0 y \otimes y)$, we get $\Phi(y \otimes y)^3 \in \mathbb{C}I \setminus \{0\}$, also impossible. So for all $A,B \in \mathscr{W}$, $ABA = 0$ implies $\Phi(A)\Phi(B)\Phi(A) = 0$. Similarly, the converse is also true. $\square$

Next we give the proof of Theorem 1.

Proof of Theorem 1. The proof will be finished by considering two cases when $\dim \mathscr{H} < \infty$ and $\dim \mathscr{H} = \infty$.

Case I: $\dim \mathscr{H} = n < \infty$.

By Lemma 7 and Lemma 4, there exist a functional $l : \mathscr{W} \to \mathbb{C} \setminus \{0\}$, an invertible matrix $U \in M_n(\mathbb{C})$, and a monomorphism $\eta : \mathbb{C} \to \mathbb{C}$ such that either $\Phi(T) = l(T)UT^{\eta}U^{-1}$ for each $T \in \mathscr{W}$ or $\Phi(T) = l(T)U(T^{\eta})^{\mathrm{tr}}U^{-1}$ for each $T \in \mathscr{W}$. Next there are two steps to check. Here the main idea comes from the proof of [2, Theorem 2.1]. In the first step, we will consider the restriction of Φ on the set of all rank-one matrices. If T is rank-one, then T is unitary similar to T^{tr}, and thus $F(T) = F(T^{\mathrm{tr}})$. So we assume that Φ has the first form; otherwise, replace Φ by $A \mapsto \Phi(A^{\mathrm{tr}})$.

Step 1.1. *U can be chosen as a unitary matrix and $|l(x\otimes f)| \equiv 1$ for any $x, f \in \mathscr{H}$.*

Let $U = V|U|$ be the polar decomposition of U. Using the property (P_1), we may assume $U > 0$. It is known that $\eta(x\otimes f) = \eta(x)\otimes \overline{\eta(\bar{f})}$ for any $x, f \in \mathbb{C}^n$, thus $\Phi(x\otimes f) = l(x\otimes f)U(\eta(x))\otimes U^{-1}(\overline{\eta(\bar{f})})$ for any $x, f \in \mathbb{C}^n$. Denote $l(I) = c_1$. First we show that

$$\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\bar{f})})\| = c_1^{-3}\|x\|\|f\| \tag{2}$$

for each $x, f \in \mathbb{C}^n$.

Noticing that $\langle U(\eta(x)), U^{-1}(\overline{\eta(\bar{f})})\rangle = \langle \eta(x), \overline{\eta(\bar{f})}\rangle = \eta(\langle x, f\rangle)$, with the property (P_3), we have

$$F(\mu U(\eta(x))\otimes U^{-1}(\overline{\eta(\bar{f})})) = F(\nu x\otimes f) \Rightarrow |\mu|\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\bar{f})})\| = |\nu|\|x\|\|f\| \tag{3}$$

for any complex μ, ν and $x, f \in \mathbb{C}^n$.

Now we consider $F(\Phi(I)\Phi(x\otimes f)\Phi(I))$ and get

$$F({c_1}^2 l(x\otimes f)U(\eta(x))\otimes U^{-1}(\overline{\eta(\bar{f})})) = F(x\otimes f)$$

for any $x, f \in \mathbb{C}^n$. Now (3) implies

$$|c_1|^2|l(x\otimes f)|\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\bar{f})})\| = \|x\|\|f\|. \tag{4}$$

By $F(\Phi(x\otimes f)\Phi(I)\Phi(x\otimes f)) = F((x\otimes f)^2)$, we obtain

$$|c_1||l(x\otimes f)|^2\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\bar{f})})\| = \|x\|\|f\|. \tag{5}$$

Comparing (4) and (5) we get

$$|l(x\otimes f)| = |c_1| \tag{6}$$

for any $x, f \in \mathbb{C}^n$. Applying (6) to (4), Equation (2) holds. Particularly, let $f = x$ in (2), it becomes

$$\|U(\eta(x))\|\|U^{-1}(\overline{\eta(\bar{x})})\| = \frac{1}{|c_1|^3}\|x\|^2 \tag{7}$$

for any $x \in \mathbb{C}^n$. Then there is a unitary matrix U_0 such that $U_0 U U_0^* = \mathrm{diag}\{u_1, u_2, \cdots, u_n\}$ with $u_i > 0$ for $i = 1, 2, \cdots, n$. Next we claim $u_1 = u_2 = \cdots = u_n$. If not, say $u_1 \neq u_2$. Substituting $x = (1, 0, \cdots, 0)^t$ in (7), we get $\frac{1}{|c_1|^3} = 1$, thus $|c_1| = 1$. Again substituting $x = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \cdots, 0\right)^t$ in (7), it yields

$$\left(\left(\frac{1}{\sqrt{2}} u_1\right)^2 + \left(\frac{1}{\sqrt{2}} u_2\right)^2\right) \cdot \left(\left(\frac{1}{\sqrt{2}} \frac{1}{u_1}\right)^2 + \left(\frac{1}{\sqrt{2}} \frac{1}{u_2}\right)^2\right) = 1.$$

This leads to $u_1 = u_2$, a contradiction. Hence $U = tI$ for some $t > 0$. Then it follows that

$$\|\eta(x)\| \|\overline{\eta(\bar{f})}\| = \|x\| \|f\| \tag{8}$$

for any $x, f \in \mathbb{C}^n$.

Substituting $x = (1, 0, \cdots, 0)^t$ and $f = (1, \bar{\lambda}, \cdots, 0)^t$ in (8), it implies $1 + |\eta(\lambda)|^2 = 1 + |\lambda|^2$. So $|\eta(\lambda)| = |\lambda|$ for all $\lambda \in \mathbb{C}$. This ensures that η is continuous and either $\eta(\lambda) = \lambda$ or $\eta(\lambda) = \bar{\lambda}$ for all complex λ.

Step 1.2. $|l(T)| = 1$ *for all* $T \in \mathscr{W}$.

For any $T \in \mathscr{W}$ with rank $T \geqslant 2$, there are $x, f \in \mathbb{C}^n$ such that $\langle Tx, f \rangle \neq 0$. Then by considering $F(\Phi(x \otimes f)\Phi(T)\Phi(x \otimes f))$, we get

$$|\langle Tx, f \rangle| = |l(x \otimes f)|^2 |l(T)| \|x\| \|f\|,$$

thus $|l(T)| = 1$.

For any $A \in \mathscr{W}$, when η is the conjugation and $\Phi(A) = l(A) U A^\eta U^*$, we have

$$U^* \Phi(A) U x = l(A) A^\eta x = l(A) \overline{A(\bar{x})} = l(A) JAJx$$

for all $x \in \mathbb{C}^n$. When η is the conjugation and $\Phi(A) = l(A) U (A^\eta)^{\mathrm{tr}} U^*$, we have

$$U^* \Phi(A) U x = l(A) (A^\eta)^{\mathrm{tr}} x = l(A) \overline{A^*(\bar{x})} = l(A) JA^*Jx$$

for all $x \in \mathbb{C}^n$. So $\Phi(A)$ has one of the forms among $l(A)UAU^*$, $l(A)UA^*U^*$, $l(A)UJAJU^*$ or $l(A)UJA^*JU^*$ for all $A \in \mathscr{W}$.

Case II: $\dim \mathscr{H} = \infty$.

By Lemma 3 and Lemma 7, there exist a functional $l : \mathscr{W} \to \mathbb{C} \setminus \{0\}$ and a bounded invertible linear or conjugate linear operator V on $\mathscr{H}$ such that $\Phi(T) = l(T) V T V^{-1}$ for all $T \in \mathscr{W}$ or $\Phi(T) = l(T) V T^* V^{-1}$ for all $T \in \mathscr{W}$.

For an arbitrary orthogonal basis $\{e_i\}_{i \in \Gamma}$ of $\mathscr{H}$ and $x = \sum_{i \in \Gamma} x_i e_i$, we have known $Jx = \sum_{i \in \Gamma} \overline{x_i} e_i$. Let

$$U = \begin{cases} V, & \text{if } V \text{ is linear,} \\ VJ, & \text{if } V \text{ is conjugate linear.} \end{cases}$$

Thus we can write

$$\Phi(x \otimes f) = l(x \otimes f) U (\eta(x) \otimes \overline{\eta(\bar{f})}) U^{-1}$$

for any $x, f \in \mathscr{H}$, where η is the identity or the conjugation. By inspecting the proof in Case I, one can see that U can be chosen as a unitary operator and $|l(T)| = 1$ for all $T \in \mathscr{W}$. Analogously, the form of Φ can be obtained. The proof is finished. $\square$

3. Preservers of the c-numerical range in $\mathfrak{B}(\mathscr{H})$

In this section, we describe the form of surjective maps preserving the c-numerical range of Jordan semi-triple products for operators in $\mathfrak{B}(\mathscr{H})$. We will present the results according to three cases of c: (1) when $c_1+c_k\neq 0$, (2) when $c_j+c_{k+1-j}=0$ for $j=1,\cdots,k$, (3) when there is an integer p $(1<p<k)$ such that $c_p+c_{k+1-p}\neq 0$ and $c_j+c_{k+1-j}=0$ for $j=1,\cdots,p-1$.

First we introduce some notations needed. Let

$$\mathscr{S}=\{S\in\mathfrak{B}(\mathscr{H}):W_c(S)=-W_c(S)\},$$

$$\mathscr{T}'=\{T\in\mathscr{S}:TAT\in\mathscr{S}\text{ and }ATA\in\mathscr{S}\ \text{ for all }\ A\in\mathfrak{B}(\mathscr{H})\},$$

$$\mathscr{S}^*=\{S\in\mathfrak{B}^s(\mathscr{H}):W_c(S)=-W_c(S)\},$$

and

$$\mathscr{T}'^*=\{T\in\mathscr{S}^*:TAT\in\mathscr{S}^*\text{ and }ATA\in\mathscr{S}^*\ \text{ for all }\ A\in\mathfrak{B}^s(\mathscr{H})\}.$$

Obviously, $\mathscr{S}^*$ (or $\mathscr{T}'^*$) is the set of all self-adjoint operators in $\mathscr{S}$ (or $\mathscr{T}'$). When the c satisfies $c_j+c_{k+1-j}=0$ for $j=1,2,\cdots,k$, we know $W_c(A)=-W_c(A)$ for all $A\in\mathfrak{B}(\mathscr{H})$. Then $\mathscr{S}=\mathscr{T}'=\mathfrak{B}(\mathscr{H})$ and $\mathscr{S}^*=\mathscr{T}'^*=\mathfrak{B}^s(\mathscr{H})$. When the c satisfies $c_1+c_k\neq 0$, each rank-one operator does not belong to $\mathscr{S}$ and $\mathscr{T}'=\{0\}$. If there is $1<p<k$ such that $c_p+c_{k+1-p}\neq 0$ and $c_j+c_{k+1-j}=0$ for $j=1,\cdots,p-1$, from Lemma 2, it is easy to verify that $\mathscr{F}_1(\mathscr{H})\in\mathscr{T}'$, but difficult to describe all the elements in $\mathscr{T}'$ and $\mathscr{S}$. For more information about $\mathscr{S}$ and $\mathscr{S}^*$, one can refer to [13, Proposition 2.6].

Next we give the main result in this section.

THEOREM 3. *Let $k\geqslant 3$ and not all c_i in $c\in\mathbb{R}^k\setminus\{0\}$ be equal. The surjective map $\Phi:\mathfrak{B}(\mathscr{H})\to\mathfrak{B}(\mathscr{H})$ satisfies*

$$W_c(\Phi(A)\Phi(B)\Phi(A))=W_c(ABA)\qquad (A,B\in\mathfrak{B}(\mathscr{H}))\tag{3.0}$$

if and only if there exist a unitary operator $U\in\mathfrak{B}(\mathscr{H})$ and $\omega\in\mathbb{C}$ with $\omega^3=1$ such that the followings hold:

1. *If $c_1+c_k\neq 0$, then $\Phi(A)=\omega UA^{\ddagger}U^*$ for all $A\in\mathfrak{B}(\mathscr{H})$.*
2. *If $c_j+c_{k+1-j}=0$ for $j=1,\cdots,k$, then there exists a functional $\varepsilon:\mathfrak{B}(\mathscr{H})\to\{-1,1\}$ such that $\Phi(A)=\omega\varepsilon(A)UA^{\ddagger}U^*$ for all $A\in\mathfrak{B}(\mathscr{H})$.*
3. *If there is $1<p<k$ such that $c_p+c_{k+1-p}\neq 0$ and $c_j+c_{k+1-j}=0$ for $j=1,\cdots,p-1$, then there are a functional $\phi:\mathscr{T}'\to\{-1,1\}$, a constant functional $\bar{\varphi}:\mathfrak{B}(\mathscr{H})\setminus\mathscr{S}\to\{-1,1\}$ and a functional $\psi:\mathscr{S}\setminus\mathscr{T}'\to\{-1,1\}$ satisfying $\psi(A_1)^2\psi(A_2)\in\operatorname{sgn}(A_1A_2A_1)$, $\psi(A_1)^2\bar{\varphi}(B)\in\operatorname{sgn}(A_1BA_2)$ and $\bar{\varphi}(B)^2\psi(A_1)\in\operatorname{sgn}(BA_1B)$ for any $A_1,A_2\in\mathscr{S}\setminus\mathscr{T}'$ and $B\in\mathfrak{B}(\mathscr{H})\setminus\mathscr{S}$, where*

$$\operatorname{sgn}(X)=\begin{cases}\{1\}, & \text{if } X\notin\mathscr{S},\\ \{-1,1\}, & \text{if } X\in\mathscr{S},\end{cases}$$

such that

$$\Phi(A) = \begin{cases} \omega\phi(A)UA^{\ddagger}U^*, & \text{if } A \in \mathscr{T}', \\ \omega\psi(A)UA^{\ddagger}U^*, & \text{if } A \in \mathscr{S} \setminus \mathscr{T}', \\ \omega\bar{\varphi}(A)UA^{\ddagger}U^*, & \text{if } A \in \mathfrak{B}(\mathscr{H}) \setminus \mathscr{S}, \end{cases}$$

for all $A \in \mathfrak{B}(\mathscr{H})$.

Here $A^{\ddagger} = A$ *for all* $A \in \mathfrak{B}(\mathscr{H})$ *or* $A^{\ddagger} = JA^*J$ *for all* $A \in \mathfrak{B}(\mathscr{H})$.

The lemma below is needed for the proof of Theorem 3.

LEMMA 8. [13, Proposition 2.5] *Let* $T, S \in \mathfrak{B}(\mathscr{H})$ *be rank-one operators. Suppose* $W_c(T) = W_c(S)$, *then one of the followings holds:*

1. *If* $c_1 + c_k \neq 0$, *then* $\mathrm{tr}(T) = \mathrm{tr}(S)$;
2. *If* $c_1 + c_k = 0$, *then* $\mathrm{tr}(T) = \mathrm{tr}(S)$ *or* $\mathrm{tr}(T) = -\mathrm{tr}(S)$.

Proof of Theorem 3. Since the surjection Φ satisfies (3.0), the equality (2.0) holds true. Then there exist a unitary operator U on $\mathscr{H}$ and a functional $l : \mathfrak{B}(\mathscr{H}) \to \mathbb{T}$ such that $\Phi(A) = l(A)UA^{\ddagger}U$ for all $A \in \mathfrak{B}(\mathscr{H})$, where $A^{\ddagger}$ is one of the forms among A, A^*, JAJ and JA^*J.

Now considering $x \otimes f$ with $\langle x, f\rangle \neq 0$, we have

$$W_c((x\otimes f)^3) = W_c(((x\otimes f)^{\ddagger})^3). \tag{9}$$

If $(x\otimes f)^{\ddagger}$ is of the form $Jx \otimes Jf$, (9) becomes

$$\langle x, f\rangle^2 W_c(x\otimes f) = l(x\otimes f)^3 \overline{\langle x, f\rangle}^2 W_c(Jx\otimes Jf). \tag{10}$$

From Lemma 2 and 8, we get $\langle x, f\rangle = l(x\otimes f)\overline{\langle x, f\rangle}$. This can not hold when $\langle x, f\rangle$ is complex with non-zero real and imaginary parts. Then $(x\otimes f)^{\ddagger} = Jx\otimes Jf$ is impossible. Also, $(x\otimes f)^{\ddagger} = f\otimes x$ is impossible. Then for any $A \in \mathfrak{B}(\mathscr{H})$, either $\Phi(A) = l(A)UAU^*$ for all $A \in \mathfrak{B}(\mathscr{H})$ or $\Phi(A) = l(A)UJA^*JU^*$ for all $A \in \mathfrak{B}(\mathscr{H})$ holds. Next the proof will be finished according to three cases of c.

Case I: when $c_1 + c_k \neq 0$.

In this case, firstly we show Φ is linear. Obviously, Φ preserves rank-one operators. For Φ satisfies (3.0), we get $\mathrm{tr}(\Phi(T)\Phi(A)\Phi(T)) = \mathrm{tr}(TAT)$ for all $A \in \mathfrak{B}(\mathscr{H})$ and all rank-one $T \in \mathfrak{B}(\mathscr{H})$. Thus for $A, B \in \mathfrak{B}(\mathscr{H})$ and rank-one $T \in \mathfrak{B}(\mathscr{H})$, we have

$$\begin{aligned}\mathrm{tr}(TAT) + \mathrm{tr}(TBT) &= \mathrm{tr}(\Phi(T)\Phi(A)\Phi(T)) + \mathrm{tr}(\Phi(T)\Phi(B)\Phi(T)) \\ &= \mathrm{tr}(\Phi(T)\Phi(A+B)\Phi(T)).\end{aligned}$$

Φ can run over all rank-one operators i.e. $\Phi(T)$ can be chosen as $x\otimes f$ for any $x, f \in \mathscr{H}$, so Φ is additive. Similarly, Φ is homogenous. In fact, the linearity of Φ implies that l is a single value functional. For any linear independent $A, B \in \mathfrak{B}(\mathscr{H})$, we have

$$l(A)UA^{\ddagger}U^* + l(B)UB^{\ddagger}U^* = l(A+B)U(A+B)^{\ddagger}U^*.$$

Then $(l(A+B)-l(A))A^{\ddagger}+(l(A+B)-l(B))B^{\ddagger}=0$. That means $l(A+B)=l(A)=l(B)$ for any linear independent $A,B\in\mathfrak{B}(\mathscr{H})$. For any complex t, it can be obtained $l(tA)=l(A)$ similarly. Thus $l:\mathfrak{B}(\mathscr{H})\to\mathbb{T}$ is a constant functional, writing as ω. Again, Equation (9) entails $\langle x,f\rangle=\omega^3\langle x,f\rangle$, then $\omega^3=1$.

Case II: when $c_j+c_{k+1-j}=0$ for $j=1,\cdots,k$.

Let $\omega_0=\frac{1+\sqrt{3}i}{2}$ and denote $\Omega_i=\{{\omega_0}^i,-{\omega_0}^i\}$ for $i=0,1,2$. For $x\otimes f$ with $\langle x,f\rangle\neq 0$, by considering the c-numerical range of $(x\otimes f)^3$, we have $l(x\otimes f)^3\in\{-1,1\}$. Then $l(x\otimes f)$ must be in some Ω_i ($i=0$, 1 or 2). For $x\otimes f$ with $\langle x,f\rangle=0$, by considering $W_c((x+f)\otimes(x+f)(x\otimes f)(x+f)\otimes(x+f))$, we get $l(x\otimes f)l((x+f)\otimes(x+f))^2\in\{-1,1\}$. This entails that $l(x\otimes f)$ and $l((x+f)\otimes(x+f))$ are in the same Ω_i for some i. In fact, among Ω_i for $i=1,2,3$, $l(x\otimes f)$ belongs to only one of them for all $x,f\in\mathscr{H}$. If not, assume that there are $x\otimes f$ and $y\otimes g$ satisfying $l(x\otimes f)\in\Omega_1$ and $l(y\otimes g)\in\Omega_2$. Then $W_c(x\otimes f)=W_c(\Phi(I)\Phi(x\otimes f)\Phi(I))$ shows $l(I)^2l(x\otimes f)\in\{-1,1\}$, similarly we can get $l(I)^2l(y\otimes g)\in\{-1,1\}$. This means $l(I)^2\in\Omega_1\cap\Omega_2=\emptyset$, a nonsense.

For any $A\in\mathfrak{B}(\mathscr{H})$ with rank $A\geqslant 2$, there must be $x,f\in\mathscr{H}$ such that $\langle Ax,f\rangle\neq 0$. Then $l(x\otimes f)^2l(A)\in\{-1,1\}$. So $l(A)$ belongs to the same Ω_i for all $A\in\mathfrak{B}(\mathscr{H})$. It means that there is a functional $\varepsilon:\mathfrak{B}(\mathscr{H})\to\{-1,1\}$ such that $\Phi(A)=\omega\varepsilon(A)UAU^*$ for all $A\in\mathfrak{B}(\mathscr{H})$ or $\Phi(A)=\omega\varepsilon(A)UJA^*JU^*$ for all $A\in\mathfrak{B}(\mathscr{H})$.

Case III: when there is $1<p<k$ such that $c_p+c_{k+1-p}\neq 0$ and $c_j+c_{k+1-j}=0$ for $j=1,\cdots,p-1$.

In the similar way as in Case II , we get that Φ has one of following forms:

$$\Phi(A)=\omega l(A)UAU^*\quad (A\in\mathfrak{B}(\mathscr{H})),$$

or

$$\Phi(A)=\omega l(A)UJA^*JU^*\quad (A\in\mathfrak{B}(\mathscr{H})),$$

where $U\in\mathfrak{B}(\mathscr{H})$ is unitary and $l:\mathfrak{B}(\mathscr{H})\to\{-1,1\}$.

For $A\in\mathfrak{B}(\mathscr{H})\setminus\mathscr{S}$, $W_c(A)=W_c(\Phi(I)\Phi(A)\Phi(I))$ shows $l(I)^2l(A)\in\{-1,1\}$. Then $l(A)\equiv l(I)$ for all $A\in\mathfrak{B}(\mathscr{H})\setminus\mathscr{S}$.

For A_1 and $A_2\in\mathscr{S}\setminus\mathscr{T}'$, the equality $W_c(A_1A_2A_1)=W_c(\Phi(A_1)\Phi(A_2)\Phi(A_1))$ shows the followings hold:

(1) If $A_1A_2A_1\in\mathscr{S}$, then $l(A_1)^2l(A_2)\in\{-1,1\}$;

(2) If $A_1A_2A_1\notin\mathscr{S}$, then $l(A_1)^2l(A_2)\in\{1\}$.

Then $l(A_1)^2l(A_2)\in\operatorname{sgn}(A_1A_2A_1)$ for all $A_1,A_2\in\mathscr{S}\setminus\mathscr{T}'$. Also for $A\in\mathscr{S}\setminus\mathscr{T}'$ and $B\in\mathfrak{B}(\mathscr{H})\setminus\mathscr{S}$, we have $l(A)^2l(B)\in\operatorname{sgn}(ABA)$ and $l(B)^2l(A)\in\operatorname{sgn}(BAB)$. Let $\bar{\varphi}$, ψ and ϕ be the restrictions of the functional l on $\mathfrak{B}(\mathscr{H})\setminus\mathscr{S}$, $\mathscr{S}\setminus\mathscr{T}'$ and $\mathscr{T}'$, respectively. The proof is finished. $\square$

4. Preservers in $\mathfrak{B}^s(\mathscr{H})$

In this section, we will characterize the surjective maps preserving the c-numerical radius of Jordan semi-triple product for operators in $\mathfrak{B}^s(\mathscr{H})$. Further, the form of maps on $\mathfrak{B}^s(\mathscr{H})$ preserving the c-numerical range will be shown according to different cases of c. Firstly, we give the following general result.

THEOREM 4. *Let $\mathscr{H}$ be a complex Hilbert space with $\dim \mathscr{H} \geqslant 3$ and $\mathscr{W}$, $\mathscr{V}$ be subsets of $\mathfrak{B}^s(\mathscr{H})$ containing all rank-one self-adjoint operators. And let $F : \mathfrak{B}(\mathscr{H}) \to [0,\infty)$ be a functional satisfying (P_1)-(P_3). Suppose $\Phi : \mathscr{W} \to \mathscr{V}$ is a surjective map satisfying*

$$F(\Phi(A)\Phi(B)\Phi(A)) = F(ABA)$$

for all $A,B \in \mathscr{W}$. Then there exist a unitary operator U on $\mathscr{H}$ and a functional $h : \mathscr{W} \to \{-1,1\}$ such that $\Phi(A) = h(A)UAU^*$ for all $A \in \mathscr{W}$, or $\Phi(A) = h(A)UJAJU^*$ for all $A \in \mathscr{W}$.

The following lemma comes from [8, Lemma 2.4].

LEMMA 9. *Let $A,B \in \mathfrak{B}^s(\mathscr{H})$. If $|\langle Ax,x\rangle| = |\langle Bx,x\rangle|$ for every $x \in \mathscr{H}$, then $A = \pm B$.*

Proof of Theorem 4. Similar as in the proof of Lemma 7, one can prove the following facts about Φ:

(1) For $A \in \mathscr{W}$, $A = 0$ if and only if $\Phi(A) = 0$.

(2) For $A,B \in \mathscr{W}$, $ABA = 0$ if and only if $\Phi(A)\Phi(B)\Phi(A) = 0$.

Then by [7, Theorems 3.1 and 3.3], there are a unitary operator U on $\mathscr{H}$ and a functional $g : \mathscr{W} \to \mathbb{R} \setminus \{0\}$ such that $\Phi(T) = g(T)UTU^*$ for all rank-one operators $T \in \mathscr{W}$ or $\Phi(T) = g(T)U\overline{T}U^*$ for all rank-one operators $T \in \mathscr{W}$. By considering $F(T^2) = F(\Phi(T)^2)$, we know $\|T\|F(T) = g(T)^2\|T\|F(UTU^*)$. So $g(T) \in \{-1,1\}$ for all rank-one $T \in \mathscr{W}$. Here let $g(A) = 1$ if A is not rank-one.

For any $A \in \mathscr{W}$, denote $\Psi(A) = U^*\Phi(A)U$ if $\Phi(T) = g(T)UTU^*$ for all rank-one operators T and $\Psi(A) = JU^*\Phi(A)UJ$ if $\Phi(T) = g(T)U\overline{T}U^*$ for all rank-one operators $T \in \mathscr{W}$. Obviously, $\Psi(x \otimes x) = g(x \otimes x)x \otimes x$ for any $x \in \mathscr{H}$. For $A \in \mathscr{W}$ with rank $A \geqslant 2$, $F(x \otimes xAx \otimes x) = F(\Psi(x \otimes x)\Psi(A)\Psi(x \otimes x)) = F(x \otimes x\Psi(A)x \otimes x)$ holds for each $x \in \mathscr{H}$. From Lemma 9, $\Psi(A) = \pm A$. Then there is a functional $h : \mathscr{W} \to \{-1,1\}$ with $h(A) = g(A)$ when rank $A = 1$ such that $\Psi(A) = h(A)A$ for all $A \in \mathscr{W}$. And $h(A) = g(A)$ when rank $A = 1$. Then Φ has the required form. $\square$

As an application, next we give the result about maps on $\mathfrak{B}^s(\mathscr{H})$ preserving the c-numerical radius of Jordan semi-triple products.

THEOREM 5. *Let $k \geqslant 3$ and not all c_i in $c \in \mathbb{R}^k \setminus \{0\}$ be equal and let $\mathscr{W}, \mathscr{V} \subseteq \mathfrak{B}^s(\mathscr{H})$ be subsets containing all rank-one self-adjoint operators. The surjective map $\Phi : \mathscr{W} \to \mathscr{V}$ satisfies*

$$r_c(ABA) = r_c(\Phi(A)\Phi(B)\Phi(A))$$

for any $A,B \in \mathscr{W}$, if and only if there are a unitary operator U on $\mathscr{H}$ and a functional $g : \mathscr{W} \to \{-1,1\}$ such that $\Phi(A) = g(A)UAU^*$ for all $A \in \mathscr{W}$ or $\Phi(A) = g(A)UJAJU^*$ for all $A \in \mathscr{W}$.

Proof. When $\sum_{j=1}^{k} c_j \neq 0$, the result can be obtained from [8, Theorem 2.3]. When $\sum_{j=1}^{k} c_j = 0$, the result is obtained from Theorem 4. $\square$

The result about maps on $\mathfrak{B}^s(\mathscr{H})$ preserving c-numerical range will be shown below. The proof is omitted as it is similar to those in Section 3.

THEOREM 6. *Let $k \geqslant 3$ and not all c_i in $c \in \mathbb{R}^k \setminus \{0\}$ be equal. The surjective map $\Phi : \mathfrak{B}^s(\mathscr{H}) \to \mathfrak{B}^s(\mathscr{H})$ satisfies*

$$W_c(\Phi(A)\Phi(B)\Phi(A)) = W_c(ABA) \qquad (A, B \in \mathfrak{B}^s(\mathscr{H})),$$

if and only if there is a unitary operator $U \in \mathfrak{B}(\mathscr{H})$ such that the followings hold:

1. *If $c_1 + c_k \neq 0$, then $\Phi(A) = \pm UA^{\ddagger}U^*$ for all $A \in \mathfrak{B}^s(\mathscr{H})$.*
2. *If $c_j + c_{k+1-j} = 0$ for $j = 1, \cdots, k$, then there exists a functional $\varepsilon : \mathfrak{B}^s(\mathscr{H}) \to \{-1, 1\}$ such that $\Phi(A) = \varepsilon(A)UA^{\ddagger}U^*$ for all $A \in \mathfrak{B}^s(\mathscr{H})$.*
3. *If there is $1 < p < k$ such that $c_p + c_{k+1-p} \neq 0$ and $c_j + c_{k+1-j} = 0$ for $j = 1, \cdots, p-1$, then there are a functional $\phi : \mathscr{T}'^* \to \{-1, 1\}$, a constant functional $\bar{\varphi} : \mathfrak{B}(\mathscr{H}) \setminus \mathscr{S}^* \to \{-1, 1\}$ and a functional $\psi : \mathscr{S}^* \setminus \mathscr{T}'^* \to \{-1, 1\}$ satisfying $\psi(A_1)^2\psi(A_2) \in \operatorname{sgn}(A_1A_2A_1)$, $\psi(A_1)^2\bar{\varphi}(B) \in \operatorname{sgn}(A_1BA_2)$ and $\bar{\varphi}(B)^2\psi(A_1) \in \operatorname{sgn}(BA_1B)$ for any $A_1, A_2 \in \mathscr{S}^* \setminus \mathscr{T}'^*$ and $B \in \mathfrak{B}(\mathscr{H}) \setminus \mathscr{S}^*$, where*

$$\operatorname{sgn}(X) = \begin{cases} \{1\}, & \text{if } X \notin \mathscr{S}^*, \\ \{-1, 1\}, & \text{if } X \in \mathscr{S}^*, \end{cases}$$

such that

$$\Phi(A) = \begin{cases} \phi(A)UA^{\ddagger}U^*, & \text{if } A \in \mathscr{T}'^*, \\ \psi(A)UA^{\ddagger}U^*, & \text{if } A \in \mathscr{S}^* \setminus \mathscr{T}'^*, \\ \bar{\varphi}(A)UA^{\ddagger}U^*, & \text{if } A \in \mathfrak{B}^s(\mathscr{H}) \setminus \mathscr{S}^*, \end{cases}$$

for all $A \in \mathfrak{B}^s(\mathscr{H})$.

Here $A^{\ddagger} = A$ for all $A \in \mathfrak{B}^s(\mathscr{H})$ or $A^{\ddagger} = JA^*J$ for all $A \in \mathfrak{B}^s(\mathscr{H})$.

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