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REFINING EIGENVALUE INEQUALITIES FOR BLOCK 2×2 POSITIVE SEMIDEFINITE MATRICES

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Abstract. In this paper, by employing a result due to Bourin, Lee and Lin for block 2×2 positive semidefinite matrices, and by using gradients of Gateaux differentiable G-increasing functions, we show refinements of some majorization inequality by Lin and Wolkowicz for the eigenvalues of these block matrices. In particular, we establish a refinement for 2×2 version of Hiroshima's inequality.

We also consider some special cases of the obtained result.

1. Introduction and summary

In this work, we study some majorization inequalities for block 2×2 positive semidefinite matrices. In Section 2 we demonstrate some preliminaries on matrix notation and terminology. For instance we introduce (weakly) unitarily invariant norms. The notions of majorization preordering and of Schur-convex functions on $\mathbb{R}^p$ are defined, too. In this context, Ky Fan's eigenvalue majorization inequality for Hermitian matrices is presented. Next, for the standard group G of unitary similarities acting on the matrix space $\mathbb{H}_p$ of Hermitian $p \times p$ matrices, we define G-majorization preordering and G-increasing functions on $\mathbb{H}_p$. Furthemore, we define the notion of G-synchronicity of two Hermitian matrices and show its relationship with the equality case of Fan's eigenvalue inequality. We complete Section 2 by quoting some relevant results by Bourin, Lee and Lin [7, 6], Lin and Wolkowicz [13] and Hiroshima [11].

Results are presented in Section 3. By applying a decomposition statement due to Bourin et al. [7, 6] for block 2×2 positive semidefinite matrices, we show a refinement of some majorization inequality of Lin and Wolkowicz [13] for the eigenvalues of these block matrices (see Theorem 1). In particular, we establish a refinement for 2×2 version of Hiroshima's inequality [11].

Our method in the proof of Theorem 1 is based on the above-mentioned Ky Fan's inequality and on G-synchronous pairs of matrices. We use some Gateaux differentiable G-increasing functions ψ related to the standard Frobenius norm on the space of block 2×2 matrices. Their gradients preserve the G-synchronicity, which yields the

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equality case the Ky Fan's inequality. Finally, this leads to the required refinement of the investigated inequality by Lin and Wolkowicz [13].

In the rest of Section 3 we give interpretations of Theorem 1 for some special forms of the function ψ or of the involved matrices A and B (see Corollaries 2–3 for details).

2. Preliminaries

In this expository section we collect some notation, terminology and basic facts related to our studies.

Throughout, for a positive integer p, the symbol $\mathbb{M}_p(\mathbb{C})$ stands for the linear space of all matrices of size $p \times p$ with entries in the field $\mathbb{C}$ of complex numbers. The associated inner product is given by $\langle X, Y \rangle = \operatorname{Re}\operatorname{tr} XY^*$ for $X, Y \in \mathbb{M}_p(\mathbb{C})$, where $\operatorname{tr} Z$ and Z^* mean the trace and the conjugate transpose, respectively, of a matrix $Z \in \mathbb{M}_n(\mathbb{C})$. The norm induced by this inner product is given by $\|X\|_2 = (\operatorname{tr} XX^*)^{1/2}$ for $X \in \mathbb{M}_p(\mathbb{C})$.

We use $\mathbb{H}_p$ to denote the real linear space of all Hermitian matrices of size $p \times p$. For a given matrix $X \in \mathbb{H}_p$, we write $0 \leqslant X$ if X is positive semidefinite. By $\mathbb{M}_p^+(\mathbb{C})$ is denoted the convex cone of all positive semidefinite matrices of size $p \times p$.

For any interval $J \subset \mathbb{R}$, by $\mathbb{H}_p(J)$ we denote the set of all Hermitian matrices of size $p \times p$ with their eigenvalues belonging to J.

For a vector $z = (z_1, z_2, \dots, z_p) \in \mathbb{R}^p$, by $\operatorname{diag} z$ we mean the diagonal matrix with the entries $z_1, z_2, \dots, z_p$ on its main diagonal.

For $X \in \mathbb{H}_p$, by $\lambda(X)$ we denote the p-tuple $(\lambda_1(X), \lambda_2(X), \dots, \lambda_p(X))$ of the eigenvalues of X ordered so that $\lambda_1(X) \geqslant \lambda_2(X) \geqslant \cdots \geqslant \lambda_p(X)$.

We say that a p-tuple $y = (y_1, y_2, \dots, y_p) \in \mathbb{R}^p$ is *majorized* by a p-tuple $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}^p$ (written as $y \prec x$), if

$$\sum_{i=1}^{k} y_{[i]} \leqslant \sum_{i=1}^{k} x_{[i]} \quad \text{for all } k = 1, 2, \dots, p, \quad \text{and} \quad \sum_{i=1}^{p} y_i = \sum_{i=1}^{p} x_i,$$

where $z_{[i]}$ means the ith largest entry of a vector $z = (z_1, z_2, \dots, z_p) \in \mathbb{R}^p$ (see [14, p. 8], [2, p. 28])).

Given an interval $J \subset \mathbb{R}$, a real function $F : J^p \to \mathbb{R}$ is said to be *Schur-convex* on J^p if for $x, y \in J^p$,

$$y \prec x \quad \text{implies} \quad F(y) \leqslant F(x)$$

(see [14, p. 80], [2, p. 40])).

Ky Fan's eigenvalue majorization inequality asserts that

$$\lambda(X+Y) \prec \lambda(X) + \lambda(Y) \quad \text{for } X, Y \in \mathbb{H}_p \tag{1}$$

(see [8], cf. [2, p. 35])).

A norm $\|\cdot\|$ on $\mathbb{M}_p(\mathbb{C})$ is said to be *a unitarily invariant norm* (u.i. norm, for short), if

$$\|U_1 X U_2\| = \|X\| \quad \text{for all } X \in \mathbb{M}_p(\mathbb{C}) \text{ and } U_1, U_2 \in \mathbb{U}_p$$

(see [2, p. 91]).

A norm $\|\cdot\|$ on $\mathbb{M}_p(\mathbb{C})$ is said to be *a weakly unitarily invariant norm* (w.u.i. norm, for short), if

$$\|U_1XU_1^*\| = \|X\| \quad \text{for all } X \in \mathbb{M}_p(\mathbb{C}) \text{ and } U_1 \in \mathbb{U}_p$$

(see [2, p. 102]).

The group of all unitary matrices of size $p \times p$ is denoted by $\mathbb{U}_p$.

In what follows, we consider the group

$$G = \{U_1(\cdot)U_1^* : U_1 \in \mathbb{U}_p\}$$

acting on $\mathbb{H}_p$.

We define preorder $\prec_G$ on $\mathbb{H}_p$, as follows. For $X,Y \in \mathbb{H}_p$,

$$Y \prec_G X \quad \text{iff} \quad \lambda(Y) \prec \lambda(X)$$

(see [1, p. 234]). For a characterization of the preorder $\prec_G$ on Hermitian matrices, see [1, Theorem 7.1, p. 235].

A set $K \subset \mathbb{H}_p$ is said to be G-invariant, if

$$U_1XU_1^* \in K \quad \text{for all } X \in K \text{ and } U_1 \in \mathbb{U}_p.$$

A set $K \subset \mathbb{H}_p$ is said to be convex, if

$$X,Y \in K \quad \text{implies} \quad tX + (1-t)Y \in K \quad \text{for all } 0 \leqslant t \leqslant 1.$$

Let $K \subset \mathbb{H}_p$ be a G-invariant convex set. A function $\psi : K \to \mathbb{R}$ is said to be *G-increasing* on K, if for $X,Y \in K$,

$$Y \prec_G X \quad \text{implies} \quad \psi(Y) \leqslant \psi(X).$$

Two matrices $X,Y \in \mathbb{H}_p$ are said to be *G-synchronous*, if the exists $U_1 \in \mathbb{U}_p$ such that $X = U_1 \operatorname{diag} \lambda(X) U_1^*$ and $Y = U_1 \operatorname{diag} \lambda(Y) U_1^*$.

As can be deduced from [10, Proposition 4.2], in general, for $X,Y \in \mathbb{H}_p$ it holds that

$$\langle X,Y \rangle \leqslant \langle \lambda(X), \lambda(Y) \rangle \tag{2}$$

with the trace inner product on $\mathbb{H}_p$ on the left-hand side of (2), and the standard inner product on $\mathbb{R}^p$ on the right-hand side of (2).

However,

$$\langle X,Y \rangle = \langle \lambda(X), \lambda(Y) \rangle \quad \text{iff} \quad X \text{ and } Y \text{ are } G\text{-synchronous}.$$

Also, if $X,Y \in \mathbb{H}_p$ are G-synchronous, then

$$\lambda(X+Y) = \lambda(X) + \lambda(Y). \tag{3}$$

Thus the G-synchronicity of two matrices leads to the equality case of the Ky Fan's eigenvalue inequality (1).

In the sequel we shall explore the following interesting results due to Bourin, Lee and Lin [6, 7] and to Lin and Wolkowicz [13].

THEOREM A [7, Theorem 2.2]. *Given any matrix in* $\mathbb{M}_{2n}^{+}(\mathbb{C})$ *partitioned into blocks in* $\mathbb{M}_n(\mathbb{C})$ *with Hermitian off-diagonal blocks, we have*

$$\begin{pmatrix} A & C \\ C & B \end{pmatrix} = \frac{1}{2}\left\{ U \begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} V^* \right\} \tag{4}$$

for some unitaries $U,V \in \mathbb{M}_{2n}(\mathbb{C})$.

THEOREM B [13, Theorem 1.1]. *Let* $\begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ *be a Hermitian positive semidefinite matrix. If, in addition, the block* C *is Hermitian then the following majorization inequality holds:*

$$\lambda \begin{pmatrix} A & C \\ C & B \end{pmatrix} \prec \lambda \begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix}. \tag{5}$$

Since all the involved matrices in (5) (except C) are positive semidefinite, inequality (5) implies that

$$\left\| \begin{pmatrix} A & C \\ C & B \end{pmatrix} \right\| \leqslant \left\| \begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix} \right\| \tag{6}$$

for any unitarily invariant norm $\|\cdot\|$ on $\mathbb{M}_{2n}(\mathbb{C})$. Inequality (6) is a 2×2 variant of a Hiroshima's result [11].

The case $C=0$ of Theorem B is shown in [1, Corollary 7.3].

The present paper aims to derive some refinements for the inequalities (5) and (6) as well as to give some corollaries.

3. Refinements of inequalities

In what follows, we study a real function ψ defined and Gateaux differentiable at least on the set $\mathbb{H}_{2n}(J)$ for a given interval $J \subset [0,\infty)$. More precisely, it is always assumed that there exists an open set $\mathbb{S} \subset \mathbb{H}_{2n}$ such that $\psi : \mathbb{S} \to \mathbb{R}$ and $\mathbb{H}_{2n}(J) \subset \mathbb{S}$, and that ψ is Gateaux differentiable on $\mathbb{S}$. Therefore there exists the gradient function $\nabla\psi(\cdot)$ on $\mathbb{S}$. For technical reasons, it is also assumed that this gradient function is continuous on $\mathbb{S}$.

Under the notation and terminology introduced in the previous section, we state and prove the following result. It gives a refinement of an inequality presented in the main result of [13].

THEOREM 1. *Let* $\begin{pmatrix} A & C \\ C & B \end{pmatrix} \in \mathbb{M}_{2n}^{+}(\mathbb{C})$ *with* $C=C^* \in \mathbb{M}_n(\mathbb{C})$ *so that* (4) *holds for some unitaries* $U,V \in \mathbb{U}_{2n}$, *and let* $A+B$ *have the eigenvalues in the interval* $J=[0,a)$ *for some* $0<a\leqslant\infty$.

Let ψ *be a real function defined and Gateaux differentiable on an open subset of* $\mathbb{H}_{2n}$ *including* $\mathbb{H}_{2n}(J)$. *Assume that* ψ *is* G*-increasing function on* $\mathbb{H}_{2n}(J)$ *with continuous gradient* $\nabla\psi(\cdot)$ *on* $\mathbb{H}_{2n}(J)$ *such that the function* $\frac{1}{2}\|\cdot\|_2^2-\psi$ *is* G*-increasing on* $\mathbb{H}_{2n}(J)$, *where* $G=\{U_1(\cdot)U_1^* : U_1 \in \mathbb{U}_{2n}\}$.

Then

$$\begin{aligned}&\lambda\begin{pmatrix}A & C\\ C & B\end{pmatrix}\\ &\prec\lambda\left(\frac{1}{2}U\begin{pmatrix}A+B & 0\\ 0 & 0\end{pmatrix}U^*+Z\right)+\lambda\left(\frac{1}{2}V\begin{pmatrix}0 & 0\\ 0 & A+B\end{pmatrix}V^*-Z\right)\\ &\prec\lambda\begin{pmatrix}A+B & 0\\ 0 & 0\end{pmatrix},\end{aligned}\tag{7}$$

where

$$Z=\nabla\psi\left(\frac{1}{2}V\begin{pmatrix}0 & 0\\ 0 & A+B\end{pmatrix}V^*\right)\quad or\quad Z=-\nabla\psi\left(\frac{1}{2}U\begin{pmatrix}A+B & 0\\ 0 & 0\end{pmatrix}U^*\right).$$

Proof. First of all, we shall prove that the matrix $\begin{pmatrix}A & C\\ C & B\end{pmatrix}$ belongs to the set $\mathbb{H}_{2n}(J)$.

It is obvious that $0\leqslant\lambda_i\begin{pmatrix}A & C\\ C & B\end{pmatrix}$ for all $i=1,\ldots,2n$. It remains to show that $\lambda_i\begin{pmatrix}A & C\\ C & B\end{pmatrix}\leqslant a$ for all $i=1,\ldots,2n$.

It follows from Theorem A via Ky Fan's inequality (1) that

$$\begin{aligned}\lambda\begin{pmatrix}A & C\\ C & B\end{pmatrix}&\prec\lambda\left(\frac{1}{2}U\begin{pmatrix}A+B & 0\\ 0 & 0\end{pmatrix}U^*\right)+\lambda\left(\frac{1}{2}V\begin{pmatrix}0 & 0\\ 0 & A+B\end{pmatrix}V^*\right)\\ &=\frac{1}{2}\lambda\begin{pmatrix}A+B & 0\\ 0 & 0\end{pmatrix}+\frac{1}{2}\lambda\begin{pmatrix}0 & 0\\ 0 & A+B\end{pmatrix}=\lambda\begin{pmatrix}A+B & 0\\ 0 & 0\end{pmatrix}.\end{aligned}$$

Hence,

$$\lambda_1\begin{pmatrix}A & C\\ C & B\end{pmatrix}\leqslant\lambda_1\begin{pmatrix}A+B & 0\\ 0 & 0\end{pmatrix}.$$

Therefore,

$$0\leqslant\lambda_{2n}\begin{pmatrix}A & C\\ C & B\end{pmatrix}\leqslant\ldots\leqslant\lambda_1\begin{pmatrix}A & C\\ C & B\end{pmatrix}\leqslant\lambda_1\begin{pmatrix}A+B & 0\\ 0 & 0\end{pmatrix}<a.$$

In consequence, $\begin{pmatrix}A & C\\ C & B\end{pmatrix}\in\mathbb{H}_{2n}(J)$, where $J=[0,a)$, as wanted.

We now consider the case $Z=\nabla\psi\left(\frac{1}{2}V\begin{pmatrix}0 & 0\\ 0 & A+B\end{pmatrix}V^*\right)$. We also denote

$$Y_1=\frac{1}{2}U\begin{pmatrix}A+B & 0\\ 0 & 0\end{pmatrix}U^*,$$

$$Y_2=\frac{1}{2}V\begin{pmatrix}0 & 0\\ 0 & A+B\end{pmatrix}V^*.$$

It is clear that $Z = \nabla\psi(Y_2)$ and

$$\nabla\left(\frac{1}{2}\|\cdot\|_2^2-\psi\right)(Y_2) = \nabla\left(\frac{1}{2}\|\cdot\|_2^2\right)(Y_2) - \nabla\psi(Y_2) = Y_2 - Z.$$

Because of the G-increase of the functions ψ and $\frac{1}{2}\|\cdot\|_2^2-\psi$, the matrices $Z = \nabla\psi(Y_2)$ and $Y_2 - Z = \nabla\left(\frac{1}{2}\|\cdot\|_2^2-\psi\right)(Y_2)$ are G-synchronous. In fact, with the notation $g = U_1(\cdot)U_1^*$ for any $U_1 \in \mathbb{U}_{2n}$, and $D = \{\operatorname{diag}(\lambda_1,\dots,\lambda_{2n}) : \lambda_1 \geqslant \dots \geqslant \lambda_{2n}\}$, we have

$$\nabla\psi(gD) = g\nabla\psi(D) \subset gD \tag{8}$$

(see [16, Theorem 2.1]). By Spectral Theorem, there exists a $g = U_1(\cdot)U_1^*$ such that $Y_2 = g\operatorname{diag}\lambda(Y_2) \in gD$. In consequence, by (8), $\nabla\psi(Y_2) \in gD$, which leads to the G-synchronicity of Y_2 and $Z = \nabla\psi(Y_2)$. Likewise, we get the G-synchronicity of Y_2 and $Y_2 - Z$. All of this yields the required G-synchronicity of Z and $Y_2 - Z$.

Therefore by (3) we have

$$\lambda(Y_2 - Z) = \lambda(Y_2) - \lambda(Z).$$

In light of Fan's inequality (1), we have

$$\lambda(Y_1+Y_2) = \lambda(Y_1+Z+Y_2-Z) \prec \lambda(Y_1+Z) + \lambda(Y_2-Z)$$

$$= \lambda(Y_1+Z) + \lambda(Y_2) - \lambda(Z) \prec \lambda(Y_1) + \lambda(Z) + \lambda(Y_2) - \lambda(Z) = \lambda(Y_1) + \lambda(Y_2).$$

Therefore we can write

$$\lambda\begin{pmatrix} A & C \\ C & B \end{pmatrix} = \lambda\left(\frac{1}{2}U\begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix}U^* + \frac{1}{2}V\begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix}V^*\right)$$

$$\prec \lambda\left(\frac{1}{2}U\begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix}U^* + Z\right) + \lambda\left(\frac{1}{2}V\begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix}V^* - Z\right)$$

$$\prec \lambda\left(\frac{1}{2}U\begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix}U^*\right) + \lambda\left(\frac{1}{2}V\begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix}V^*\right)$$

$$= \frac{1}{2}\lambda\left(U\begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix}U^*\right) + \frac{1}{2}\lambda\left(V\begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix}V^*\right)$$

$$= \lambda\begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix}.$$

This completes the proof of inequality (7) for $Z = \nabla\psi\left(\frac{1}{2}V\begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix}V^*\right)$.

The proof of (7) for $Z = -\nabla\psi\left(\frac{1}{2}U\begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix}U^*\right)$ is analogous as above. $\square$

COROLLARY 1. *Under the assumptions of Theorem 1, the following refinement of Hiroshima's inequality (6) holds:*

$$\begin{aligned}
&\left\| \begin{pmatrix} A & C \\ C & B \end{pmatrix} \right\| \\
&\leqslant \left\| \operatorname{diag}\left(\lambda\left(\frac{1}{2} U \begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix} U^* + Z \right) + \lambda\left(\frac{1}{2} V \begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} V^* - Z \right) \right) \right\| \\
&\leqslant \left\| \begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix} \right\| \qquad (9)
\end{aligned}$$

for any weakly unitarily invariant norm $\|\cdot\|$ *on* $\mathbb{H}_{2n}$, *where*

$$Z = \nabla\psi\left(\frac{1}{2} V \begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} V^* \right) \quad \text{or} \quad Z = -\nabla\psi\left(\frac{1}{2} U \begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix} U^* \right).$$

Proof. Inequality (9) is a corollary to (7) via the Spectral Theorem for Hermitian matrices and the weak unitary invariance of $\|\cdot\|$. □

COROLLARY 2. *Let* $\begin{pmatrix} A & C \\ C & B \end{pmatrix} \in \mathbb{M}_{2n}^{+}(\mathbb{C})$ *with* $C = C^* \in \mathbb{M}_n(\mathbb{C})$ *so that (4) holds for some unitaries* $U, V \in \mathbb{U}_{2n}$.

Let $0 \leqslant t \leqslant 1$. *Then*

$$\begin{aligned}
&\lambda \begin{pmatrix} A & C \\ C & B \end{pmatrix} \\
&\prec \lambda\left(\begin{pmatrix} A & C \\ C & B \end{pmatrix} - \frac{1-t}{2} V \begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} V^* \right) + \frac{1-t}{2} \lambda \begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} \\
&\prec \lambda \begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix}. \qquad (10)
\end{aligned}$$

Proof. We set $J = [0, \infty)$ and $\psi = t\frac{1}{2}\|\cdot\|_2^2$ on $\mathbb{H}_{2n}$. This is a G-increasing function on $\mathbb{H}_{2n}$. It is not hard to check that $\nabla\psi(T) = tT$ for $T \in \mathbb{H}_{2n}$. Therefore,

$$Z = \nabla\psi\left(\frac{1}{2} V \begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} V^* \right) = \frac{t}{2} V \begin{pmatrix} 0 & 0 \\ 0 & A+B \end{pmatrix} V^*.$$

By making use inequality (9) in Theorem 1, we get (10). □

Let $X, Y \in \mathbb{M}_n(\mathbb{C})$ with $X^*Y = Y^*X$. Then the matrix

$$\begin{pmatrix} X^* & 0 \\ Y^* & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} X^*X & X^*Y \\ X^*Y & Y^*Y \end{pmatrix}$$

is positive semidefinite with X^*Y Hermitian.

By Theorem A, there exist unitaries $U, V \in \mathbb{U}_{2n}$ such that

$$\begin{pmatrix} X^*X & X^*Y \\ X^*Y & Y^*Y \end{pmatrix} = \frac{1}{2} U \begin{pmatrix} X^*X + Y^*Y & 0 \\ 0 & 0 \end{pmatrix} U^* + \frac{1}{2} V \begin{pmatrix} 0 & 0 \\ 0 & X^*X + Y^*Y \end{pmatrix} V^*. \qquad (11)$$

The below result is related to [13, Corollary 2.2].

COROLLARY 3. *Let $X, Y \in \mathbb{M}_n(\mathbb{C})$ with $X^*Y = Y^*X$ so that (11) holds for some unitaries $U, V \in \mathbb{U}_{2n}$.*

Let $\psi : \mathbb{H}_{2n} \to \mathbb{R}$ be a Gateaux differentiable G-increasing function with continuous gradient $\nabla\psi(\cdot)$ on $\mathbb{H}_{2n}$ such that the function $\frac{1}{2}\|\cdot\|_2^2 - \psi$ is G-increasing, where $G = \{U_1(\cdot)U_1^* : U_1 \in \mathbb{U}_{2n}\}$.

Then

$$\lambda\begin{pmatrix} XX^*+YY^* & 0 \\ 0 & 0 \end{pmatrix}$$
$$\prec \lambda\left(\frac{1}{2}U\begin{pmatrix} X^*X+Y^*Y & 0 \\ 0 & 0 \end{pmatrix}U^* + Z\right) + \lambda\left(\frac{1}{2}V\begin{pmatrix} 0 & 0 \\ 0 & X^*X+Y^*Y \end{pmatrix}V^* - Z\right)$$
$$\prec \lambda\begin{pmatrix} X^*X+Y^*Y & 0 \\ 0 & 0 \end{pmatrix}, \tag{12}$$

where

$$Z = \nabla\psi\left(\frac{1}{2}V\begin{pmatrix} 0 & 0 \\ 0 & X^*X+Y^*Y \end{pmatrix}V^*\right) \text{ or } Z = -\nabla\psi\left(\frac{1}{2}U\begin{pmatrix} X^*X+Y^*Y & 0 \\ 0 & 0 \end{pmatrix}U^*\right).$$

Proof. We put $J = [0,\infty)$. It is sufficient to employ inequality (7) in Theorem 1 for $A = X^*X$, $B = Y^*Y$ and $C = X^*Y$ with the equalities

$$\lambda\begin{pmatrix} X^*X & X^*Y \\ X^*Y & Y^*Y \end{pmatrix} = \lambda\left(\begin{pmatrix} X^* & 0 \\ Y^* & 0 \end{pmatrix}\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}\right)$$
$$= \lambda\left(\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} X^* & 0 \\ Y^* & 0 \end{pmatrix}\right) = \lambda\begin{pmatrix} XX^*+YY^* & 0 \\ 0 & 0 \end{pmatrix}. \quad \square$$

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REFERENCES

[1] T. ANDO, *Majorization, doubly stochastic matrices, and comparison of eigenvalues*, Linear Algebra Appl. **118**, (1989), 163–248.

[2] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[3] J. C. BOURIN AND F. HIAI, *Norm and anti-norm inequalities for positive semi-definite matrices*, Internat. J. Math. **22**, (2011), 1121–1138.

[4] J. C. BOURIN AND E.-T. LEE, *Unitary orbits of Hermitian operators with convex and concave functions*, Bull. London Math. Soc. **44**, (2012), 1085–1102.

[5] J. C. BOURIN AND E.-T. LEE, *Decomposition and partial trace of positive matrices with Hermitian blocks*, Internat. J. Math. **24**, (01) (2013), 1350010.

[6] J. C. BOURIN, E.-T. LEE AND M. LIN, *On a decomposition lemma for positive semi-definite block-matrices*, Linear Algebra Appl. **437**, (2012), 1906–1912.

[7] J. C. BOURIN, E.-T. LEE AND M. LIN, *Positive matrices partitioned into a small number of Hermitian blocks*, Linear Algebra Appl. **438**, (2013), 2591–2598.

[8] K. FAN, *On a theorem of Weyl concerning eigenvalues of linear transformations I*, Proc. Nat. Acad. Sci. U.S.A. **35**, (1949), 652–655.

[9] K. FAN, *Maximum properties and inequalities for the eigenvalues of completely continuous operators*, Proc. Nat. Acad. Sci. U.S.A. **37**, (1951), 760–766.

[10] M. S. GOWDA, J. JEONG, *Commutation principles in Euclidean Jordan algebras and normal decomposition systems*, SIAM J. Optim. **27**, (2017), 1390–1402.

[11] T. HIROSHIMA, *Majorization criterion for distillability of a bipartite quantum state*, Phys. Rev. Lett. **91**, 5 (2003), 057902.

[12] M. LIN AND H. WOLKOWICZ, *Hiroshima's theorem and matrix norm inequalities*, Acta Sci. Math. (Szeged) **81**, 12 (2015), 45–53.

[13] M. LIN AND H. WOLKOWICZ, *An eigenvalue majorization inequality for positive semidefinite block matrices*, Linear Multilinear Algebra **60**, 11–12 (2012), 1365–1368.

[14] A. W. MARSHALL, I. OLKIN AND B. C. ARNOLD, *Inequalities: Theory of Majorization and Its Applications*, 2nd ed., Springer, New York, 2011.

[15] M. NIEZGODA, *Extended commutation principles for normal decomposition systems*, Linear Algebra Appl. **539**, (2018), 251–273.

[16] M. NIEZGODA, *An extension of Schur-Ostrowski's condition, weak Eaton triples and generalized AI functions*, Linear Algebra Appl. **580**, (2019), 212–235.

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