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CYLINDRICAL HARDY TYPE INEQUALITIES WITH BESSEL PAIRS

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Abstract. Using factorizations of suitable partial differential operators and the notion of Bessel pairs, we establish several cylindrical Hardy's type identities and inequaltities in the sense of Badiale-Tarantello [2].

1. Introduction

In this paper, we concern the celebrated Hardy inequality in $\mathbb{R}^N$, $N \geqslant 3$: for all $u \in C_0^\infty\left(\mathbb{R}^N\right)$:

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx. \tag{1.1}$$

Here $\left(\frac{N-2}{2}\right)^2$ is the best possible constant.

It is well-known that $\left(\frac{N-2}{2}\right)^2$ in (1.1) is never achieved by nontrivial functions. Therefore, many efforts have been devoted to enhance the Hardy inequalities. One way to do so is to add extra nonnegative terms to the RHS of (1.1). On the whole space $\mathbb{R}^N$, Ghoussoub and Moradifam showed in [22] that there is no strictly positive $W \in C^1(0,\infty)$ such that the inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx + \int_{\mathbb{R}^N} W(|x|)\,|u|^2 \, dx$$

holds for all $u \in C_0^\infty\left(\mathbb{R}^N\right)$. However, the situation on bounded domain is different. Indeed, let Ω be a bounded domain in $\mathbb{R}^N$, $N \geqslant 3$, with $0 \in \Omega$. Then Brezis and Vázquez proved in [7] that for all $u \in W_0^{1,2}(\Omega)$:

$$\int_{\Omega} |\nabla u|^2 \, dx \geqslant \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{|u|^2}{|x|^2} dx + z_0^2 \omega_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |u|^2 \, dx \tag{1.2}$$

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where ω_N is the volume of the unit ball and $z_0 = 2.4048\ldots$ is the first zero of the Bessel function $J_0(z)$. The constant $z_0^2\omega_N^{\frac{2}{N}}|\Omega|^{-\frac{2}{N}}$ is optimal when Ω is a ball. However, $z_0^2\omega_N^{\frac{2}{N}}|\Omega|^{-\frac{2}{N}}$ is not attained in $W_0^{1,2}(\Omega)$. Hence, it is conjectured by Brezis and Vázquez that $z_0^2\omega_N^{\frac{2}{N}}|\Omega|^{-\frac{2}{N}}\int\limits_{\Omega}|u|^2\,dx$ is just a first term of an infinite series of extra terms that can be added to the RHS of (1.2). This question was addressed by many authors. We refer the interested reader to [1, 4, 5, 14, 15, 18, 23, 32, 40, 41], to name just a few. See also the monographs [3, 21, 25, 26, 34], for instance, that are excellent references on the topic.

In [20, 21], Ghoussoub and Moradifam proved the following result to improve, extend and unify several results about the Hardy type inequalities:

THEOREM A. *Let $0 < R \leqslant \infty$, V and W be positive C^1-functions on $(0,R)$ such that $\int\limits_0^R \frac{1}{r^{N-1}V(r)}dr = \infty$ and $\int\limits_0^R r^{N-1}V(r)\,dr < \infty$. Then the following are equivalent:*

(1) (V,W) is a N-dimensional Bessel pair on $(0,R)$.

(2) $\int\limits_{B_R} V(|x|)|\nabla u|^2\,dx \geqslant \beta(V,W;R)\int\limits_{B_R} W(|x|)|u|^2\,dx$ for all $u \in C_0^\infty(B_R)$ with $\beta(V,W;R)$ being the best constant.

Here we say that a couple of C^1-functions (V,W) is a N-dimensional Bessel pair on $(0,R)$ if there exists $c > 0$ such that the ordinary differential equation

$$y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)}\right)y'(r) + \frac{cW(r)}{V(r)}y(r) = 0$$

has a positive solution on the interval $(0,R)$. Also, $\beta(V,W;R)$ is defined as the supremum of such c. It can be also verified that (V,W) is a N-dimensional Bessel pair on $(0,R)$ if and only if $\left(r^{N-1}V, r^{N-1}W\right)$ is a 1-dimensional Bessel pair on $(0,R)$. See the book [21] for more properties and examples about the N-dimensional Bessel pair.

Recently, the Hardy type inequalities and other functional and geometric inequalities have been improved by replacing the usual ∇ by $\frac{x}{|x|}\cdot\nabla$. It can be noted that $\frac{x}{|x|}\cdot\nabla u$ is the radial gradient of u. Indeed, in the polar coordinate, $\left|\frac{x}{|x|}\cdot\nabla u\right| = |\partial_r u(r\sigma)|$ while $|\nabla u| = \left(|\partial_r u(r\sigma)|^2 + \frac{|\nabla_{\mathbb{S}^{N-1}}u(r\sigma)|^2}{r^2}\right)^{\frac{1}{2}}$. Actually, the operator $\mathscr{R} = \frac{x}{|x|}\cdot\nabla$ play important roles in the literature. The interested reader is referred to [36] for the applications of the radial derivation $\mathscr{R}$ in the literature. We also mention here that the Hardy type inequalities with radial gradient have been intensively studied recently. See [9, 10, 11, 12, 24, 27, 28, 31, 33, 35, 36, 38], for example.

In an effort to unify many results about the Hardy type inequalities with radial derivation, and to compute the exact remainders of the Hardy type inequalities, the authors in [13] have proved the following result:

THEOREM B. *Let* $0 < R \leqslant \infty$, V *and* W *be positive* C^1*-functions on* $(0,R)$. *Assume that* $\left(r^{N-1}V, r^{N-1}W\right)$ *is a Bessel pair on* $(0,R)$. *Then for all* $u \in C_0^\infty(B_R \setminus \{0\})$*:*

$$\int_{B_R} V(|x|)\,|\mathscr{R}u|^2\,dx - \int_{B_R} W(|x|)\,|u|^2\,dx$$
$$= \int_{B_R} V(|x|)\left|\mathscr{R}\left(\frac{u}{\varphi_{r^{N-1}V,r^{N-1}W;R}}\right)\right|^2 \varphi^2_{r^{N-1}V,r^{N-1}W;R}dx$$

and

$$\int_{B_R} V(|x|)\,|\nabla u|^2\,dx - \int_{B_R} W(|x|)\,|u|^2\,dx$$
$$= \int_{B_R} V(|x|)\left|\nabla\left(\frac{u}{\varphi_{r^{N-1}V,r^{N-1}W;R}}\right)\right|^2 \varphi^2_{r^{N-1}V,r^{N-1}W;R}dx$$

where $\varphi_{r^{N-1}V,r^{N-1}W;R}$ *is the positive solution of*

$$y''(r) + \left(\frac{N-1}{r} + \frac{V_r(r)}{V(r)}\right) y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

on the interval $(0,R)$.

Here (V,W) is a Bessel pair on $(0,R)$ if the ordinary differential equation

$$y''(r) + \frac{V_r(r)}{V(r)} y'(r) + \frac{W(r)}{V(r)} y(r) = 0$$

(equivalently, $(Vy')' + Wy = 0$) has a positive solution on the interval $(0,R)$. Bessel pair can be considered as *normalized* 1-dimensional Bessel pair.

It is worth mentioning that the results in Theorem B also hold for $u \in C_0^\infty(B_R)$ if we impose extra assumptions on the pair (V,W) such as $\int_0^R \frac{1}{r^{N-1}V(r)}dr = \infty$ and $\int_0^R r^{N-1}V(r)\,dr < \infty$.

The method has been used in [13] is the factorizations of suitable differential operators. We note here that factorizations of singular partial differential operators has been applied in [19] to give a simple approach to the classical Hardy inequality and in [17] for the radial and logarithmic refinements of the Hardy inequality. Recently, in [19], the factorization method was used to obtain Hardy, Hardy-Rellich and refined Hardy inequalities on general stratified groups and weighted Hardy inequalities on general homogeneous groups in [37]. More recently, several Hardy's type identities on half-spaces were established in [29, 30] using factorizations. For a thorough review, the history and properties of the factorization method, we refer the interested reader to [16].

In [2], Badiale and Tarantello studied the existence and nonexistence of cylindrical solutions for the following nonlinear elliptic equation in $\mathbb{R}^3$:

$$\begin{cases} -\Delta u = \phi\left(\sqrt{x_1^2+x_2^2}\right)|u|^{p-2}u & \text{in } \mathbb{R}^3 \\ u(x) > 0 & \text{in } \mathbb{R}^3 \\ \int\limits_{\mathbb{R}^3} \phi\left(\sqrt{x_1^2+x_2^2}\right)|u|^{p-1}dx < \infty \end{cases}$$

with $p > 1$. This equation has been proposed by Bertin and Ciotti as a model describing the dynamics of elliptic galaxies. See [6, 8]. Badiale and Tarantello then investigated the following cylindrical Hardy type inequalities: for $1 < p < k \leqslant N$ and $u \in C_0^\infty\left(\mathbb{R}^N\right)$

$$\int\limits_{\mathbb{R}^N} |\nabla u(x)|^p dx \geqslant C_{N,k,p} \int\limits_{\mathbb{R}^N} \frac{|u(x)|^p}{|y|^p} dx \tag{1.3}$$

where $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. The optimal constant $C_{N,k,p} = \left(\frac{k-p}{p}\right)^p$ was also conjectured in [2] and then verified in [39].

Recently, in [28], the following result about the cylindrical Hardy type inequalities with Bessel pairs has been set up:

THEOREM C. *Let V and W be positive C^1-functions on $(0,R)$ such that* $\int\limits_0^\infty \frac{1}{r^{k-1}V(r)}dr = \infty$ *and* $\int\limits_0^\infty r^{k-1}V(r)\,dr < \infty$. *Then the following are equivalent:*

(1) $\left(r^{k-1}V, r^{k-1}W\right)$ *is a 1-dimensional Bessel pair on* $(0,\infty)$.

(2) $\int\limits_{\mathbb{R}^{N-k}}\int\limits_{\mathbb{R}^k} V(|y|)\left|\frac{y}{|y|}\cdot\nabla_y u(x)\right|^2 dydz \geqslant c \int\limits_{\mathbb{R}^{N-k}}\int\limits_{\mathbb{R}^k} W(|y|)|u(x)|^2 dydz$ *for all* $u \in C_0^\infty\left(\mathbb{R}^N\right)$ *for some* $c > 0$.

(3) $\int\limits_{\mathbb{R}^{N-k}}\int\limits_{\mathbb{R}^k} V(|y|)|\nabla_y u(x)|^2 dydz \geqslant c \int\limits_{\mathbb{R}^{N-k}}\int\limits_{\mathbb{R}^k} W(|y|)|u(x)|^2 dydz$ *for all* $u \in C_0^\infty\left(\mathbb{R}^N\right)$ *for some* $c > 0$.

(4) $\int\limits_{\mathbb{R}^{N-k}}\int\limits_{\mathbb{R}^k} V(|y|)|\nabla u(x)|^2 dydz \geqslant c \int\limits_{\mathbb{R}^{N-k}}\int\limits_{\mathbb{R}^k} W(|y|)|u(x)|^2 dydz$ *for all* $u \in C_0^\infty\left(\mathbb{R}^N\right)$ *for some* $c > 0$.

Moreover, $\beta\left(r^{k-1}V, r^{k-1}W;\infty\right)$ *is the optimal constant.*

Motivated by the cylindrical Hardy type inequalities studied in [2, 28], and the method and results in [13], our principal goal of this paper is to use the factorization method to investigate the cylindrical Hardy type inequalities with Bessel pairs and with exact remainder terms. Let $x=(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$, $1\leqslant k\leqslant N$. Our main result can be read as follows:

THEOREM 1.1. *Let $0<R\leqslant\infty$, V and W be positive C^1-functions on $(0,R)$. Assume that $\left(r^{k-1}V,r^{k-1}W\right)$ is a Bessel pair on $(0,R)$. Then for $u\in C_0^\infty(\{0<|y|<R\})$:*

$$\int\limits_{0<|y|<R} V(|y|)\,|\nabla u(x)|^2\,dydz-\int\limits_{0<|y|<R} W(|y|)\,|u(x)|^2\,dydz$$
$$=\int\limits_{0<|y|<R} V(|y|)\,\varphi^2(|y|)\left|\nabla\left(\frac{u(x)}{\varphi(|y|)}\right)\right|^2 dydz.$$

and

$$\int\limits_{0<|y|<R} V(|y|)\left|\frac{y}{|y|}\cdot\nabla_y u(x)\right|^2 dydz-\int\limits_{0<|y|<R} W(|y|)\,|u(x)|^2\,dydz$$
$$=\int\limits_{0<|y|<R} V(|y|)\,\varphi^2(|y|)\left|\frac{y}{|y|}\cdot\nabla_y\left(\frac{u(x)}{\varphi(|y|)}\right)\right|^2 dydz.$$

Here φ is the positive solution of

$$\left(r^{k-1}V(r)\,y'(r)\right)'+r^{k-1}W(r)\,y(r)=0$$

on the interval $(0,R)$.

We will list here a few applications of our result. First, since $\left(r^{k-1},\frac{(k-2)^2}{4}r^{k-3}\right)$ is a Bessel pair on $(0,\infty)$ with $\varphi=r^{-\frac{k-2}{2}}$, we deduce from Theorem 1.1 that

COROLLARY 1.1. *For $u\in C_0^\infty\left(\mathbb{R}^N\setminus\{0\}\right)$:*

$$\int\limits_{\mathbb{R}^N}|\nabla u|^2\,dx-\left(\frac{k-2}{2}\right)^2\int\limits_{\mathbb{R}^N}\frac{|u|^2}{|y|^2}dx$$
$$=\int\limits_{\mathbb{R}^N}\frac{1}{|y|^{k-2}}\left|\nabla\left(|y|^{\frac{k-2}{2}}u\right)\right|^2 dx$$

and

$$\int\limits_{\mathbb{R}^N}\left|\frac{y}{|y|}\cdot\nabla_y u\right|^2 dx-\left(\frac{k-2}{2}\right)^2\int\limits_{\mathbb{R}^N}\frac{|u|^2}{|y|^2}dx$$
$$=\int\limits_{\mathbb{R}^N}\frac{1}{|y|^{k-2}}\left|\frac{y}{|y|}\cdot\nabla_y\left(|y|^{\frac{k-2}{2}}u\right)\right|^2 dx.$$

Obviously, our Corollary 1.1 provides a direct understanding and precise information on the cylindrical Hardy inequality to (1.3).

More generally, since $\left(r^{k-1-\alpha}, \frac{(k-2-\alpha)^2}{4} r^{k-3-\alpha}\right)$ is a Bessel pair on $(0,\infty)$ with $\varphi = r^{-\frac{k-2-\alpha}{2}}$, we obtain

COROLLARY 1.2. *For* $u \in C_0^\infty\left(\mathbb{R}^N \setminus \{0\}\right)$:

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|y|^\alpha} dx - \left(\frac{k-2-\alpha}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|y|^{2+\alpha}} dx = \int_{\mathbb{R}^N} \frac{1}{|y|^{k-2}} \left|\nabla\left(|y|^{\frac{k-2-\alpha}{2}} u\right)\right|^2 dx$$

and

$$\int_{\mathbb{R}^N} \frac{\left|\frac{y}{|y|} \cdot \nabla_y u\right|^2}{|y|^\alpha} dx - \left(\frac{k-2-\alpha}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|y|^{2+\alpha}} dx = \int_{\mathbb{R}^N} \frac{1}{|y|^{k-2}} \left|\frac{y}{|y|} \cdot \nabla_y \left(|y|^{\frac{k-2-\alpha}{2}} u\right)\right|^2 dx.$$

Now, since $\left(r^{k-1}\frac{1}{r^{k-2}}, r^{k-1}\frac{1}{4r^k\left|\log\frac{r}{R}\right|^2}\right)$ is a Bessel pair on $(0,R)$ with $\varphi = \sqrt{\left|\log\frac{r}{R}\right|}$. By Theorem 1.1, we get the cylindrical critical Hardy inequalities:

COROLLARY 1.3. *For* $u \in C_0^\infty(\{0<|y|<R\})$:

$$\int_{0<|y|<R} \frac{|\nabla u(x)|^2}{|y|^{k-2}} dx - \frac{1}{4} \int_{0<|y|<R} \frac{|u(x)|^2}{|y|^k \left|\log\frac{R}{|y|}\right|^2} dx = \int_{0<|y|<R} \frac{1}{|y|^{k-2}} \log\frac{R}{|y|} \left|\nabla\left(\frac{u(x)}{\sqrt{\log\frac{R}{|y|}}}\right)\right|^2 dx$$

and

$$\int_{0<|y|<R} \frac{\left|\frac{y}{|y|} \cdot \nabla_y u(x)\right|^2}{|y|^{k-2}} dx - \frac{1}{4} \int_{0<|y|<R} \frac{|u(x)|^2}{|y|^k \left|\log\frac{R}{|y|}\right|^2} dx = \int_{0<|y|<R} \frac{1}{|y|^{k-2}} \log\frac{R}{|y|} \left|\frac{y}{|y|} \cdot \nabla_y \left(\frac{u(x)}{\sqrt{\log\frac{R}{|y|}}}\right)\right|^2 dx.$$

Actually, we can obtain as many cylindrical Hardy type inequalities as we can set up Bessel pairs. For more examples of the Bessel pairs, the interested reader is referred to [21].

2. Proof of Theorem 1.1

Proof of Theorem 1.1. For $u \in C_0^\infty(\{0<|y|<R\})$, one denotes

$$Su = \sqrt{V(|y|)}\nabla u - \sqrt{V(|y|)}\frac{\varphi'(|y|)}{\varphi(|y|)}\frac{(y,0)}{|y|}u.$$

Then a direct computation shows that its formal adjoint is

$$S^*\overrightarrow{v} = -\operatorname{div}\left(\sqrt{V(|y|)}\overrightarrow{v}\right) - \sqrt{V(|y|)}\frac{\varphi'(|y|)}{\varphi(|y|)}\frac{(y,0)}{|y|}\cdot\overrightarrow{v}.$$

As a consequence, one gets

$$\begin{aligned}
&\int\limits_{0<|y|<R} \overline{u(x)}\left(S^*Su\right)(x)\,dx \\
&= \int\limits_{0<|y|<R} |Su(x)|^2\,dx \\
&= \int\limits_{0<|y|<R} V(|y|)\,\varphi^2(|y|)\left|\nabla\left(\frac{u(x)}{\varphi(|y|)}\right)\right|^2 dx. \qquad (2.1)
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
S^*Su(x) &= -\operatorname{div}\left(V(|y|)\left[\nabla u(x) - \frac{\varphi'(|y|)}{\varphi(|y|)}u(x)\frac{(y,0)}{|y|}\right]\right) \\
&\quad - V(|y|)\frac{\varphi'(|y|)}{\varphi(|y|)}\frac{(y,0)}{|y|}\cdot\nabla u(x) + V(|y|)\left(\frac{\varphi'(|y|)}{\varphi(|y|)}\right)^2 u(x) \\
&= -V(|y|)\Delta u(x) - V'(|y|)\frac{(y,0)}{|y|}\cdot\nabla u(x) + V'(|y|)\frac{\varphi'(|y|)}{\varphi(|y|)}u(x) \\
&\quad + V(|y|)\frac{\varphi''(|y|)}{\varphi(|y|)}u(x) + V(|y|)\frac{\varphi'(|y|)}{\varphi(|y|)}u(x)\frac{k-1}{|y|}.
\end{aligned}$$

Hence, one obtains

$$\begin{aligned}
&\int\limits_{0<|y|<R} \overline{u(x)}\left(S^*Su\right)(x)\,dx \\
&= \int\limits_{0<|y|<R} V(|y|)\,|\nabla u(x)|^2\,dx \\
&\quad + \int\limits_{0<|y|<R} \frac{V(|y|)}{\varphi(|y|)}\left[\varphi''(|y|) + \frac{V'(|y|)}{V(|y|)}\varphi'(|y|) + \varphi'(|y|)\frac{k-1}{|y|}\right]|u(x)|^2\,dx \\
&= \int\limits_{0<|y|<R} V(|y|)\,|\nabla u(x)|^2\,dx - \int\limits_{0<|y|<R} W(|y|)\,|u(x)|^2\,dx. \qquad (2.2)
\end{aligned}$$

Thus, from (2.1) and (2.2), one deduces that

$$\begin{aligned}&\int\limits_{0<|y|<R} V(|y|)\,|\nabla u(x)|^2\,dx-\int\limits_{0<|y|<R} W(|y|)\,|u(x)|^2\,dx\\&=\int\limits_{0<|y|<R} V(|y|)\,\varphi^2(|y|)\left|\nabla\left(\frac{u(x)}{\varphi(|y|)}\right)\right|^2 dx.\end{aligned}$$

Now, $u\in C_0^\infty(\{0<|y|<R\})$, one denotes

$$Tu(x)=\sqrt{V(|y|)}\frac{(y,0)}{|y|}\cdot\nabla u(x)-\sqrt{V(|y|)}\frac{\varphi'(|y|)}{\varphi(|y|)}u(x)$$

Then its formal adjoint is

$$T^*v(x)=-\operatorname{div}\left(\sqrt{V(|y|)}\frac{(y,0)}{|y|}v(x)\right)-\sqrt{V(|y|)}\frac{\varphi'(|y|)}{\varphi(|y|)}v(x).$$

As a consequence

$$\begin{aligned}&\int\limits_{0<|y|<R}\overline{u(x)}\,(T^*Tu)(x)\,dx\\&=\int\limits_{0<|y|<R}|Tu(x)|^2\,dx\\&=\int\limits_{0<|y|<R} V(|y|)\left|\frac{(y,0)}{|y|}\cdot\nabla u(x)-\left(\frac{\varphi'(|y|)}{\varphi(|y|)}\right)u(x)\right|^2 dx\\&=\int\limits_{0<|y|<R} V(|y|)\,\varphi^2(|y|)\left|\frac{(y,0)}{|y|}\cdot\nabla\left(\frac{u(x)}{\varphi(|y|)}\right)\right|^2 dx.\end{aligned}\tag{2.3}$$

Also, by direct computation, one gets

$$\begin{aligned}T^*Tu(x)&=-\operatorname{div}\left(V(|y|)\,\frac{(y,0)}{|y|}\left[\frac{(y,0)}{|y|}\cdot\nabla u(x)-\frac{\varphi'(|y|)}{\varphi(|y|)}u(x)\right]\right)\\&\quad-V(|y|)\,\frac{\varphi'(|y|)}{\varphi(|y|)}\frac{(y,0)}{|y|}\cdot\nabla u(x)+V(|y|)\left(\frac{\varphi'(|y|)}{\varphi(|y|)}\right)^2u(x)\\&=-V(|y|)\,\frac{(y,0)}{|y|}\cdot\nabla\left(\frac{(y,0)}{|y|}\cdot\nabla u(x)\right)-V'(|y|)\,\frac{(y,0)}{|y|}\cdot\nabla u(x)\\&\quad-V(|y|)\,\frac{(y,0)}{|y|}\cdot\nabla u(x)\,\frac{k-1}{|y|}+V'(|y|)\,\frac{\varphi'(|y|)}{\varphi(|y|)}u(x)\\&\quad+V(|y|)\,\frac{\varphi''(|y|)}{\varphi(|y|)}u(x)+V(|y|)\,\frac{\varphi'(|y|)}{\varphi(|y|)}u(x)\,\frac{k-1}{|y|}.\end{aligned}$$

Hence

$$\begin{aligned}
&\int\limits_{0<|y|<R} \overline{u(x)}\,(T^*Tu)(x)\,dx \\
&= \int\limits_{0<|y|<R} V(|y|)\left|\frac{(y,0)}{|y|}\cdot\nabla u(x)\right|^2 dx \\
&\quad + \int\limits_{0<|y|<R} \frac{V(|y|)}{\varphi(|y|)}\left[\varphi''(|y|)+\frac{V'(|y|)}{V(|y|)}\varphi'(|y|)+\varphi'(|y|)\frac{k-1}{|y|}\right]|u(x)|^2 dx \\
&= \int\limits_{0<|y|<R} V(|y|)\left|\frac{(y,0)}{|y|}\cdot\nabla u(x)\right|^2 dx - \int\limits_{0<|y|<R} W(|y|)\,|u(x)|^2 dx. \qquad (2.4)
\end{aligned}$$

From (2.3) and (2.4), one obtains

$$\begin{aligned}
&\int\limits_{0<|y|<R} V(|y|)\left|\frac{y}{|y|}\cdot\nabla_y u(x)\right|^2 dx - \int\limits_{0<|y|<R} W(|y|)\,|u(x)|^2 dx \\
&= \int\limits_{0<|y|<R} V(|y|)\,\varphi^2(|y|)\left|\frac{y}{|y|}\cdot\nabla_y\left(\frac{u(x)}{\varphi(|y|)}\right)\right|^2 dx. \quad \square
\end{aligned}$$

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