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A BISHOP–PHELPS–BOLLOBÁS TYPE PROPERTY FOR MINIMUM ATTAINING OPERATORS

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Abstract. In this article, we study the Bishop-Phelps-Bollobás type theorem for minimum attaining operators. More explicitly, if we consider a bounded linear operator T on a Hilbert space H and a unit vector $x_0 \in H$ such that $\|Tx_0\|$ is very close to the minimum modulus of T, then T and x_0 are simultaneously approximated by a minimum attaining operator S on H and a unit vector $y \in H$ for which $\|Sy\|$ is equal to the minimum modulus of S. Further, we extend this result to a more general class of densely defined closed operators (need not be bounded) in Hilbert space. As a consequence, we get the denseness of the set of minimum attaining operators in the class of densely defined closed operators with respect to the gap metric.

1. Introduction

The renowned Bishop-Phelps theorem states that the space of norm attaining functionals on a Banach space is dense in the dual of the Banach space. Bollobás gave a quantitative version of the Bishop-Phelps theorem, which is known as the Bishop-Phelps-Bollobás theorem.

The operator version of the Bishop-Phelps theorem asks whether the class of all norm attaining operators between any two Banach spaces is dense in the space of all bounded linear operators between the Banach spaces with respect to the operator norm. There are several authors who have studied the operator version of Bishop-Phelps theorem on various Banach spaces, for example [1, 3, 10]. In general, the operator version of the Bishop-Phelps theorem need not hold. Lindenstrauss [10] gave a counter example which illustrated this fact. He also proved that the answer is affirmative if the domain space is reflexive.

Acosta et. al. [1] defined the notion of the Bishop-Phelps-Bollobás property (BPBP), which asserts that a pair of Banach spaces (X,Y) is said to have BPBP if for every $\varepsilon > 0$, there are $\alpha(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that for every bounded linear operator T from X into Y with $\|T\| = 1$, if $x_0 \in X$ with $\|x_0\| = 1$ such that $\|Tx_0\| > 1-\alpha(\varepsilon)$, then there exist $x_\varepsilon \in X$, $\|x_\varepsilon\| = 1$ and a bounded linear operator S from X into Y with $\|S\| = 1$ such that

$$\|Sx_\varepsilon\| = 1,\ \|x_\varepsilon - x_0\| < \beta(\varepsilon) \text{ and } \|T-S\| < \varepsilon.$$

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It is proved by Chang and Dong [3] that for every Hilbert space H, (H,H) have the BPB property.

If T is a bounded linear operator on a Hilbert space H, then the minimum modulus of T is defined by $m(T) = \inf\{\|Tx\| : x \in H, \|x\| = 1\}$. In this article, we introduce the minimum attaining analog of BPBP on Hilbert spaces. In particular, we show that:

Let T be a bounded linear operator on a Hilbert space H with $m(T) > 0$. Then for all $\varepsilon \in (0, m(T))$ and a unit vector x_0 in H satisfying

$$\|Tx_0\| < m(T) + \varepsilon, \tag{1.1}$$

there exist a bounded linear operator T_ε on H and a unit vector x_ε in H satisfying the following;

1. $\|T_\varepsilon x_\varepsilon\| = m(T_\varepsilon) = m(T)$,
2. $\|T - T_\varepsilon\| < \eta(\varepsilon, T)$,
3. $\|x_0 - x_\varepsilon\| < \gamma(\varepsilon, T)$,

where $\eta(\varepsilon, T)$, $\gamma(\varepsilon, T) \to 0$ as $\varepsilon \to 0$.

In case, if $m(T) = 0$ then for all $\varepsilon > 0$ and a unit vector x_0 satisfying (1.1), there exists a bounded operator T_ε on H satisfying all the conditions (1), (2) and (3).

Later we extend this notion to a more general class of densely defined closed operators defined between Hilbert spaces.

This article is divided into four sections. In section 2, we set up some notations and terminologies. In section 3, we deal with the BPBP analog of bounded minimum attaining operators in the space of all bounded linear operators on a Hilbert space. In section 4, we extend the results of section 3 to the class of densely defined closed operators.

2. Preliminaries

In this article, we deal with complex Hilbert spaces, which are denoted by H, H_1, H_2 etc. If M is a subspace of H, then the unit sphere in M is defined by $S_M := \{x \in M : \|x\| = 1\}$.

By a linear operator from H_1 to H_2, we mean a linear mapping T whose domain $D(T)$ and range $R(T)$ are subspaces of H_1 and H_2, respectively. It is called densely defined, if $\overline{D(T)} = H_1$. For every densely defined linear operator T, there exist a unique linear operator T^* called the *adjoint* of T, which satisfies

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \text{ for } x \in D(T), y \in D(T^*),$$

where $D(T^*) = \{y \in H_2 : x \to \langle Tx, y\rangle \text{ is a continuous functional on } D(T)\}$.

The graph $\mathscr{G}(T)$ of a linear operator T from H_1 to H_2 is the subspace $\{(x, Tx) : x \in D(T)\}$ of $H_1 \oplus H_2$. A linear operator T is said to be *closed* if $\mathscr{G}(T)$ is a closed subspace of $H_1 \oplus H_2$. We denote the class of closed linear operators from H_1 to H_2 by $\mathscr{C}(H_1, H_2)$. In particular, $\mathscr{C}(H) := \mathscr{C}(H, H)$. By the closed graph theorem, a linear operator T is bounded if and only if T is closed and $D(T) = H$. We denote the class of bounded linear operators from H_1 to H_2 by $\mathscr{B}(H_1,H_2)$ and $\mathscr{B}(H,H)$ is simply denoted by $\mathscr{B}(H)$.

Let $T \in \mathscr{C}(H_1,H_2)$ be a densely defined injective operator. Then the inverse of T is the linear map from $R(T)$ into H_1, satisfying $T^{-1}Tx = x$ for all $x \in D(T)$. In addition, if T is onto, then $T^{-1} \in \mathscr{B}(H_2,H_1)$ and in addition satisfy $TT^{-1}y = y$ for all $y \in H_2$.

An operator $A \in \mathscr{B}(H_1,H_2)$ is called an *isometry*, if $\|Ax\| = \|x\|$, for every $x \in H_1$ and is a *partial isometry*, if $A|_{N(A)^\perp}$ is an isometry, where $N(A)$ denotes the null space of A. For the partial isometry A, $N(A)^\perp$ is called the *initial space* and $R(A)$ is called the *final space*.

A linear operator S is called an *extension* of T, if $D(T) \subseteq D(S)$ and $Sx = Tx$, for all $x \in D(T)$. It is denoted by $T \subseteq S$. In addition if $D(S) = D(T)$, then $S = T$. A linear operator T in H is said to be *normal* if T is densely defined, closed and $T^*T = TT^*$. If $T = T^*$, then it is called *self-adjoint*. If T is self-adjoint and $\langle Tx,x\rangle \geqslant 0$, for every $x \in D(T)$, then T is called a *positive* operator.

THEOREM 2.1. [12, Theorem 13.31] *If $T \in \mathscr{C}(H)$ is a densely defined positive operator, then there exists a unique positive operator $S \in \mathscr{C}(H)$ such that $S^2 = T$. This unique S is denoted by $\sqrt{T}$.*

THEOREM 2.2. [2, Theorem 2, Page 184] *Let $T \in \mathscr{C}(H_1,H_2)$ be a densely defined operator. Then there exists a unique partial isometry $V : H_1 \to H_2$ with the initial space $N(T)^\perp$ and the final space $\overline{R(T)}$ such that*

$$T = V|T|, \text{ where } |T| = \sqrt{T^*T}. \tag{2.1}$$

Note that $D(T) = D(|T|)$. The Equation (2.1) is called the *polar decomposition* of T.

Let Σ be a σ-algebra of subsets of a set X and H be a Hilbert space. A *spectral measure* for (X,Σ,H) is a map $E : \Sigma \to \mathscr{B}(H)$ such that

1. For each $\omega \in \Sigma$, $E(\omega)$ is an orthogonal projection.
2. $E(\emptyset) = 0, E(X) = I$.
3. $E(\omega_1 \cap \omega_2) = E(\omega_1)E(\omega_2)$, for all $\omega_1, \omega_2 \in \Sigma$.
4. If $\{\omega_n\}_{n=1}^\infty$ is a sequence of mutually disjoint sets in Σ, then

$$E\left(\bigcup_{n=1}^{\infty} \omega_n\right) = \sum_{n=1}^{\infty} E(\omega_n),$$

where the series on the right hand side converges in the strong operator topology.

THEOREM 2.3. [12, Theorem 13.30] *To every self-adjoint operator A in H, there corresponds a unique spectral measure E on the Borel subsets of real line, such that*

$$A=\int_{-\infty}^{\infty}\lambda dE.$$

Moreover, E is concentrated on $\sigma(A)\subset(-\infty,\infty)$, in the sense that $E(\sigma(A))=I$.

The above theorem is called the spectral theorem for self-adjoint operators. For more detail about spectral theory, we refer [4, 12].

If T is a linear operator from H_1 to H_2, then the *minimum modulus* of T is defined by $m(T)=\inf\{\|Tx\|:x\in S_{D(T)}\}$.

It is well known that T has bounded inverse if and only if $m(T)>0$. In this case $\|T^{-1}\|=1/m(T)$. For more details about minimum modulus, we refer to [13].

DEFINITION 2.4. [9, Definition 2.3] Let $T\in\mathscr{C}(H_1,H_2)$ be a densely defined operator. Then T is called *minimum attaining,* if there exists $x_0\in S_{D(T)}$ such that $\|Tx_0\|=m(T)$.

Among bounded operators, finite rank operators, partial isometries, all non injective operators are always minimum attaining. In fact, the set of all bounded minimum attaining operators is dense in the space all bounded operators with respect to the operator norm. For more details of this class, we refer to [5, 9].

3. Bounded operators

This section is dedicated to the Bishop-Phelps type theorem for the minimum attaining operators in $\mathscr{B}(H)$. First, we prove a quantitative version of the Bishop-Phelps theorem for norm attaining operators. To some extent, this result is same as the one proved in [3, Theorem 3.1]. We need a few observations from this result which we use in proving our further results.

THEOREM 3.1. *Let $0<\varepsilon<1/2$. For every self adjoint operator $T\in\mathscr{B}(H)$ and $x_0\in S_H$ such that $\|Tx_0\|>\|T\|(1-\varepsilon)$, there exist a self adjoint operator $S\in\mathscr{B}(H)$ and $x_\varepsilon\in S_H$ such that*

1. $\|Sx_\varepsilon\|=\|S\|=\|T\|$,
2. $\|S-T\|<C\sqrt{2\varepsilon}$, *for some constant* $C>2\|T\|$,
3. $\|x_0-x_\varepsilon\|<\sqrt{2\varepsilon}+\sqrt[4]{2\varepsilon}$.

Moreover, we have the following;

(a) If T is positive, then S is positive.

(b) $N(T)=N(S)$.

(c) $m(S)\geqslant m(T)$.

Proof. Without loss of generality, we assume that $\|T\| = 1$. Suppose E is the spectral measure associated with T. Define $\omega_1 = \sigma(T) \cap [-1, -(1-\sqrt{2\varepsilon})]$, $\omega_2 = \sigma(T) \cap [(1-\sqrt{2\varepsilon}), 1]$ and $\omega_3 = \sigma(T) \cap \big(-(1-\sqrt{2\varepsilon}), (1-\sqrt{2\varepsilon})\big)$. Note that ω_1, ω_2 and ω_3 are mutually disjoint. Next, define

$$S = [-E(\omega_1) + E(\omega_2)] + TE(\omega_3). \tag{3.1}$$

Clearly, S is self-adjoint, as it is the sum of self-adjoint operators.

Let $x_0 = x_1 + x_2$, where $x_1 \in R(E(\omega_1 \cup \omega_2))$ and $x_2 \in R(E(\omega_3))$. Let $x_\varepsilon = x_1/\|x_1\|$. Observe that $\|Sx_\varepsilon\| = 1$ and

$$S - T = \int_{\omega_1} (-1-\lambda)\, dE(\lambda) + \int_{\omega_2} (1-\lambda)\, dE(\lambda).$$

Note that if $\lambda \in \omega_1$, then $-1 \leqslant \lambda \leqslant -(1-\sqrt{2\varepsilon})$ so that $\sup_{\lambda\in\omega_1} |1+\lambda| = \sqrt{2\varepsilon}$. Similarly $\sup_{\lambda\in\omega_2} |1-\lambda| = \sqrt{2\varepsilon}$, so that

$$\|S-T\| \leqslant \sup_{\lambda\in\omega_1} |1+\lambda| + \sup_{\lambda\in\omega_2} |1-\lambda| \leqslant 2\sqrt{2\varepsilon}.$$

Thus in (2) we can choose $C > 2$. Observe that $\|T|_{R(E(\omega_1\cup\omega_2))}\| \leqslant 1$ and $\|T|_{R(E(\omega_3))}\| \leqslant (1-\sqrt{2\varepsilon})$, thus we get

$$\begin{aligned}(1-\varepsilon)^2 &< \|Tx_0\|^2 \leqslant \|x_1\|^2 + \Big((1-\sqrt{2\varepsilon})\|x_2\|\Big)^2 \\ &= \left(\|x_1\|^2 + \|x_2\|^2\right) + \left(2\varepsilon - 2\sqrt{2\varepsilon}\right)\|x_2\|^2 \\ &= 1 + \left(2\varepsilon - 2\sqrt{2\varepsilon}\right)\|x_2\|^2.\end{aligned}$$

That is, $\varepsilon^2 - 2\varepsilon < (2\varepsilon - 2\sqrt{2\varepsilon})\|x_2\|^2$. From this inequality, on simplification we obtain,

$$\|x_2\|^2 < \frac{2\varepsilon - \varepsilon^2}{2(\sqrt{2\varepsilon} - \varepsilon)} = \frac{\sqrt{2\varepsilon} + \varepsilon}{2} \leqslant \sqrt{2\varepsilon}.$$

Consequently, we have

$$\|x_1\| = \sqrt{1 - \|x_2\|^2} > \sqrt{1 - \sqrt{2\varepsilon}} \geqslant 1 - \sqrt{2\varepsilon},$$

and

$$\|x_0 - x_\varepsilon\| = \|x_1 - (x_1/\|x_1\|) + x_2\| \leqslant 1 - \|x_1\| + \|x_2\| < \sqrt{2\varepsilon} + \sqrt[4]{2\varepsilon}.$$

Proof of (a): Suppose T is positive. Then $\sigma(T) \subseteq [0,1]$ and the operator in Equation (3.1) takes the form

$$S = E(\omega_2) + TE\Big(\sigma(T) \cap [0, (1-\sqrt{2\varepsilon}))\Big). \tag{3.2}$$

For every $x \in H$, we have $x = x_1 + x_2$, where $x_1 \in R[E(\omega_2)]$, $x_2 \in R[E(\sigma(T) \cap [0,(1-\sqrt{2\varepsilon}))]$. Hence

$$\langle Sx, x\rangle = \|x_1\|^2 + \langle Tx_2, x_2\rangle \geqslant \langle Tx, x\rangle. \tag{3.3}$$

The above inequality implies that S is positive, whenever T is positive. By the definition of the minimum modulus, we can easily verify that $m(S) \geqslant m(T)$.

Proof of (b): Let $x \in N(S)$. Then $x = x_1 + x_2 + x_3$, where $x_1 \in R(E(\omega_1))$, $x_2 \in R(E(\omega_2))$ and $x_3 \in R(E(\omega_3))$. For $i = 1,2$, we have $(-1)^i\|x_i\|^2 = \langle Sx, x_i\rangle = 0$, which implies $x_i = 0$. Thus we get $Tx = Tx_3 = Sx_3 = 0$ and consequently $N(S) \subseteq N(T)$.

Conversely, if $y \in N(T)$, then $y \in R(E\{0\}) \subseteq R(E(\omega_3))$, by [2, Theorem 4, Page 155]. This gives $Sy = Ty = 0$. Hence $N(T) \subseteq N(S)$.

Proof of (c): Let T be an arbitrary element of $\mathscr{B}(H)$ and $T = W|T|$ be its polar decomposition. Let S_1 be the operator defined in (3.2) corresponding to the operator $|T|$. That is,

$$S_1 = E\left(\sigma(|T|) \setminus [0,(1-\sqrt{2\varepsilon}))\right) + |T|E\left(\sigma(|T|) \cap [0,(1-\sqrt{2\varepsilon}))\right).$$

Let $S = WS_1$. Then $m(S_1) \geqslant m(|T|) = m(T)$. By part (b), we have $N(S_1) = N(|T|) = N(T)$. It can be easily verified that $N(S) = N(S_1)$.

For $y \in H$, we have $y = y_1 + y_2$, where $y_1 \in N(T)$ and $y_2 \in N(T)^{\perp}$. Hence

$$\|Sy\| = \|WS_1y_1 + WS_1y_2\| = \|WS_1y_2\| = \|S_1y\|.$$

The above equality implies that $m(S) = m(S_1) \geqslant m(T)$. □

REMARK 3.2.

1. Given $\varepsilon > 0$, it is possible to find a unit vector x_0 such that $\|Tx_0\| > \|T\|(1-\varepsilon)$ by the definition of the norm

2. If we do not assume $\|T\| = 1$ in Theorem 3.1, we have to define S as $S = \|T\|[E(\omega_2) - E(\omega_1)] + TE(\omega_3)$.

The following result is a Bishop-Phelps type theorem for minimum attaining operators.

THEOREM 3.3. *Let $T \in \mathscr{B}(H)$ be a positive operator, $0 < \varepsilon < m(T)$ and $x_0 \in S_H$ with*

$$\|Tx_0\| < m(T) + \varepsilon. \tag{3.4}$$

Then there exist a positive operator $T_\varepsilon \in \mathscr{B}(H)$ and $x_\varepsilon \in S_H$ satisfying the following.

1. $\|T_\varepsilon x_\varepsilon\| = m(T_\varepsilon) = m(T)$,

2. $\|T - T_\varepsilon\| < \eta(\varepsilon, T)$,

3. $\|x_0 - x_\varepsilon\| < \gamma(\varepsilon, T)$,

where $\eta(\varepsilon, T), \gamma(\varepsilon, T) \to 0$ *as* $\varepsilon \to 0$.

Proof. Note that T is invertible and $T^{-1} \in \mathscr{B}(H)$, hence $Tx_0 \neq 0$. By inequality (3.4), and the fact that $m(T) = 1/\|T^{-1}\|$, T^{-1} satisfies the following condition;

$$\left\| T^{-1} \frac{Tx_0}{\|Tx_0\|} \right\| > \|T^{-1}\|(1-\delta), \text{ where } \delta = \frac{\varepsilon}{m(T)+\varepsilon}. \tag{3.5}$$

As $0 < \varepsilon < m(T)$ we get $0 < \delta < 1/2$. By Theorem 3.1, there exist a positive operator $S_\varepsilon \in \mathscr{B}(H)$ and $x_\varepsilon^1 \in S_H$, such that

$$\|S_\varepsilon x_\varepsilon^1\| = \|S_\varepsilon\| = \|T^{-1}\|, \tag{3.6}$$

$$\|T^{-1} - S_\varepsilon\| < C\sqrt{2\delta} \text{ for some constant } C > 0 \tag{3.7}$$

and

$$\left\| x_\varepsilon^1 - \frac{Tx_0}{\|Tx_0\|} \right\| < \sqrt{2\delta} + \sqrt[4]{2\delta}. \tag{3.8}$$

As a consequence of part (c) of Theorem 3.1, S_ε^{-1} exists. Define $T_\varepsilon := S_\varepsilon^{-1}$ and $x_\varepsilon := \frac{S_\varepsilon x_\varepsilon^1}{\|S_\varepsilon x_\varepsilon^1\|}$. It can be easily seen from Equation (3.6) that

$$\begin{aligned} \left\| T_\varepsilon \frac{S_\varepsilon x_\varepsilon^1}{\|S_\varepsilon x_\varepsilon^1\|} \right\| &= \frac{\|x_\varepsilon^1\|}{\|S_\varepsilon x_\varepsilon^1\|} = \frac{1}{\|S_\varepsilon\|} \\ &= m\left(S_\varepsilon^{-1}\right) = m(T_\varepsilon) = m(T). \end{aligned}$$

We know that the inverse of a positive operator is positive, hence T_ε is positive.

By part (c) of Theorem 3.1, we have $m(S_\varepsilon) \geqslant m(T^{-1})$. Using this inequality and relations (3.6), (3.7) we get that

$$\begin{aligned} \|T_\varepsilon - T\| = \|T_\varepsilon(T^{-1} - T_\varepsilon^{-1})T\| &\leqslant \|S_\varepsilon^{-1}\| \|T^{-1} - S_\varepsilon\| \|T\| \\ &\leqslant \frac{1}{m(S_\varepsilon)} \|T^{-1} - S_\varepsilon\| \|T\| \\ &< C\|T\|^2\sqrt{2\delta} \\ &= \eta(\varepsilon, T), \end{aligned}$$

where $\eta(\varepsilon, T) = \|T\|^2 C\sqrt{2\delta}$. From the inequality (3.5), it is easy to see that $\delta \to 0$ as $\varepsilon \to 0$. Consequently, $\eta(\varepsilon, T) \to 0$ as $\varepsilon \to 0$.

From inequalities (3.4), (3.6), (3.7) and (3.8) we have the following estimate;

$$
\begin{aligned}
\|x_\varepsilon - x_0\| &= \left\| \frac{S_\varepsilon x_\varepsilon^1}{\|S_\varepsilon x_\varepsilon^1\|} - x_0 \right\| \\
&\leqslant \left\| \frac{S_\varepsilon x_\varepsilon^1}{\|S_\varepsilon x_\varepsilon^1\|} - \frac{T^{-1} x_\varepsilon^1}{\|S_\varepsilon x_\varepsilon^1\|} \right\| + \left\| \frac{T^{-1} x_\varepsilon^1}{\|S_\varepsilon x_\varepsilon^1\|} - \frac{x_0}{\|Tx_0\| \|S_\varepsilon x_\varepsilon^1\|} \right\| \\
&\quad + \left\| \frac{x_0}{\|Tx_0\| \|S_\varepsilon x_\varepsilon^1\|} - x_0 \right\| \\
&\leqslant \frac{\|S_\varepsilon - T^{-1}\|}{\|S_\varepsilon x_\varepsilon^1\|} + \frac{\|T^{-1}\|}{\|S_\varepsilon x_\varepsilon^1\|} \left\| x_\varepsilon^1 - \frac{Tx_0}{\|Tx_0\|} \right\| \\
&\quad + \frac{\|x_0\|}{\|Tx_0\| \|S_\varepsilon x_\varepsilon^1\|} \left| 1 - \|Tx_0\| \|S_\varepsilon x_\varepsilon^1\| \right| \\
&< m(T) C \sqrt{2\delta} + \sqrt{2\delta} + \sqrt[4]{2\delta} + \frac{m(T)}{\|Tx_0\|} \frac{|m(T) - \|Tx_0\||}{m(T)} \\
&\leqslant m(T) C \sqrt{2\delta} + \sqrt{2\delta} + \sqrt[4]{2\delta} + \frac{|m(T) - \|Tx_0\||}{m(T)}, \text{ as } \|Tx_0\| \geqslant m(T), \\
&= \gamma(\varepsilon . T),
\end{aligned}
$$

where $\gamma(\varepsilon, T) = Cm(T)\sqrt{2\delta} + \sqrt{2\delta} + \sqrt[4]{2\delta} + \varepsilon/m(T)$. Again using the fact from inequality (3.5) that $\delta \to 0$ as $\varepsilon \to 0$, we conclude that $\gamma(\varepsilon, T) \to 0$ as $\varepsilon \to 0$. □

REMARK 3.4. Here we explain how to get explicitly T_ε satisfying the conclusions of Theorem 3.1. By (2) of Remark 3.2, we have

$$
S_\varepsilon = \begin{bmatrix} \|T^{-1}\| E(\Delta_1) & 0 \\ 0 & T^{-1}|_{R[E(\Delta_2)]} \end{bmatrix},
$$

where $\Delta_1 = \sigma(T^{-1}) \cap (\|T^{-1}\|(1 - \sqrt{2\delta}), \|T^{-1}\|]$, $\Delta_2 = \sigma(T^{-1}) \cap [m(T), \|T^{-1}\|(1 - \sqrt{2\delta})]$ and E is the spectral measure corresponding to T. Note that

$$
\begin{aligned}
\Delta_1 &= \sigma(T^{-1}) \cap (\|T^{-1}\|(1 - \sqrt{2\delta}), \|T^{-1}\|] \\
&= \left\{ \mu \in \sigma(T) : \|T^{-1}\|(1 - \sqrt{2\delta}) < \mu^{-1} \leqslant \|T^{-1}\| \right\} \\
&= \left\{ \mu \in \sigma(T) : m(T) \leqslant \mu < \frac{m(T)}{1 - \sqrt{2\delta}} = m(T) + \frac{m(T)\sqrt{2\delta}}{1 - \sqrt{2\delta}} \right\} \\
&= \sigma(T) \cap \left[m(T), m(T) + m(T) \frac{\sqrt{2\delta}}{1 - \sqrt{2\delta}} \right).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\Delta_2 &= \sigma(T^{-1}) \cap [m(T^{-1}), \|T^{-1}\|(1 - \sqrt{2\delta})] \\
&= \sigma(T) \cap \left[m(T) + m(T) \frac{\sqrt{2\delta}}{1 - \sqrt{2\delta}}, \|T\| \right].
\end{aligned}
$$

Thus

$$T_\varepsilon = \begin{bmatrix} m(T)E\left(\sigma(T)\cap[m(T),m(T)+\delta_1]\right) & 0 \\ 0 & T|_{R[E(\sigma(T)\setminus[m(T),m(T)+\delta_1])]} \end{bmatrix}, \tag{3.9}$$

where $\delta_1 = (m(T)\sqrt{2\delta})/(1-\sqrt{2\delta})$.

THEOREM 3.5. *Let $T \in \mathscr{B}(H)$, $0 < \varepsilon < m(T)$ and $x_0 \in S_H$ with*

$$\|Tx_0\| < m(T) + \varepsilon. \tag{3.10}$$

Then there exist $T_\varepsilon \in \mathscr{B}(H)$ and $x_\varepsilon \in S_H$ satisfying the following.

1. $\|T_\varepsilon x_\varepsilon\| = m(T_\varepsilon) = m(T)$,
2. $\|T - T_\varepsilon\| < \eta(\varepsilon, T)$,
3. $\|x_0 - x_\varepsilon\| < \gamma(\varepsilon, T)$,

where $\eta(\varepsilon,T)$, $\gamma(\varepsilon,T) \to 0$ as $\varepsilon \to 0$.

Proof. Let $T = V|T|$ be the polar decomposition of T. As $m(T) > 0$, T must be bounded below. In this case $N(V) = N(T) = \{0\}$. Hence V is an isometry. As $m(T) = m(|T|)$ and $\||T|x_0\| = \|Tx_0\|$, by applying Theorem 3.3 to $|T|$, we can find $x_\varepsilon \in S_H$ and $S_\varepsilon \in \mathscr{B}(H)$ satisfying the conditions stated in Theorem 3.3.

Next, let $T_\varepsilon = VS_\varepsilon$. Since V is an isometry, we have $m(S_\varepsilon) = m(T_\varepsilon)$ and

$$\|T_\varepsilon x_\varepsilon\| = \|VS_\varepsilon x_\varepsilon\| = \|S_\varepsilon x_\varepsilon\| = m(T_\varepsilon) = m(T).$$

Next, $\|T_\varepsilon - T\| = \|V(S_\varepsilon - |T|)\| = \|S_\varepsilon - |T|\| < \eta(\varepsilon, T)$. This completes the proof. □

Next we study the case when $m(T) = 0$.

THEOREM 3.6. *Let $\varepsilon > 0$. Suppose $T \in \mathscr{B}(H)$ is a positive operator, $m(T) = 0$ and $x_0 \in S_H$ with*

$$\|Tx_0\| < m(T) + \varepsilon. \tag{3.11}$$

Then there exist a positive operator $T_\varepsilon \in \mathscr{B}(H)$ and $x_\varepsilon \in S_H$ satisfying the following.

1. $\|T_\varepsilon x_\varepsilon\| = m(T_\varepsilon) = m(T)$,
2. $\|T - T_\varepsilon\| < \eta(\varepsilon, T)$,
3. $\|x_0 - x_\varepsilon\| < \gamma(\varepsilon, T)$,

where $\eta(\varepsilon,T)$, $\gamma(\varepsilon,T) \to 0$ as $\varepsilon \to 0$.

Proof. Consider $S_\varepsilon = T + 2\varepsilon I$. It is easy to see that S_ε is a positive operator and $m(S_\varepsilon) = 2\varepsilon$. From the condition (3.11), we get

$$\begin{aligned}\|S_\varepsilon x_0\| &\leqslant \|Tx_0\| + 2\varepsilon \\ &< \varepsilon + 2\varepsilon = \varepsilon + m(S_\varepsilon).\end{aligned}$$

Note that $0 < \varepsilon < m(S_\varepsilon)$. By Theorem 3.3, there exist a positive operator $T_\varepsilon^1 \in \mathscr{B}(H)$ and $x_\varepsilon \in S_H$ such that

$$\|T_\varepsilon^1 x_\varepsilon\| = 2\varepsilon = m(T_\varepsilon^1),\ \|T_\varepsilon^1 - S_\varepsilon\| < \eta(\varepsilon, T) \text{ and } \|x_0 - x_\varepsilon\| < \gamma(\varepsilon, T), \tag{3.12}$$

with the condition that $\eta(\varepsilon, T), \gamma(\varepsilon, T) \to 0$ as $\varepsilon \to 0$.

Since T_ε^1 is a positive operator and $m(T_\varepsilon^1) = 2\varepsilon$, it follows that $T_\varepsilon^1 x_\varepsilon = (2\varepsilon)x_\varepsilon$ by [7, Proposition 3.9].

Take $T_\varepsilon := T_\varepsilon^1 - (2\varepsilon)I$. Note that T_ε is a positive operator and $\|T_\varepsilon x_\varepsilon\| = \|T_\varepsilon^1 x_\varepsilon - (2\varepsilon)I x_\varepsilon\| = 0 = m(T_\varepsilon) = m(T)$. By (3.12), we have that

$$\begin{aligned}\|T_\varepsilon - T\| &= \left\|T_\varepsilon^1 - 2\varepsilon I - S_\varepsilon + 2\varepsilon I\right\| \\ &= \|T_\varepsilon^1 - S_\varepsilon\| \\ &< \eta(\varepsilon, T). \quad \square\end{aligned}$$

REMARK 3.7. Here we indicate a procedure to get T_ε satisfying conclusions of Theorem 3.6. By Remark 3.4, we have

$$T_\varepsilon^1 = \begin{bmatrix} m(S_\varepsilon)E\left(\sigma(S_\varepsilon) \cap [2\varepsilon, 2\varepsilon + \alpha(\varepsilon)]\right) & 0 \\ 0 & S_\varepsilon|_{R[E(\sigma(S_\varepsilon)\setminus[2\varepsilon,2\varepsilon+\alpha(\varepsilon)])]} \end{bmatrix},$$

for some function $\alpha(\varepsilon)$ of ε. Observe that

$$2\varepsilon I = 2\varepsilon E\left(\sigma(S_\varepsilon) \cap [2\varepsilon, 2\varepsilon + \alpha(\varepsilon)]\right) + 2\varepsilon E\left(\sigma(S_\varepsilon) \setminus [2\varepsilon, 2\varepsilon + \alpha(\varepsilon)]\right).$$

We know that $(S_\varepsilon - 2\varepsilon I)|_{R[E(\sigma(S_\varepsilon)\setminus[2\varepsilon,2\varepsilon+\alpha(\varepsilon)])]} = T|_{R[E(\sigma(S_\varepsilon)\setminus[2\varepsilon,2\varepsilon+\alpha(\varepsilon)])]}$ and $\sigma(S_\varepsilon) \setminus [2\varepsilon, 2\varepsilon + \alpha(\varepsilon)] = \sigma(T) \setminus [0, \alpha(\varepsilon)]$. Thus

$$T_\varepsilon = \begin{bmatrix} 0 & 0 \\ 0 & T|_{R[E(\sigma(T)\setminus[0,\alpha(\varepsilon)])]} \end{bmatrix}. \tag{3.13}$$

THEOREM 3.8. *Let $\varepsilon > 0$. Suppose $T \in \mathscr{B}(H)$ with $m(T) = 0$ and $x_0 \in S_H$ with*

$$\|Tx_0\| < m(T) + \varepsilon. \tag{3.14}$$

Then there exist $T_\varepsilon \in \mathscr{B}(H)$ and $x_\varepsilon \in S_H$ satisfying the following.

1. $\|T_\varepsilon x_\varepsilon\| = m(T_\varepsilon) = m(T)$,
2. $\|T - T_\varepsilon\| < \eta(\varepsilon, T)$,
3. $\|x_0 - x_\varepsilon\| < \gamma(\varepsilon, T)$,

where $\eta(\varepsilon,T)$, $\gamma(\varepsilon,T)\to 0$ *as* $\varepsilon\to 0$.

Proof. Let $T=V|T|$ be the polar decomposition of T. Using the fact that $\||T|x_0\| = \|Tx_0\| < m(T)+\varepsilon$ and earlier arguments, we conclude that there exist a positive operator $\tilde{T}_\varepsilon \in \mathscr{B}(H)$ and $x_\varepsilon \in S_H$ such that

$$\|\tilde{T}_\varepsilon x_\varepsilon\| = m(\tilde{T}_\varepsilon),\ \|\tilde{T}_\varepsilon - |T|\| < \eta(\varepsilon,T) \text{ and } \|x_0 - x_\varepsilon\| < \gamma(\varepsilon,T).$$

Define $T_\varepsilon := V\tilde{T}_\varepsilon$. From Equations (3.9), (3.13) and [2, Theorem 4, Page 155], we observe that $N(T)\subseteq R(E\{0\})\subseteq N(\tilde{T}_\varepsilon)$, where E is the spectral measure associated with $|T|$. That is, $N(\tilde{T}_\varepsilon)^\perp \subseteq N(T)^\perp$.

Observe that $\|T_\varepsilon x_\varepsilon\| = \|V\tilde{T}_\varepsilon x_\varepsilon\| = \|\tilde{T}_\varepsilon x_\varepsilon\| = m(\tilde{T}_\varepsilon) = m(T_\varepsilon)$. Here we used the fact that $V|_{N(\tilde{T}_\varepsilon)^\perp}$ is an isometry.

Next, $\|T_\varepsilon - T\| \leqslant \|V\|\|\tilde{T}_\varepsilon - |T|\| < \eta(\varepsilon,T)$. This proves the result. □

REMARK 3.9. Given $\varepsilon>0$ it is possible to find a unit vector x_0 such that $\|Tx_0\| < m(T)+\varepsilon$ by the definition of the minimum modulus.

We illustrate Theorem 3.8 with a few examples.

EXAMPLE 3.10. Let $0<\varepsilon<1$. Consider the operator $M: L^2[-1,1]\to L^2[-1,1]$ defined by

$$Mf(t) = tf(t) \text{ for } t\in[-1,1],\ f\in L^2[-1,1].$$

It is easy to check that $m(M)=0$. We define a function $g=(1/2\varepsilon^2)\chi_{(-\varepsilon^2,\varepsilon^2)}$. Then $g\in L^2[-1,1]$ and it satisfy

$$\|Mg\|_2 = \frac{\varepsilon}{\sqrt{6}} < \varepsilon = m(M)+\varepsilon.$$

Now, we show that M and g can be approximated by an operator $M_\varepsilon \in \mathscr{B}\left(L^2[-1,1]\right)$ and $g_\varepsilon \in L^2[-1,1]$, respectively. To deduce this, we define $M_\varepsilon = MP_\omega$, where

$$\omega = \left\{h\in L^2[-1,1]:\ \text{support of } h \subseteq [-1,1]\setminus(-\varepsilon^2,\varepsilon^2)\right\},$$

P_ω is orthogonal projection onto ω and $g_\varepsilon := g$. We observe that $\|M_\varepsilon g_\varepsilon\|=0$ and

$$\|g-g_\varepsilon\|_2 = 0 < \varepsilon,$$
$$\|M-M_\varepsilon\| \leqslant \left(\sup_{t\in(-\varepsilon^2,\varepsilon^2)} |t|^2\right)^{\frac{1}{2}} < \sqrt{2}\varepsilon^2.$$

EXAMPLE 3.11. Let $0<\varepsilon<1$. Consider the operator $T:\ell^2(\mathbb{N})\to\ell^2(\mathbb{N})$ defined by

$$T(x_1,x_2,x_3,\dots) = \left(x_1,\frac{x_2}{2},\frac{x_3}{3},\dots\right) \text{ for } (x_1,x_2,x_3,\dots)\in\ell^2(\mathbb{N}).$$

Note that $\sigma(T) = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ and $m(T) = d(0, \sigma(T)) = \inf\{|\lambda| : \lambda \in \sigma(T)\} = 0$.

Suppose $x_0 = \sum_{i=1}^{\infty} \alpha_i e_i \in \ell^2(\mathbb{N})$ satisfying $\|Tx_0\| < \varepsilon = m(T) + \varepsilon$, where $\{e_i\}_{i\in\mathbb{N}}$ is the standard othonormal basis for $\ell^2(\mathbb{N})$ and α_i is a scalar for every $i \in \mathbb{N}$. As $\left(\frac{\alpha_i}{i}\right) \in \ell^2(\mathbb{N})$, we get an $n_\varepsilon \in \mathbb{N}$ such that $\frac{1}{n_\varepsilon} < \varepsilon$ and $\sum_{i=1}^{n_\varepsilon} \alpha_i^2 < \varepsilon^2$.

Now we choose $x_\varepsilon = \sum_{i=n_\varepsilon+1}^{\infty} \alpha_i e_i$ and define $T_\varepsilon \in \mathscr{B}\left(\ell^2(\mathbb{N})\right)$ by

$$T_\varepsilon(x_1, x_2, \dots x_{n_\varepsilon}, x_{n_\varepsilon+1} \dots) = \left(x_1, \frac{x_2}{2}, \dots \frac{x_{n_\varepsilon}}{n_\varepsilon}, 0, \dots\right), \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$

It is easy to observe that $\|T_\varepsilon x_\varepsilon\| = 0$, $\|x_0 - x_\varepsilon\| < \varepsilon$ and

$$\|T - T_\varepsilon\| = \sup_{i \geqslant n_\varepsilon+1} |1/i| = 1/(n_\varepsilon + 1) < \varepsilon.$$

It can be easily shown that T is not minimum attaining.

Let us take $\varepsilon = \frac{1}{3}$, $x_0 = e_4$. Then

$$\|Te_4\| = \frac{1}{4} < \frac{1}{3} = m(T) + \varepsilon.$$

Now take $x_\varepsilon = e_4$. For $n \geqslant n_0 = 4$ we have $\frac{1}{n} < \varepsilon$. Define

$$T_\varepsilon(x_1, x_2, x_3, \dots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, 0, \dots\right) \text{ for } (x_n) \in \ell^2(\mathbb{N}).$$

Then

$$(T - T_\varepsilon)(x_1, x_2, \dots) = \left(0, 0, 0, \frac{x_4}{4}, \frac{x_5}{5}, \dots\right).$$

Hence $\|T - T_\varepsilon\| = \frac{1}{4} < \varepsilon$. Clearly $\|x_0 - x_\varepsilon\| = 0 < \varepsilon$.

If we take $\varepsilon = \frac{1}{3}$, $x_0 = e_5$. Then

$$\|Te_5\| = \frac{1}{5} < \frac{1}{3} = m(T) + \varepsilon.$$

For $n \geqslant n_0 = 4$, $\frac{1}{n} < \varepsilon$. In this case, take $x_\varepsilon = e_5$. Define

$$T_\varepsilon(x_1, x_2, x_3, x_4, \dots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, 0, \dots\right) \text{ for } (x_n) \in \ell^2(\mathbb{N}).$$

Then

$$(T - T_\varepsilon)(x_1, x_2, \dots) = \left(0, 0, 0, 0, \frac{x_5}{5}, \dots\right).$$

Hence $\|T - T_\varepsilon\| = \frac{1}{5} < \varepsilon$. Clearly $\|x_0 - x_\varepsilon\| = 0 < \varepsilon$.

4. Unbounded operators

In this section, we generalize the results of the earlier section to densely defined closed operators, which are not necessarily bounded. In this case, we have to discuss the approximation of operators in the gap topology. For this purpose first we define the gap between two closed subspaces of a Hilbert space.

Let M,N be two closed subspaces of a Hilbert space H. Define $d(M,N) = \sup\limits_{x\in S_M} \operatorname{dist}(x,S_N)$. The gap between M and N is defined by

$$\theta(M,N) = \max\{d(M,N),d(N,M)\}.$$

For $T_1,T_2 \in \mathscr{C}(H_1,H_2)$, the gap between T_1 and T_2 is defined by the gap between the corresponding graphs. That is,

$$\theta(T_1,T_2) = \theta(\mathscr{G}(T_1),\mathscr{G}(T_2)).$$

It is well known that $\theta(\cdot,\cdot)$ is a metric on $\mathscr{C}(H_1,H_2)$ and is called the gap metric. For more details about this metric we refer to [6, 8, 11].

PROPOSITION 4.1. [6, Theorem 2.20, Page 205] *Let $S,T \in \mathscr{C}(H)$. Assume that both S^{-1} and T^{-1} exists. Then $\theta(S,T) = \theta(T^{-1},S^{-1})$.*

PROPOSITION 4.2. [9, Theorem 3.1(2)] *Let $S,T \in \mathscr{C}(H_1,H_2)$ with $D(S) = D(T)$. If $S-T \in \mathscr{B}(H_1,H_2)$, then $\theta(S,T) \leqslant \|S-T\|$.*

Next, we prove our main theorem in this section.

PROPOSITION 4.3. *Let $T \in \mathscr{C}(H)$ be positive. Let ε be such that $\varepsilon \in (0,m(T))$ if $m(T) > 0$ and, $\varepsilon > 0$ when $m(T) = 0$. Let $x_0 \in S_{D(T)}$ with*

$$\|Tx_0\| < m(T)+\varepsilon. \tag{4.1}$$

Then there exist a densely defined operator $T_\varepsilon \in \mathscr{C}(H)$ which is positive and $x_\varepsilon \in S_{D(T_\varepsilon)}$ satisfying the following.

1. *$T_\varepsilon x_\varepsilon = m(T_\varepsilon)x_\varepsilon = m(T)x_\varepsilon$,*
2. *$\|x_0 - x_\varepsilon\| < \gamma(\varepsilon,T)$,*
3. *$\theta(T,T_\varepsilon) < \eta(\varepsilon,T)$,*

where $\eta(\varepsilon,T), \gamma(\varepsilon,T) \to 0$ as $\varepsilon \to 0$.

Proof. *Case* (1): $m(T) > 0$: then T^{-1} exists and $T^{-1} \in \mathscr{B}(H)$. From the given condition (4.1), we deduce that

$$\left\|\frac{T^{-1}Tx_0}{\|Tx_0\|}\right\| > \|T^{-1}\|(1-\delta), \text{ where } \delta = \frac{\varepsilon}{m(T)+\varepsilon}.$$

As $0<\varepsilon<m(T)$, we have $0<\delta<\frac{1}{2}$. Hence by Theorem 3.1, there exist $S_\varepsilon\in\mathscr{B}(H)$ and $y_\varepsilon\in S_H$ such that

$$\|S_\varepsilon y_\varepsilon\|=\|S_\varepsilon\|=\|T^{-1}\|, \tag{4.2}$$

$$\|T^{-1}-S_\varepsilon\|<C\sqrt{2\delta},\ \text{for some constant } C, \tag{4.3}$$

$$\left\|\frac{Tx_0}{\|Tx_0\|}-y_\varepsilon\right\|<\sqrt{2\delta}+\sqrt[4]{2\delta}. \tag{4.4}$$

Also $N(S_\varepsilon)=N(T^{-1})=\{0\}$. Hence $S_\varepsilon^{-1}:R(S_\varepsilon)\to H$ exists. We define $T_\varepsilon:=S_\varepsilon^{-1}$ and $x_\varepsilon:=\frac{S_\varepsilon y_\varepsilon}{\|S_\varepsilon y_\varepsilon\|}$. We have $D(T_\varepsilon)=R(S_\varepsilon)$ and as S_ε is injective, we have $\overline{R(S_\varepsilon)}=N(S_\varepsilon)^\perp=H$. Hence T_ε is a densely defined operator. It is clear that $T_\varepsilon\in\mathscr{C}(H)$. Using the positivity it can be shown that T_ε is a positive operator. By similar explanation as given in the Proof of Theorem 3.3, we get

$$\begin{aligned}\|T_\varepsilon x_\varepsilon\|&=m(T_\varepsilon)\,(=m(T))\ \text{and}\\ \|x_\varepsilon-x_0\|&<Cm(T)\sqrt{2\delta}+\sqrt{2\delta}+\sqrt[4]{2\delta}+\frac{\varepsilon}{m(T)}\\ &<\gamma(\varepsilon,T),\end{aligned}$$

where $\gamma(\varepsilon,T)=Cm(T)\sqrt{2\delta}+\sqrt{2\delta}+\sqrt[4]{2\delta}+\frac{\varepsilon}{m(T)}$. As T_ε is a positive operator, the equation $\|T_\varepsilon x_\varepsilon\|=m(T_\varepsilon)=m(T)$ implies that

$$T_\varepsilon x_\varepsilon=m(T_\varepsilon)x_\varepsilon=m(T)x_\varepsilon,\ \text{by [7, Proposition 3.9]}.$$

Since $N(T_\varepsilon)=\{0\},R(T_\varepsilon)=D(S_\varepsilon)=H$, we get that $T_\varepsilon^{-1}:H\to R(S_\varepsilon)$ exists and $T_\varepsilon^{-1}=S_\varepsilon$. By Proposition 4.1, we have the following inequality;

$$\theta(T_\varepsilon,T)=\theta(S_\varepsilon,T^{-1})\leqslant\|S_\varepsilon-T^{-1}\|<C\sqrt{2\delta}=:\eta(\varepsilon,T).$$

Case (2): Let $m(T)=0$. Define $\hat{T}:=T+2\varepsilon I$. Note that $\hat{T}$ is positive, $D(\hat{T})=D(T)$ and $m(\hat{T})=2\varepsilon$. Also $\|\hat{T}x_0\|\leqslant\|Tx_0\|+2\varepsilon<\varepsilon+2\varepsilon=\varepsilon+m(\hat{T})$. By Case (1), there exist a positive operator $T_2\in\mathscr{C}(H)$, $x_\varepsilon\in S_{D(T_2)}$ such that

$$T_2x_\varepsilon=m(T_2)x_\varepsilon=m(\hat{T})x_\varepsilon,\ \theta(\hat{T},T_2)<\eta_1(\varepsilon,T)\ \text{and}\ \|x-x_\varepsilon\|<\gamma(\varepsilon,T).$$

Define $T_\varepsilon:=T_2-2\varepsilon I$. Clearly $D(T_\varepsilon)=D(T_2)$, $m(T_\varepsilon)=m(T_2)-2\varepsilon=0$ and by [7, Proposition 3.8] T_ε is positive. Also $T_\varepsilon x_\varepsilon=T_2x_\varepsilon-2\varepsilon x_\varepsilon=0=m(T_\varepsilon)x_\varepsilon=m(T)x_\varepsilon$. We have the following approximation;

$$\begin{aligned}\theta(T,T_\varepsilon)=&\theta\left(\hat{T}-2\varepsilon I,T_2-2\varepsilon I\right),\\ \leqslant&\theta\left(\hat{T}-2\varepsilon I,\hat{T}\right)+\theta(\hat{T},T_2)+\theta\left(T_2,T_2-2\varepsilon I\right),\\ \leqslant&2\varepsilon+\eta_1(\varepsilon,T)+2\varepsilon,\\ =&4\varepsilon+\eta_1(\varepsilon,T)(:=\eta(\varepsilon,T)).\end{aligned}$$

This completes the proof. □

REMARK 4.4. In Proposition 4.3, more precisely T_ε has the following structure.

$$T_\varepsilon = m(T)E\left(\sigma(T)\cap[0,m(T)+\alpha(\varepsilon)]\right) + TE\left(\sigma(T)\setminus[0,m(T)+\alpha(\varepsilon)]\right), \tag{4.5}$$

where $\alpha(\varepsilon)\to 0$ as $\varepsilon\to 0$ and E is the spectral measure corresponding to T. Moreover, $N(T)\subseteq N(T_\varepsilon)$.

Proof. Without loss of generality, we assume that $N(T)\neq\{0\}$. Then $m(T)=0$ and

$$T_\varepsilon = TE\left(\sigma(T)\setminus[0,\alpha(\varepsilon)]\right).$$

By [2, Theorem 4, Page 155], we know that

$$N(T)\subseteq R(E(\{0\}))\subseteq R(E(\sigma(T)\cap[0,\alpha(\varepsilon)]))\subseteq N(T_\varepsilon). \quad \square$$

THEOREM 4.5. *Let $T\in\mathscr{C}(H)$ be densely defined. Let $\varepsilon\in(0,m(T))$ if $m(T)>0$ and, $\varepsilon>0$ when $m(T)=0$. Let $x_0\in S_{D(T)}$ be such that*

$$\|Tx_0\| < m(T)+\varepsilon. \tag{4.6}$$

Then there exist a densely defined operator $T_\varepsilon\in\mathscr{C}(H)$ and $x_\varepsilon\in S_{D(T_\varepsilon)}$ satisfying the following.

1. $\|T_\varepsilon x_\varepsilon\| = m(T_\varepsilon) = m(T)$,
2. $\|x_0 - x_\varepsilon\| < \gamma(\varepsilon,T)$,
3. $\theta(T,T_\varepsilon) < \eta(\varepsilon,T)$,

where $\eta(\varepsilon,T)$, $\gamma(\varepsilon,T)\to 0$ as $\varepsilon\to 0$.

Proof. Let $T=W|T|$ be the polar decomposition of T. From the given condition (4.6), we have $\||T|x_0\| = \|Tx_0\| < m(T)+\varepsilon = m(|T|)+\varepsilon$. As a result of Proposition 4.3, there exist a densely defined positive operator $S_\varepsilon\in\mathscr{C}(H)$, $x_\varepsilon\in S_{D(S_\varepsilon)}$ such that

$$S_\varepsilon x_\varepsilon = m(S_\varepsilon)x_\varepsilon = m(T)x_\varepsilon,\ \theta(S_\varepsilon,|T|)<\eta(\varepsilon,T) \text{ and } \|x_0-x_\varepsilon\|<\gamma(\varepsilon,T), \tag{4.7}$$

$\eta(\varepsilon,T)$, $\gamma(\varepsilon,T)\to 0$ as $\varepsilon\to 0$.

Define $T_\varepsilon = WS_\varepsilon$. Note that

1. $D(T_\varepsilon) = \{x\in D(S_\varepsilon) : S_\varepsilon x\in D(W)=H\} = D(S_\varepsilon)$,
2. $N(T_\varepsilon) = N(S_\varepsilon)$,
3. $\|T_\varepsilon y\| = \|S_\varepsilon y\|$, for every $y\in D(T_\varepsilon)$.

Clearly $N(S_\varepsilon) \subseteq N(T_\varepsilon)$. To get the reverse containment, let $x \in N(T_\varepsilon)$. So $S_\varepsilon x \in N(W) = N(T) \subseteq N(S_\varepsilon)$, this implies $S_\varepsilon^2 x = 0$. Since S_ε is a positive operator, we see that $S_\varepsilon x = 0$, that is $x \in N(S_\varepsilon)$. Hence $N(T_\varepsilon) \subseteq N(S_\varepsilon)$ and consequently $N(S_\varepsilon) = N(T_\varepsilon)$.

Every $x \in D(S_\varepsilon) = D(T_\varepsilon)$ can be written as $x = x_1 + x_2$, where $x_1 \in N(S_\varepsilon)$ and $x_2 \in N(S_\varepsilon)^\perp \cap D(S_\varepsilon)$. From Remark 4.4, $N(S_\varepsilon)^\perp \subseteq N(T)^\perp$. Thus $\|WS_\varepsilon x_2\| = \|S_\varepsilon x_2\|$. Consequently, we have the following equality;

$$\|T_\varepsilon x\| = \|WS_\varepsilon(x_1 + x_2)\| = \|WS_\varepsilon x_2\| = \|S_\varepsilon x_2\| = \|S_\varepsilon x\|.$$

Thus we conclude that $\|T_\varepsilon x_\varepsilon\| = \|S_\varepsilon x_\varepsilon\| = m(S_\varepsilon) = m(T_\varepsilon)\,(= m(T))$.

We proceed to show that $\theta(T_\varepsilon, T) = \theta(S_\varepsilon, |T|)$. First we claim that $\|Tx - T_\varepsilon y\| = \||T|x - S_\varepsilon y\|$ for every $x \in D(T)$ and $y \in D(T_\varepsilon)$. Assuming the claim, we have

$$\begin{aligned}
\operatorname{dist}\big((x,Tx), S_{\mathscr{G}(T_\varepsilon)}\big) &= \inf_{\substack{y\in D(T_\varepsilon)\\ \|y\|^2+\|T_\varepsilon y\|^2=1}} \|(x,Tx)-(y,T_\varepsilon y)\| \\
&= \inf_{\substack{y\in D(S_\varepsilon)\\ \|y\|^2+\|S_\varepsilon y\|^2=1}} \|(x,Tx)-(y,T_\varepsilon y)\| \\
&= \inf_{\substack{y\in D(S_\varepsilon)\\ \|y\|^2+\|S_\varepsilon y\|^2=1}} \sqrt{\|x-y\|^2+\|Tx-T_\varepsilon y\|^2} \\
&= \inf_{\substack{y\in D(S_\varepsilon)\\ \|y\|^2+\|S_\varepsilon y\|^2=1}} \sqrt{\|x-y\|^2+\||T|x-S_\varepsilon y\|^2} \\
&=\operatorname{dist}\big((x,|T|x), S_{\mathscr{G}(S_\varepsilon)}\big),\ \forall x \in D(T).
\end{aligned}$$

By simple computation, we get $\theta(T_\varepsilon, T) = \theta(S_\varepsilon, |T|) < \eta(\varepsilon, T)$.

To prove our claim, suppose $x \in D(T)$ and $y \in D(T_\varepsilon)$. Then $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1 \in N(T)$, $x_2 \in N(T)^\perp \cap D(T)$, $y_1 \in N(T_\varepsilon)$ and $y_2 \in N(T_\varepsilon)^\perp \cap D(T_\varepsilon)$. Using the fact that $N(T_\varepsilon)^\perp \subseteq N(T)^\perp$, we have

$$\begin{aligned}
\|Tx - T_\varepsilon y\| &= \|W|T|x_2 - WS_\varepsilon y_2\| \\
&= \|W(|T|x_2 - S_\varepsilon y_2)\| \\
&= \||T|x_2 - S_\varepsilon y_2\| \\
&= \||T|x - S_\varepsilon y\|.
\end{aligned}$$

This completes the proof. □

As a consequence of Theorem 4.5, we conclude that the set of all minimum attaining operators is dense in the class of all densely defined closed operators with respect to the gap metric.

COROLLARY 4.6. *Let $T \in \mathscr{C}(H_1, H_2)$ be densely defined. Then for $\varepsilon > 0$ there exists a minimum attaining densely defined operator $S \in \mathscr{C}(H_1, H_2)$ such that $\theta(S,T) \leqslant \varepsilon$.*

A more sharpened version of the above corollary can be found in [9].

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