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A NON–UNITAL GENERALIZED TRACE AND LINEAR COMPLEX STRUCTURES

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Abstract. A basis-free formula for the generalized trace of a linear map between tensor products of vector spaces is proposed which does not refer to scalar multiplication or scalar valued functions. The main application is to real vector spaces with complex structure operators.

1. Introduction

Defining the trace of a square matrix as the sum of its diagonal entries is simple and useful, and generalizes to the contraction operation on multi-indexed tensors as a sum over repeated indices. It is then well-known that the trace is an invariant quantity under change of basis. So, for a finite dimensional vector space V, the trace $Tr_V(A)$ of a linear map $A : V \to V$ is independent of any matrix representation for A. There are other ways to compute, or define, $Tr_V(A)$ that do not require any initial choice of basis; we will recall one such formula in Proposition 2.11, after developing some notation for a more abstract approach to linear algebra as in [4]. An abstract analogue of tensor contraction is the generalized trace, which takes as input a linear map between tensor products, $A : V \otimes U \to V \otimes W$, and returns as output a linear map $U \to W$; Proposition 2.12 shows how a basis-free definition coincides with the repeated index summation.

The abstract, basis-free approach to the trace and generalized trace is well-known in category theory, because it can be adapted to define a trace of a morphism, in categories that have enough structure in common with the category of finite dimensional vector spaces. For example, a generalized trace can be defined in some monoidal categories ([2], [7], [9]), where for two objects U and V there is another object $U \otimes V$, subject to certain properties including a notion of associativity and the existence of a unit object $\mathbb{K}$ so that $V \otimes \mathbb{K}$ is isomorphic to V.

This article, in Section 3, will propose new abstract formulas for the generalized trace in the category of vector spaces (Theorem 3.7, Proposition 3.8). The novelty is that the formulas do not rely on the existence of, or a choice of, any unit object for tensor products. So in addition to being basis-free, the formulation will be scalar-free, and adaptable to some other categories without a unit object for $\otimes$ (a "semigroup category" as in Remark 2.2.9 of [7], or [10]), although we are going to focus on linear

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algebra rather than going into any depth in category theory, tensor analysis, or other applications. We will consider vector spaces over a fixed field; the main results of Section 3 show that the claimed formulas for the generalized trace coincide with the usual coordinate-based notion.

Section 4 presents the main motivating example of the general framework set up in Section 3, a category where the objects are real vector spaces with complex structure operators (real linear maps J with $J \circ J = -Id$), and the morphisms are real linear maps that are compatible with the complex structures. Applying the ideas of Sections 2 and 3 to this category was originally motivated by differential-geometric calculations in vector bundles (as in [15], [17]), where J can vary continuously from point to point, and wanting to find a way to define a complex trace in almost complex geometry that relies only on J and the real linear structure at each point.

A previously known matrix formula that will be generalized in Section 5 is:

$$Tr_{\mathbf{V}}(\mathbf{A}) = \frac{1}{2}\left(Tr_V(A) - \sqrt{-1}Tr_V(A \circ J)\right), \tag{1.1}$$

from ([15], [17] Chapter 5). The LHS is a complex number valued trace of a complex linear ($\mathbf{A} \circ J = J \circ \mathbf{A}$) map $V \to V$ with a size $N \times N$ complex matrix representation, while the real valued RHS traces are applied to the same transformation considered as a real linear map A, with $2N \times 2N$ real matrix representations for A and $A \circ J$. Equation (1.1) is the last time any complex number appears in this article; in Sections 2 and 3, the scalars are from an arbitrary field $\mathbb{K}$, and in Sections 4 and 5, the scalar field is $\mathbb{R}$ and all maps are $\mathbb{R}$-linear.

As a remark to conclude the Introduction, the methods used here are elementary but maybe becoming old-fashioned; commutative diagrams, sums over repeated indices, and in particular calculations involving a trace, can be more graphically presented by various types of "string diagrams" as in [3], [5], [9], [10], [12], [13], [14], [16].

2. Notation

In this Section we recall some already known formulas for the trace, after developing enough notation to state them, and check that they coincide with the classical summations after choosing a basis. Fix a field $\mathbb{K}$, and consider vector spaces with scalars $\mathbb{K}$. The same symbol $\mathbb{K}$ denotes itself considered as a one-dimensional vector space.

NOTATION 2.1. For vector spaces U and V over $\mathbb{K}$, the vector space of all $\mathbb{K}$-linear maps from U to V is denoted $\mathrm{Hom}(U,V)$, and the term *map* will always refer to a $\mathbb{K}$-linear map. As a special case, we abbreviate $\mathrm{Hom}(V,\mathbb{K})$, the dual space of V, by V^*. For maps $A : U' \to U$ and $B : V \to V'$, the map denoted $\mathrm{Hom}(A,B) : \mathrm{Hom}(U,V) \to \mathrm{Hom}(U',V')$ acts on $F : U \to V$ so that $\mathrm{Hom}(A,B) : F \mapsto B \circ F \circ A : U' \to V'$.

NOTATION 2.2. The space $U \otimes V$ is the tensor product over $\mathbb{K}$, spanned by elements of the form $\vec{u} \otimes \vec{v}$ for $\vec{u} \in U$ and $\vec{v} \in V$. The products $U \otimes (V \otimes W)$ and

$(U \otimes V) \otimes W$ will be identified with each other and with the triple $U \otimes V \otimes W$, so that these elements are equal:

$$(\vec{u} \otimes \vec{v}) \otimes \vec{w} = \vec{u} \otimes (\vec{v} \otimes \vec{w}) = \vec{u} \otimes \vec{v} \otimes \vec{w}.$$

NOTATION 2.3. The invertible maps $V \otimes \mathbb{K} \to V$ and $\mathbb{K} \otimes V \to V$ corresponding to scalar multiplication are denoted ℓ. The space $\mathbb{K}$, taken together with these isomorphisms, is a (two-sided) *unit* for the tensor product operation.

NOTATION 2.4. For any product $V \otimes U$, the *switching* map $s : V \otimes U \to U \otimes V$ is linear and defined on elements of the form $\vec{v} \otimes \vec{u}$ by the formula $s : \vec{v} \otimes \vec{u} \mapsto \vec{u} \otimes \vec{v}$.

NOTATION 2.5. For maps $A : U_1 \to U_2$, $B : V_1 \to V_2$, denote (with square brackets) the map

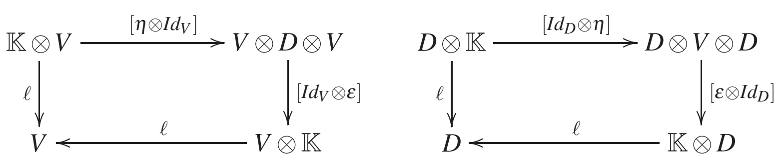
$$[A \otimes B] : U_1 \otimes V_1 \to U_2 \otimes V_2$$

defined by acting on elements of the form $\vec{u} \otimes \vec{v}$ by $[A \otimes B] : \vec{u} \otimes \vec{v} \mapsto (A(\vec{u})) \otimes (B(\vec{v}))$.

NOTATION 2.6. For any vector space V, there is a canonical identity map $Id_V \in \mathrm{Hom}(V,V)$, so that $Id_V(\vec{v}) = \vec{v}$.

NOTATION 2.7. For any vector space V, the operation of applying a linear map $\phi \in V^*$ to a vector $\vec{v} \in V$ to get the scalar $\phi(\vec{v}) \in \mathbb{K}$ is bilinear in the pair $(\phi, \vec{v})$. This bilinear pairing defines a canonical evaluation map $Ev_V : V^* \otimes V \to \mathbb{K}$, acting on elements of the form $\phi \otimes \vec{v}$ by $Ev_V : \phi \otimes \vec{v} \mapsto \phi(\vec{v})$.

DEFINITION 2.8. A vector space V is *dualizable* means: there exists (D, ε, η), where D is a vector space, and $\varepsilon : D \otimes V \to \mathbb{K}$ and $\eta : \mathbb{K} \to V \otimes D$ are linear maps such that the following diagrams are commutative.



REMARK 2.9. In category theory and other areas using versions of this construction ([12], [16]), ε is called an *evaluation* map and η a *coevaluation* map. A more general notion, with left and right duals, is considered by [11].

PROPOSITION 2.10. *If V is finite dimensional then V is dualizable.*

Proof. An example of a triple of duality data is $D = V^*$, $\varepsilon = Ev_V$, and η chosen in the following way. Let $(\vec{v}_1, \ldots, \vec{v}_N)$ be a basis of V, and let $(\phi_1, \ldots, \phi_N)$ be the dual basis so that $\phi_i(\vec{v}_j) = \delta_{ij}$. Then consider the following specific candidate for a coevaluation, η_V, defined for $\alpha \in \mathbb{K}$ by

$$\eta_V : \alpha \mapsto \alpha \cdot (\vec{v}_1 \otimes \phi_1 + \cdots + \vec{v}_N \otimes \phi_N). \tag{2.1}$$

Checking that if $\eta = \eta_V$, then the identities from Definition 2.8 are satisfied is then straightforward, using methods similar to the sums in the next Proof. □

PROPOSITION 2.11. *If V is finite dimensional, then for any map $A : V \to V$, and any duality data (D, ε, η), the image of $1 \in \mathbb{K}$ under the following composite map:*

$$\mathbb{K} \xrightarrow{\eta} V \otimes D \xrightarrow{s} D \otimes V \xrightarrow{[Id_D \otimes A]} D \otimes V \xrightarrow{\varepsilon} \mathbb{K} \tag{2.2}$$

is an element of $\mathbb{K}$ that depends only on A and not on the choice of (D, ε, η).

Proof. There exists some (D, ε, η) by Proposition 2.10, but the triple need not be unique. Let $(\vec{v}_1, \ldots, \vec{v}_N)$ be a basis of V, and let $\{\varphi_j : j \in \mathbf{J}\}$ be a basis for D, with index set $\mathbf{J}$, so that the set $\{\vec{v}_i \otimes \varphi_j : i = 1, \ldots, N, j \in \mathbf{J}\}$ is a basis of $V \otimes D$. There are finitely many coefficients η_{ij} so that for $\alpha \in \mathbb{K}$,

$$\eta : \alpha \mapsto \alpha \cdot \sum_{i=1}^{N} \sum_{j=1}^{M} \eta_{ij} \vec{v}_i \otimes \varphi_j.$$

For any $i = 1, \ldots, N$, $j \in \mathbf{J}$, there is some scalar ε_{ji} so that $\varepsilon : \varphi_j \otimes \vec{v}_i \mapsto \varepsilon_{ji}$. The hypothesis that ε and η are an evaluation and coevaluation implies that for any $\alpha \in \mathbb{K}$ and basis element $\vec{v}_{i'}$,

$$\begin{aligned}
\ell(\alpha \otimes \vec{v}_{i'}) &= (\ell \circ [Id_V \otimes \varepsilon] \circ [\eta \otimes Id_V])(\alpha \otimes \vec{v}_{i'}) \\
\alpha \vec{v}_{i'} &= \ell\left([Id_V \otimes \varepsilon]\left(\left(\alpha \sum_{i=1}^{N} \sum_{j=1}^{M} \eta_{ij} \vec{v}_i \otimes \varphi_j\right) \otimes \vec{v}_{i'}\right)\right) \\
&= \ell\left(\alpha \sum_{i=1}^{N} \sum_{j=1}^{M} \eta_{ij} \vec{v}_i \otimes \varepsilon_{ji'}\right) = \alpha \sum_{i=1}^{N} \sum_{j=1}^{M} \eta_{ij} \varepsilon_{ji'} \vec{v}_i,
\end{aligned}$$

so

$$\sum_{j=1}^{M} \eta_{ij} \varepsilon_{ji'} = \delta_{ii'}. \tag{2.3}$$

This uses only the first diagram from Definition 2.8. Equation (2.3) can be interpreted as a one-sided matrix inverse property.

With respect to the chosen basis for V, there are coefficients $A_{i'i}$ so that

$$A(\vec{v}_i) = \sum_{i'=1}^{N} A_{i'i} \vec{v}_{i'},$$

and the image of 1 under the map (2.2) is:

$$\begin{aligned}
\varepsilon([Id_D \otimes A](s(\eta(1)))) &= \varepsilon\left([Id_D \otimes A]\left(\sum_{i=1}^{N}\sum_{j=1}^{M}\eta_{ij}\varphi_j \otimes \vec{v}_i\right)\right) \\
&= \varepsilon\left(\sum_{i=1}^{N}\sum_{j=1}^{M}\eta_{ij}\varphi_j \otimes \left(\sum_{i'=1}^{n} A_{i'i}\vec{v}_{i'}\right)\right) \\
&= \sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{i'=1}^{N}\eta_{ij}A_{i'i}\varepsilon_{ji'} \\
&= \sum_{i=1}^{N}\sum_{i'=1}^{N}\delta_{ii'}A_{i'i} = \sum_{i=1}^{N}A_{ii},
\end{aligned}$$

which is the trace of $A : V \to V$, denoted $Tr_V(A)$, and, as mentioned previously, does not depend on the choice of basis for V. $\square$

The above formula (2.2) for the trace, and the following formula (2.4) for the generalized trace, are well-known, although different authors ([9], [13]) may use different orderings for the products of spaces or composites of maps.

PROPOSITION 2.12. *If V is finite dimensional, then for any vector spaces U and W, and any map $A : V \otimes U \to V \otimes W$, and any duality data (D, ε, η) for V, the following composite map from U to W:*

$$\begin{array}{ccccc}
V \otimes D \otimes U & \xrightarrow{[s \otimes Id_U]} & D \otimes V \otimes U & \xrightarrow{[Id_D \otimes A]} & D \otimes V \otimes W \\
\uparrow{\scriptstyle [\eta \otimes Id_U]} & & & & \downarrow{\scriptstyle [\varepsilon \otimes Id_W]} \\
\mathbb{K} \otimes U & & & & \mathbb{K} \otimes W \\
\uparrow{\scriptstyle \ell^{-1}} & & & & \downarrow{\scriptstyle \ell} \\
U & & & & W
\end{array} \tag{2.4}$$

is an element of $\mathrm{Hom}(U, W)$ *that depends only on A and not on the choice of* (D, ε, η).

Proof. Using the same notation and basis as in the previous Proof for V and D, let $\{\vec{u}_l : l \in \mathbf{L}\}$ be a basis for U and let $\{\vec{w}_k : k \in \mathbf{K}\}$ be a basis for W. For each basis element $\vec{v}_i \otimes \vec{u}_l$ of $V \otimes U$, there are coefficients $A_{i'kil}$ (finitely many non-zero for each l) so that

$$A(\vec{v}_i \otimes \vec{u}_l) = \sum_{k \in \mathbf{K}}\sum_{i'=1}^{N} A_{i'kil}\vec{v}_{i'} \otimes \vec{w}_k. \tag{2.5}$$

The above composite map (2.4) then maps basis element $\vec{u}_l$ to:

$$\begin{aligned}
&\ell\circ[\varepsilon\otimes Id_W]\circ[Id_D\otimes A]\circ[s\otimes Id_U]\circ[\eta\otimes Id_U]\circ\ell^{-1}:\\
\vec{u}_l \mapsto\ &\ell([\varepsilon\otimes Id_W]([Id_D\otimes A]([(s\circ\eta)\otimes Id_U](1\otimes\vec{u}_l))))\\
&=\ell\left([\varepsilon\otimes Id_W]\left([Id_D\otimes A]\left(\left(\sum_{i=1}^{N}\sum_{j=1}^{M}\eta_{ij}\varphi_j\otimes\vec{v}_i\right)\otimes\vec{u}_l\right)\right)\right)\\
&=\ell\left([\varepsilon\otimes Id_W]\left(\sum_{i=1}^{N}\sum_{j=1}^{M}\eta_{ij}\varphi_j\otimes\left(\sum_{k\in\mathbf{K}}\sum_{i'=1}^{N}A_{i'kil}\vec{v}_{i'}\otimes\vec{w}_k\right)\right)\right)\\
&=\ell\left(\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k\in\mathbf{K}}\sum_{i'=1}^{N}\eta_{ij}A_{i'kil}\left(\varepsilon_{ji'}\otimes\vec{w}_k\right)\right)\\
&=\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{k\in\mathbf{K}}\sum_{i'=1}^{N}\eta_{ij}A_{i'kil}\varepsilon_{ji'}\vec{w}_k=\sum_{i=1}^{N}\sum_{k\in\mathbf{K}}\sum_{i'=1}^{N}\delta_{ii'}A_{i'kil}\vec{w}_k\\
&=\sum_{k\in\mathbf{K}}\sum_{i=1}^{N}A_{ikil}\vec{w}_k. \qquad (2.6)
\end{aligned}$$

The last sum (2.6) is an element of W depending on A and the input $\vec{u}_l$ from U but not on (D,ε,η). The map (2.4) is denoted $Tr_{V;U,W}(A):U\to W$, the generalized trace of A. $\square$

Neither of the above formulas (2.4) nor (2.6) for the generalized trace requires U or W to have finite dimension. This generalized trace $Tr_{V;U,W}$ is, in some applications, also called a partial trace ([5], [8]) or twisted trace ([13]). As mentioned in the Introduction, both the abstract formulation of the trace $Tr_{V;U,W}$ from the statement of Proposition 2.12, and multi-indexed summations such as those in its Proof, appear in calculations in local differential geometry and other applications of tensor analysis.

EXAMPLE 2.13. The scalar valued trace of a map $A:V\to V$ is related to a generalized trace by the following formula, using $\ell:V\otimes\mathbb{K}\to V$:

$$Tr_{V;\mathbb{K},\mathbb{K}}(\ell^{-1}\circ A\circ\ell)=Tr_V(A)\cdot Id_{\mathbb{K}}:1\mapsto Tr_V(A). \qquad (2.7)$$

It is easy to check, without choosing any basis, that the map (2.2) from Proposition 2.11 is the same as the special case $U=W=\mathbb{K}$ of the map (2.4) from Proposition 2.12 applied to $\ell^{-1}\circ A\circ\ell$.

3. A new formula for the generalized trace

Our main goal is to state an expression equal to the generalized trace, (2.6), in a basis-free way analogous to the expression (2.4), but which uses an abstract notion of dualizability that does not refer to the space of scalars $\mathbb{K}$.

One approach might be to just replace each column of vertical arrows in the diagram (2.4) by the corresponding composite to get abstract maps $U\to V\otimes D\otimes U$ and

$D \otimes V \otimes W \to W$; this bypasses the steps where $\mathbb{K}$ appears, but the notion of dualizability in Definition 2.8 would then need to be adjusted to take into account U and W, in addition to V and D. In fact, this is the general idea, but the construction in Theorem 3.7 will be organized differently, motivated in part by the existence and convenience of the following canonical map.

NOTATION 3.1. For any vector spaces V and W, there is a distinguished map

$$Ev_{VW} : \mathrm{Hom}(V,W) \otimes V \to W,$$

defined on elements of the form $A \otimes \vec{v}$ by evaluation:

$$Ev_{VW} : A \otimes \vec{v} \mapsto A(\vec{v}).$$

This generalizes the construction from Notation 2.7: in the $W = \mathbb{K}$ case, $Ev_{V\mathbb{K}}$ is the distinguished element $Ev_V \in (V^* \otimes V)^*$. We want to generalize further, from the canonical map Ev_{VW} to a more abstract evaluation map $\varepsilon : \mathrm{Hom}(X,W) \otimes V \to W$, where $\mathrm{Hom}(X,-)$ plays the role of $D \otimes -$ appearing in Definition 2.8 and Proposition 2.12. The canonical evaluation maps have some elementary properties as in the following Lemmas, one of which (Lemma 3.6) we will also want to generalize to the abstract evaluation maps.

LEMMA 3.2. *For any vector spaces U, V, W, and any map $B : U \to W$, the following diagram is commutative.*

$$\begin{array}{ccc} U & \xrightarrow{\quad B \quad} & W \\ \uparrow{\scriptstyle Ev_{VU}} & & \uparrow{\scriptstyle Ev_{VW}} \\ \mathrm{Hom}(V,U) \otimes V & \xrightarrow[{[\mathrm{Hom}(Id_V,B) \otimes Id_V]}]{} & \mathrm{Hom}(V,W) \otimes V \end{array}$$

Proof. Both paths take $A \otimes \vec{v} \in \mathrm{Hom}(V,U) \otimes V$ to $B(A(\vec{v}))$. $\square$

One more bit of notation will be needed for Theorem 3.7.

NOTATION 3.3. For any vector spaces U, V, X, there is a canonical map

$$n_U : V \otimes \mathrm{Hom}(X,U) \to \mathrm{Hom}(X, V \otimes U) \tag{3.1}$$

defined on elements of the form $\vec{v} \otimes A \in V \otimes \mathrm{Hom}(X,U)$ and $\vec{x} \in X$ by:

$$n_U(\vec{v} \otimes A) : \vec{x} \mapsto \vec{v} \otimes (A(\vec{x})).$$

Analogously (and equally canonically except the ordering of spaces is different), define for any W,

$$n_W : \mathrm{Hom}(X,W) \otimes V \to \mathrm{Hom}(X, V \otimes W) \tag{3.2}$$

so that

$$n_W(B \otimes \vec{v}) : \vec{x} \mapsto \vec{v} \otimes (B(\vec{x})).$$

LEMMA 3.4. *If X or V is finite dimensional, then n_U and n_W are invertible.*

Proof. We refer to [1] §20, or [4] §II.7.7. □

LEMMA 3.5. *For any U, V, W, the following diagram is commutative.*

$$\begin{array}{ccc} U\otimes \mathrm{Hom}(V,W)\otimes V & \xrightarrow{[Id_U\otimes Ev_{VW}]} & U\otimes W \\ \Big\downarrow{\scriptstyle [n_1\otimes Id_V]} & \nearrow{\scriptstyle Ev_{V,U\otimes W}} & \\ \mathrm{Hom}(V,U\otimes W)\otimes V & & \end{array}$$

Proof. The n_1 map is a version of (3.1) from Notation 3.3. Both paths in the diagram take an element of the form $\vec{u}\otimes A\otimes\vec{v}\in U\otimes\mathrm{Hom}(V,W)\otimes V$ to $\vec{u}\otimes(A(\vec{v}))\in U\otimes W$. □

LEMMA 3.6. *For any vector spaces V, U, W, and any map $F: V\otimes U\to V\otimes W$, if V is finite dimensional then the n maps in the following diagram are invertible:*

$$\begin{array}{ccc} V\otimes U & \xrightarrow{F} & V\otimes W \\ \Big\uparrow{\scriptstyle [Id_V\otimes Ev_{VU}]} & & \Big\uparrow{\scriptstyle [Id_V\otimes Ev_{VW}]} \\ V\otimes\mathrm{Hom}(V,U)\otimes V & & V\otimes\mathrm{Hom}(V,W)\otimes V \\ \Big\downarrow{\scriptstyle [n_2\otimes Id_V]} & & \Big\downarrow{\scriptstyle [n_3\otimes Id_V]} \\ \mathrm{Hom}(V,V\otimes U)\otimes V & \xrightarrow[{[\mathrm{Hom}(Id_V,F)\otimes Id_V]}]{} & \mathrm{Hom}(V,V\otimes W)\otimes V \end{array}$$

and the diagram is commutative, in the sense that

$$\begin{aligned} & F\circ[Id_V\otimes Ev_{VU}]\circ[n_2\otimes Id_V]^{-1} \\ = & [Id_V\otimes Ev_{VW}]\circ[n_3\otimes Id_V]^{-1}\circ[\mathrm{Hom}(Id_V,F)\otimes Id_V]. \end{aligned}$$

Proof. The n_2, n_3 maps are versions of (3.1) from Notation 3.3; they are invertible by Lemma 3.4, and of course the inverse of $[n_2\otimes Id_V]$ is $[(n_2)^{-1}\otimes Id_V]$. By Lemma 3.5, the upward composite on the left, $[Id_V\otimes Ev_{VU}]\circ[n_2\otimes Id_V]^{-1}$, is equal to $Ev_{V,V\otimes U}$, and similarly the upward composite on the right is equal to $Ev_{V,V\otimes W}$. The claim then follows from Lemma 3.2. □

The following Theorem is the main result of this Section. It gives a formula for the generalized trace, in terms of abstractly defined evaluation and coevaluation maps. There is no reference to the scalar field $\mathbb{K}$, but the trade-off is that instead of one evaluation map ε for a given space V as in Definition 2.8, there are different evaluation maps ε^U and ε^W corresponding to pairs (V,U) and (V,W), and that satisfy a certain compatibility condition (3.4) analogous to the property of Ev_{VU} and Ev_{VW} from Lemma 3.6. There is also a twisted coevaluation map η^U which is a generalization of the η from Definition 2.8.

THEOREM 3.7. *Given vector spaces U, V, W, suppose there exist a vector space X and maps*

$$\begin{aligned} \eta^U &: U \to V \otimes \mathrm{Hom}(X,U) \\ \varepsilon^U &: \mathrm{Hom}(X,U) \otimes V \to U \\ \varepsilon^W &: \mathrm{Hom}(X,W) \otimes V \to W \end{aligned}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} U \otimes V & \xrightarrow{[\eta^U \otimes Id_V]} & V \otimes \mathrm{Hom}(X,U) \otimes V \\ & \searrow^{s} & \downarrow [Id_V \otimes \varepsilon^U] \\ & & V \otimes U \end{array} \tag{3.3}$$

meaning that the composite is equal to a switching map:

$$[Id_V \otimes \varepsilon^U] \circ [\eta^U \otimes Id_V] = s : U \otimes V \to V \otimes U.$$

Suppose further that V and X are both finite dimensional, so that the n maps in the following diagram are invertible, and that the diagram is commutative for any $F : V \otimes U \to V \otimes W$,

$$\begin{array}{ccc} V \otimes U & \xrightarrow{F} & V \otimes W \\ \uparrow [Id_V \otimes \varepsilon^U] & & \uparrow [Id_V \otimes \varepsilon^W] \\ V \otimes \mathrm{Hom}(X,U) \otimes V & & V \otimes \mathrm{Hom}(X,W) \otimes V \\ \downarrow [n_U \otimes Id_V] & & \downarrow [n_4 \otimes Id_V] \\ \mathrm{Hom}(X, V \otimes U) \otimes V & \xrightarrow[{[\mathrm{Hom}(Id_X,F) \otimes Id_V]}]{} & \mathrm{Hom}(X, V \otimes W) \otimes V \end{array} \tag{3.4}$$

in the sense that

$$F \circ [Id_V \otimes \varepsilon^U] \circ [n_U \otimes Id_V]^{-1} = [Id_V \otimes \varepsilon^W] \circ [n_4 \otimes Id_V]^{-1} \circ [\mathrm{Hom}(Id_X, F) \otimes Id_V].$$

Then the canonical map

$$n_W : \mathrm{Hom}(X,W) \otimes V \to \mathrm{Hom}(X, V \otimes W)$$

is also invertible, and for any $A : V \otimes U \to V \otimes W$, the composite map clockwise from U to W in the following diagram depends only on A and not on $(X, \eta^U, \varepsilon^U, \varepsilon^W)$.

$$\begin{array}{ccc} \mathrm{Hom}(X, V \otimes U) & \xrightarrow{\mathrm{Hom}(Id_X, A)} & \mathrm{Hom}(X, V \otimes W) \\ \uparrow n_U & & \downarrow n_W^{-1} \\ V \otimes \mathrm{Hom}(X,U) & & \mathrm{Hom}(X,W) \otimes V \\ \uparrow \eta^U & & \downarrow \varepsilon^W \\ U & \xrightarrow[Tr_{V;U,W}(A)]{} & W \end{array} \tag{3.5}$$

The diagram is commutative, so the composite map is equal to the generalized trace:

$$Tr_{V;U,W}(A) = \varepsilon^W \circ n_W^{-1} \circ \mathrm{Hom}(Id_X, A) \circ n_U \circ \eta^U : U \to W. \tag{3.6}$$

Proof. We start with some remarks before the calculation proving the claims. The diagram (3.3) with the abstract evaluation ε^U and abstract coevaluation η^U is analogous to the first diagram from Definition 2.8, and like the Proof of Proposition 2.11, this Theorem does not need an analogue of the second diagram from Definition 2.8. The diagram (3.4) is a generalization of the property of the canonical evaluation maps from Lemma 3.6, so that the two abstract evaluations are suitably compatible. All the n maps are invertible by Lemma 3.4.

The following steps use the same notation for the basis sets of V, U, and W as in Proposition 2.11 and Proposition 2.12, which assumed only that V has finite dimension. Now assume X has finite dimension, with basis $\{\vec{x}_q, q = 1, \ldots, Q\}$; the finite dimension also allows the existence of a basis set for $\mathrm{Hom}(X,U)$ of the form $\{\Phi_{ql} : q = 1, \ldots, Q, l \in \mathbf{L}\}$ where each basis element is defined by

$$\Phi_{ql} : \vec{x}_{q'} \mapsto \delta_{q'q}\vec{u}_l. \tag{3.7}$$

Then, for each basis element $\vec{u}_{l'}$ of U, there are coefficients $\eta^U_{iqll'}$ (finitely many non-zero for each l') so that

$$\eta^U : \vec{u}_{l'} \mapsto \sum_{i=1}^{N} \sum_{q=1}^{Q} \sum_{l \in \mathbf{L}} \eta^U_{iqll'} \vec{v}_i \otimes \Phi_{ql}.$$

For each basis element $\Phi_{ql} \otimes \vec{v}_i \in \mathrm{Hom}(X,U) \otimes V$, there are coefficients $\varepsilon^U_{l'qli}$ (finitely many non-zero for each l) so that

$$\varepsilon^U : \Phi_{ql} \otimes \vec{v}_i \mapsto \sum_{l' \in \mathbf{L}} \varepsilon^U_{l'qli} \vec{u}_{l'}.$$

The hypothesis (3.3) then gives this equality for any basis element $\vec{u}_{l'} \otimes \vec{v}_{i'}$ of $U \otimes V$:

$$\begin{aligned} s(\vec{u}_{l'} \otimes \vec{v}_{i'}) &= ([Id_V \otimes \varepsilon^U] \circ [\eta^U \otimes Id_V])(\vec{u}_{l'} \otimes \vec{v}_{i'}) \\ \vec{v}_{i'} \otimes \vec{u}_{l'} &= [Id_V \otimes \varepsilon^U] \left(\left(\sum_{i=1}^{N} \sum_{q=1}^{Q} \sum_{l \in \mathbf{L}} \eta^U_{iqll'} \vec{v}_i \otimes \Phi_{ql} \right) \otimes \vec{v}_{i'} \right) \\ &= \sum_{i=1}^{N} \sum_{q=1}^{Q} \sum_{l \in \mathbf{L}} \eta^U_{iqll'} \vec{v}_i \otimes \left(\sum_{l'' \in \mathbf{L}} \varepsilon^U_{l''qli'} \vec{u}_{l''} \right), \end{aligned}$$

so for any $i' = 1, \ldots, N$, $l' \in \mathbf{L}$, this sum has finitely many non-zero terms:

$$\sum_{q=1}^{Q} \sum_{l \in \mathbf{L}} \eta^U_{iqll'} \varepsilon^U_{l''qli'} = \delta_{ii'} \delta_{l'l''}. \tag{3.8}$$

This is analogous to Equation (2.3).

Similarly for hypothesis (3.4), let $\mathrm{Hom}(X,W)$ have basis set

$$\{\Psi_{qk} : q = 1,\dots,Q, k \in \mathbf{K}\},$$

with $\Psi_{qk} : \vec{x}_{q'} = \delta_{qq'}\vec{w}_k$. Then for each basis element $\Psi_{qk}\otimes\vec{v}_i \in \mathrm{Hom}(X,W)\otimes V$, there are coefficients $\varepsilon^W_{k'qki}$ (finitely many non-zero for each k) so that

$$\varepsilon^W : \Psi_{qk}\otimes\vec{v}_i \mapsto \sum_{k'\in\mathbf{K}} \varepsilon^W_{k'qki}\vec{w}_{k'}.$$

For $F : V\otimes U \to V\otimes W$, and each basis element $\vec{v}_i\otimes\vec{u}_l$, there are coefficients $F_{i'kil}$ (finitely many non-zero for each l) so that

$$F : \vec{v}_i\otimes\vec{u}_l \mapsto \sum_{i'=1}^{N}\sum_{k\in\mathbf{K}} F_{i'kil}\vec{v}_{i'}\otimes\vec{w}_k.$$

A basis for $\mathrm{Hom}(X,V\otimes U)$ can be chosen in the same way as (3.7), with maps $\vec{x}_{q'} \mapsto \delta_{qq'}\vec{v}_i\otimes\vec{u}_l$, but this map is exactly the same as $n_U(\vec{v}_i\otimes\Phi_{ql})$. Similarly, the maps $n_4(\vec{v}_i\otimes\Psi_{qk}) : \vec{x}_{q'} \mapsto \delta_{qq'}\vec{v}_i\otimes\vec{w}_k$ form a basis for $\mathrm{Hom}(X,V\otimes W)$.

To calculate the composites in the diagram (3.4), start with:

$$\begin{aligned}
&(\mathrm{Hom}(Id_X,F)\circ n_U)(\vec{v}_i\otimes\Phi_{ql}) :\\
\vec{x}_{q'} \mapsto\ &(F\circ(n_U(\vec{v}_i\otimes\Phi_{ql})))(\vec{x}_{q'}) = F(\vec{v}_i\otimes(\Phi_{ql}(\vec{x}_{q'}))) = F(\vec{v}_i\otimes(\delta_{qq'}\vec{u}_l))\\
=\ &\delta_{qq'}\sum_{i''=1}^{N}\sum_{k\in\mathbf{K}} F_{i''kil}\vec{v}_{i''}\otimes\vec{w}_k\\
=\ &\sum_{i''=1}^{N}\sum_{k\in\mathbf{K}} F_{i''kil}(n_4(\vec{v}_{i''}\otimes\Psi_{qk}))(\vec{x}_{q'}).
\end{aligned}$$

It follows that

$$n_4^{-1}\circ\mathrm{Hom}(Id_X,F)\circ n_U : \vec{v}_i\otimes\Phi_{ql} \mapsto \sum_{i''=1}^{N}\sum_{k\in\mathbf{K}} F_{i''kil}\vec{v}_{i''}\otimes\Psi_{qk}, \tag{3.9}$$

and

$$\begin{aligned}
&[Id_V\otimes\varepsilon^W]\circ[(n_4^{-1}\circ\mathrm{Hom}(Id_X,F)\circ n_U)\otimes Id_V] :\\
\vec{v}_i\otimes\Phi_{ql}\otimes\vec{v}_{i'} \mapsto\ &[Id_V\otimes\varepsilon^W]\left(\left(\sum_{i''=1}^{N}\sum_{k\in\mathbf{K}} F_{i''kil}\vec{v}_{i''}\otimes\Psi_{qk}\right)\otimes\vec{v}_{i'}\right)\\
=\ &\sum_{i''=1}^{N}\sum_{k\in\mathbf{K}}\sum_{k'\in\mathbf{K}} F_{i''kil}\varepsilon^W_{k'qki'}\vec{v}_{i''}\otimes\vec{w}_{k'}.
\end{aligned}\tag{3.10}$$

The hypothesis (3.4) is that for any F, the expressions (3.10) and (3.11) are equal:

$$\begin{aligned}
F\circ[Id_V\otimes\varepsilon^U] : \vec{v}_i\otimes\Phi_{ql}\otimes\vec{v}_{i'} \mapsto\ &F\left(\vec{v}_i\otimes\sum_{l'\in\mathbf{L}}\varepsilon^U_{l'qli'}\vec{u}_{l'}\right)\\
=\ &\sum_{l'\in\mathbf{L}}\sum_{i''=1}^{N}\sum_{k'\in\mathbf{K}}\varepsilon^U_{l'qli'}F_{i''k'il'}\vec{v}_{i''}\otimes\vec{w}_{k'},
\end{aligned}\tag{3.11}$$

so for any i, q, l, i', i'', k', these finite sums are equal:

$$\sum_{k\in\mathbf{K}} F_{i''kil}\varepsilon^{W}_{k'qki'} = \sum_{l'\in\mathbf{L}} \varepsilon^{U}_{l'qli'}F_{i''k'il'}. \tag{3.12}$$

For the RHS composite from (3.6) in the conclusion of the Theorem, a calculation analogous to (3.9) gives, for any A as in (2.5):

$$n_W^{-1}\circ \mathrm{Hom}(Id_X, A)\circ n_U : \vec{v}_i\otimes \Phi_{ql} \mapsto \sum_{i'=1}^{N}\sum_{k\in\mathbf{K}} A_{i'kil}\Psi_{qk}\otimes \vec{v}_{i'}. \tag{3.13}$$

The next steps use (3.8), (3.12), and (3.13).

$$\begin{aligned}
&\varepsilon^{W}\circ n_W^{-1}\circ \mathrm{Hom}(Id_X, A)\circ n_U\circ \eta^{U} : \\
\vec{u}_l \mapsto{}& \varepsilon^{W}\left(\left(n_W^{-1}\circ \mathrm{Hom}(Id_X, A)\circ n_U\right)\left(\sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{l'\in\mathbf{L}} \eta^{U}_{iql'l}\vec{v}_i\otimes \Phi_{ql'}\right)\right)\\
={}& \varepsilon^{W}\left(\sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{l'\in\mathbf{L}} \eta^{U}_{iql'l}\sum_{i'=1}^{N}\sum_{k\in\mathbf{K}} A_{i'kil'}\Psi_{qk}\otimes \vec{v}_{i'}\right)\\
={}& \sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{l'\in\mathbf{L}}\sum_{i'=1}^{N}\sum_{k\in\mathbf{K}}\sum_{k'\in\mathbf{K}} \eta^{U}_{iql'l}A_{i'kil'}\varepsilon^{W}_{k'qki'}\vec{w}_{k'}\\
={}& \sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{l'\in\mathbf{L}}\sum_{i'=1}^{N}\sum_{k'\in\mathbf{K}}\sum_{l''\in\mathbf{L}} \eta^{U}_{iql'l}\varepsilon^{U}_{l''ql'i'}A_{i'k'il''}\vec{w}_{k'} &\quad (3.14)\\
={}& \sum_{i=1}^{N}\sum_{i'=1}^{N}\sum_{k'\in\mathbf{K}}\sum_{l''\in\mathbf{L}} \delta_{ii'}\delta_{ll''}A_{i'k'il''}\vec{w}_{k'}\\
={}& \sum_{i=1}^{N}\sum_{k'\in\mathbf{K}} A_{ik'il}\vec{w}_{k'}. &\quad (3.15)
\end{aligned}$$

The last sum (3.15) is the same as (2.6), the generalized trace. □

Theorem 3.7 is still true even without the assumption that X has finite dimension, and can be given a proof without choosing a basis for everything, although this turns out to be more complicated than the above proof. Also, the property (3.4) of ε^U and ε^W is only used in step (3.14), so to compute the trace of a particular map A, one could assume the commutativity of (3.4) only for $F = A$, instead of for all F. However, our goal is to find a formula (3.6) for the operator $Tr_{V;U,W}$ that works for any input.

The following result on the generalized trace has a conclusion analogous to that of Theorem 3.7 (Equations (3.6) and (3.17) are the same), but replaces its assumption (3.4) about two evaluation maps with a dual statement about two compatible coevaluation maps. The proof is omitted but would be very similar to the previous proof.

PROPOSITION 3.8. *Given vector spaces U, V, W, suppose there exist a vector space X and maps*

$$\begin{aligned} \eta^U &: U \to V \otimes \mathrm{Hom}(X,U) \\ \eta^W &: W \to V \otimes \mathrm{Hom}(X,W) \\ \varepsilon^W &: \mathrm{Hom}(X,W) \otimes V \to W \end{aligned}$$

such that this composite is a switching map:

$$[Id_V \otimes \varepsilon^W] \circ [\eta^W \otimes Id_V] = s : W \otimes V \to V \otimes W,$$

and the following diagram is commutative for any $F : U \otimes V \to U \otimes W$.

$$\begin{array}{ccc} V \otimes U & \xrightarrow{F} & V \otimes W \\ \downarrow{\scriptstyle [Id_V \otimes \eta^U]} & & \downarrow{\scriptstyle [Id_V \otimes \eta^W]} \\ V \otimes V \otimes \mathrm{Hom}(X,U) & & V \otimes V \otimes \mathrm{Hom}(X,W) \\ \downarrow{\scriptstyle [Id_V \otimes s_1]} & & \downarrow{\scriptstyle [Id_V \otimes s_2]} \\ V \otimes \mathrm{Hom}(X,U) \otimes V & & V \otimes \mathrm{Hom}(X,W) \otimes V \\ \downarrow{\scriptstyle [n_U \otimes Id_V]} & & \downarrow{\scriptstyle [n_4 \otimes Id_V]} \\ \mathrm{Hom}(X, V \otimes U) \otimes V & \xrightarrow[{[\mathrm{Hom}(Id_X,F) \otimes Id_V]}]{} & \mathrm{Hom}(X, V \otimes W) \otimes V \end{array} \tag{3.16}$$

If V and X are finite dimensional, then for any $A : V \otimes U \to V \otimes W$,

$$Tr_{V;U,W}(A) = \varepsilon^W \circ n_W^{-1} \circ \mathrm{Hom}(Id_X, A) \circ n_U \circ \eta^U : U \to W, \tag{3.17}$$

so the RHS composite does not depend on $(X, \eta^U, \eta^W, \varepsilon^W)$.

The following Corollary is another scalar-free formula for the generalized trace of $A : V \otimes U \to V \otimes W$, in the special case where A can be factored into the form $[B \otimes C]$.

COROLLARY 3.9. *If V, U, W, X, η^U, and ε^W satisfy the hypothesis of either Theorem 3.7 or Proposition 3.8, and $s : V \otimes \mathrm{Hom}(X,U) \to \mathrm{Hom}(X,U) \otimes V$ is a switching map, then for any $B : V \to V$ and $C : U \to W$,*

$$Tr_{V;U,W}([B \otimes C]) = \varepsilon^W \circ [\mathrm{Hom}(Id_X, C) \otimes B] \circ s \circ \eta^U.$$

Proof. The following diagram is copied from (3.5) with $A = [B \otimes C]$, and the arrows added in the middle correspond to the maps in the claimed formula, so the lower block with the switching map is analogous to the diagram (2.4), but without the scalars.

$$\begin{array}{ccccc}
\mathrm{Hom}(X,V\otimes U) & & \xrightarrow{\mathrm{Hom}(Id_X,[B\otimes C])} & & \mathrm{Hom}(X,V\otimes W) \\
\uparrow{\scriptstyle n_U} & & & & \uparrow{\scriptstyle n_W} \\
V\otimes\mathrm{Hom}(X,U) & \xrightarrow{s} & \mathrm{Hom}(X,U)\otimes V & \xrightarrow{[\mathrm{Hom}(Id_X,C)\otimes B]} & \mathrm{Hom}(X,W)\otimes V \\
\uparrow{\scriptstyle \eta^U} & & & & \downarrow{\scriptstyle \varepsilon^W} \\
U & & \xrightarrow[Tr_{V;U,W}([B\otimes C])]{} & & W
\end{array}$$

The upper block is easily checked to be commutative for any B and C: both paths from $V\otimes\mathrm{Hom}(X,U)$ to $\mathrm{Hom}(X,V\otimes W)$ take an element of the form $\vec{v}\otimes D$ to a map $\vec{x}\mapsto(B(\vec{v}))\otimes(C(D(\vec{x})))$. The commutativity around the outside of the diagram is the conclusion from either Theorem 3.7 or Proposition 3.8, so the commutativity of the lower block follows, and this is the claim of the Corollary.

We remark that the assumption about the finite dimension of V is sufficient for the invertibility of the canonical map from Notation 2.5:

$$\mathrm{Hom}(V,V)\otimes\mathrm{Hom}(U,W)\to\mathrm{Hom}(V\otimes U,V\otimes W):B\otimes C\mapsto[B\otimes C]$$

([4] §II.7.7.), so any map $A:V\otimes U\to V\otimes W$ can be written as a finite sum of maps of the form $[B\otimes C]$. $\square$

Calculations similar to the commutativity of the upper block from the above diagram will appear again in the next Sections, so we state a general result as the following Lemma. The canonical maps n and n' are of the form (3.1), but there are analogous results for other versions of n maps such as (3.2), or for composites with switching maps as in Corollary 3.9. (The Lemma can be interpreted as a statement about the naturality of the n maps, in a technical sense of category theory.)

LEMMA 3.10. *For any vector spaces X, X', V, V', U, U', and maps $F:X'\to X$, $B:V\to V'$, $C:U\to U'$, this diagram is commutative.*

$$\begin{array}{ccc}
V\otimes\mathrm{Hom}(X,U) & \xrightarrow{n} & \mathrm{Hom}(X,V\otimes U) \\
{\scriptstyle [B\otimes\mathrm{Hom}(F,C)]}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}(F,[B\otimes C])} \\
V'\otimes\mathrm{Hom}(X',U') & \xrightarrow{n'} & \mathrm{Hom}(X',V'\otimes U')
\end{array}$$

Proof. For elements in the domain of the form $\vec{v}\otimes A$ with $A\in\mathrm{Hom}(X,U)$ and $\vec{v}\in V$, and for $\vec{x}\in X'$,

$$\begin{aligned}
\vec{v}\otimes A &\mapsto (\mathrm{Hom}(F,[B\otimes C])\circ n)(\vec{v}\otimes A)=[B\otimes C]\circ(n(\vec{v}\otimes A))\circ F: \\
&\quad \vec{x}\mapsto[B\otimes C](\vec{v}\otimes(A(F(\vec{x}))))=(B(\vec{v}))\otimes((C\circ A\circ F)(\vec{x})), \\
\vec{v}\otimes A &\mapsto (n'\circ[B\otimes\mathrm{Hom}(F,C)])(\vec{v}\otimes A)=n'((B(\vec{v}))\otimes(C\circ A\circ F)): \\
&\quad \vec{x}\mapsto(B(\vec{v}))\otimes((C\circ A\circ F)(\vec{x})). \quad\square
\end{aligned}$$

EXAMPLE 3.11. Let V be finite dimensional. A specific example of a coevaluation η^U corresponding to $X=V$ and the canonical evaluation map $\varepsilon^U = Ev_{VU}$: $\mathrm{Hom}(V,U)\otimes V \to U$ is the following map $\eta_{VU}: U \to V\otimes \mathrm{Hom}(V,U)$, defined on basis elements of U by:

$$\eta_{VU}: \vec{u}_l \mapsto \sum_{i=1}^{N} \vec{v}_i \otimes \Phi_{il}, \tag{3.18}$$

where $\Phi_{il}\in \mathrm{Hom}(V,U)$ is as in (3.7) with basis elements $\vec{x}_q$ replaced by $\vec{v}_i$. (This is analogous to the example (2.1) from Proposition 2.10.) Property (3.3) is satisfied with $\eta^U = \eta_{VU}$ and $\varepsilon^U = Ev_{VU}$:

$$\begin{aligned}
\vec{u}_l\otimes\vec{v}_i &\mapsto ([Id_V\otimes Ev_{VU}]\circ[\eta_{VU}\otimes Id_V])(\vec{u}_l\otimes\vec{v}_i)\\
&= [Id_V\otimes Ev_{VU}]\left(\left(\sum_{i'=1}^{N}\vec{v}_{i'}\otimes\Phi_{i'l}\right)\otimes\vec{v}_i\right) = \sum_{i'=1}^{N}\vec{v}_{i'}\otimes(Ev_{VU}(\Phi_{i'l}\otimes\vec{v}_i))\\
&= \sum_{i'=1}^{N}\vec{v}_{i'}\otimes(\Phi_{i'l}(\vec{v}_i)) = \sum_{i'=1}^{N}\vec{v}_{i'}\otimes(\delta_{i'i}\vec{u}_l) = \vec{v}_i\otimes\vec{u}_l.
\end{aligned}$$

The coevaluations $\eta^U = \eta_{VU}$ and $\eta^W = \eta_{VW}$ also satisfy the compatibility condition (3.16). Using (3.9),

$$\begin{aligned}
& [(n_4^{-1}\circ \mathrm{Hom}(Id_X,F)\circ n_U)\otimes Id_V]\circ[Id_V\otimes(s_1\circ\eta_{VU})]:\\
\vec{v}_i\otimes\vec{u}_l &\mapsto [(n_4^{-1}\circ \mathrm{Hom}(Id_X,F)\circ n_U)\otimes Id_V]\left(\vec{v}_i\otimes\left(\sum_{i'=1}^{N}\Phi_{i'l}\otimes\vec{v}_{i'}\right)\right)\\
&= \sum_{i'=1}^{N}\left(\sum_{i''=1}^{N}\sum_{k\in\mathbf{K}}F_{i''kil}\vec{v}_{i''}\otimes\Psi_{i'k}\right)\otimes\vec{v}_{i'},\\
& [Id_V\otimes(s_2\circ\eta_{VW})]\circ F:\\
\vec{v}_i\otimes\vec{u}_l &\mapsto [Id_V\otimes(s_2\circ\eta_{VW})]\left(\sum_{i''=1}^{N}\sum_{k\in\mathbf{K}}F_{i''kil}\vec{v}_{i''}\otimes\vec{w}_k\right)\\
&= \sum_{i''=1}^{N}\sum_{k\in\mathbf{K}}F_{i''kil}\vec{v}_{i''}\otimes\left(\sum_{i'=1}^{N}\Psi_{i'k}\otimes\vec{v}_{i'}\right).
\end{aligned}$$

The following Lemma, which will be used in Section 4, states an identity for the above coevaluation map which does not depend on choices of basis, but which does need one more canonical n map,

$$n_5: \mathrm{Hom}(V,V)\otimes U \to \mathrm{Hom}(V,V\otimes U),$$

defined as in Notation 3.3 by $n_5(A\otimes\vec{u}): \vec{v}\mapsto (A(\vec{v}))\otimes\vec{u}$.

LEMMA 3.12. *For any* $\vec{u}\in U$, $\eta_{VU}(\vec{u}) = n_U^{-1}(n_5(Id_V\otimes\vec{u}))$.

Proof. The map n_U is as in (3.1) with $X = V$, so it is the same as n_2 from Lemma 3.5. It is enough to check that for basis elements $\vec{u}_l$, $n_U(\eta_{VU}(\vec{u}_l)) = n_5(Id_V \otimes \vec{u}_l)$.

$$\begin{aligned} n_U(\eta_{VU}(\vec{u}_l)) : \vec{v}_{i'} \mapsto & \left(n_U \left(\sum_{i=1}^{N} \vec{v}_i \otimes \Phi_{il} \right) \right) (\vec{v}_{i'}) \\ &= \sum_{i=1}^{N} \vec{v}_i \otimes (\Phi_{il}(\vec{v}_{i'})) = \sum_{i=1}^{N} \vec{v}_i \otimes (\delta_{ii'} \vec{u}_l) = \vec{v}_{i'} \otimes \vec{u}_l \\ &= (n_5(Id_V \otimes \vec{u}_l))(\vec{v}_{i'}). \quad \square \end{aligned}$$

The following Lemma is a coevaluation version of Lemma 3.2.

LEMMA 3.13. *For V with finite dimension and any map $B : U \to W$,*

$$\eta_{VW} \circ B = [Id_V \otimes \mathrm{Hom}(Id_V, B)] \circ \eta_{VU} : U \to V \otimes \mathrm{Hom}(V, W).$$

Proof. The claim is that the upper block in the following diagram is commutative. Three of the n maps have appeared previously, the map n_6 is analogous to n_5 as indicated in the diagram, and all the n maps are invertible.

$$\begin{array}{ccc} U & \xrightarrow{\quad B \quad} & W \\ \downarrow{\scriptstyle \eta_{VU}} & & \downarrow{\scriptstyle \eta_{VW}} \\ V \otimes \mathrm{Hom}(V,U) & \xrightarrow{[Id_V \otimes \mathrm{Hom}(Id_V, B)]} & V \otimes \mathrm{Hom}(V,W) \\ \downarrow{\scriptstyle n_U} & & \downarrow{\scriptstyle n_3} \\ \mathrm{Hom}(V, V \otimes U) & \xrightarrow{\mathrm{Hom}(Id_V, [Id_V \otimes B])} & \mathrm{Hom}(V, V \otimes W) \\ \uparrow{\scriptstyle n_5} & & \uparrow{\scriptstyle n_6} \\ \mathrm{Hom}(V,V) \otimes U & \xrightarrow{[\mathrm{Hom}(Id_V, Id_V) \otimes B]} & \mathrm{Hom}(V,V) \otimes W \end{array}$$

The lower two blocks are commutative by Lemma 3.10. Using Lemma 3.12, for $\vec{u} \in U$,

$$\begin{aligned} \vec{u} \mapsto & ([\mathrm{Hom}(Id_V, Id_V) \otimes B] \circ n_5^{-1} \circ n_U \circ \eta_{VU})(\vec{u}) \\ &= Id_V \otimes (B(\vec{u})) \\ &= (n_6^{-1} \circ n_3 \circ \eta_{VW} \circ B)(\vec{u}). \end{aligned}$$

So, the two paths from U to $\mathrm{Hom}(V,V) \otimes W$ are equal composites, which is enough to show that the upper block is commutative as claimed. $\square$

The formula (3.6) from Theorem 3.7 can be used to prove some well-known elementary properties of the generalized trace (as in [8], [9]). We will state just one such result, which will be used later.

THEOREM 3.14. *For V with finite dimension, and maps $A: V\otimes U \to V\otimes W$, $B: U' \to U$ and $G: W \to W'$, the composite $[Id_V\otimes G]\circ A\circ[Id_V\otimes B]: V\otimes U' \to V\otimes W'$ has trace*

$$Tr_{V;U',W'}([Id_V\otimes G]\circ A\circ[Id_V\otimes B]) = G\circ(Tr_{V;U,W}(A))\circ B.$$

Proof. Theorem 3.7 showed that any evaluation and coevaluation maps satisfying its hypotheses can be used to calculate the trace, so we use the canonical evaluation and the coevaluation from Example 3.11 with $X = V$.

$$\begin{aligned}
LHS &= Ev_{VW'}\circ n_{W'}^{-1}\circ \mathrm{Hom}(Id_V,[Id_V\otimes G]\circ A\circ[Id_V\otimes B])\circ n_{U'}\circ\eta_{VU'}\\
&= Ev_{VW'}\circ n_{W'}^{-1}\circ \mathrm{Hom}(Id_V,[Id_V\otimes G])\circ \mathrm{Hom}(Id_V,A)\circ\\
&\quad \mathrm{Hom}(Id_V,[Id_V\otimes B])\circ n_{U'}\circ\eta_{VU'}\\
&= Ev_{VW'}\circ[\mathrm{Hom}(Id_V,G)\otimes Id_V]\circ n_W^{-1}\circ \mathrm{Hom}(Id_V,A)\circ\\
&\quad n_U\circ[Id_V\otimes \mathrm{Hom}(Id_V,B)]\circ\eta_{VU'} &(3.19)\\
&= G\circ Ev_{VW}\circ n_W^{-1}\circ \mathrm{Hom}(Id_V,A)\circ n_U\circ\eta_{VU}\circ B = RHS. &(3.20)
\end{aligned}$$

Line (3.19) follows from the previous by Lemma 3.10, and line (3.20) uses Lemma 3.2 and Lemma 3.13. ☐

4. Complex linear algebra without complex numbers

Formula (3.6) could be taken as a definition of the trace $Tr_{V;U,W}$ in categories of vector spaces that do not include $\mathbb{K}$ as an object, but otherwise have enough structure (including some natural transformations n), to support the hypotheses of either Theorem 3.7 or Proposition 3.8. Examples of such categories include some subcategories of the category of finite dimensional vector spaces which are closed under $-\otimes-$ and $\mathrm{Hom}(-,-)$, e.g., where the objects are just the vector spaces with dimensions N satisfying $N>1$, or $N=2K>1$, or $N=2^K>1$, etc.

In categories where there is a unit object for $\otimes$ but it is not unique, then using (3.6) to define the trace shows that the trace does not depend on any choice of unit object or scalar multiplication morphisms.

An example of such a category, and the original motivation for this approach, is the category $\mathscr{C}$ of real vector spaces with linear complex structures. Each object of $\mathscr{C}$ is a pair (V,J), where V is a real vector space, and J is a real linear map $V\to V$ such that $J\circ J = -Id_V$, called a *complex structure operator* (CSO). The morphisms from (U,J_U) to (V,J_V) are real linear maps $A: U\to V$ such that $A\circ J_U = J_V\circ A$. Clearly, Id_V is the identity morphism for any object (V,J), and the composite of morphisms is a morphism. We call such maps *c-linear*, and it may be useful to think of them as "complex linear," commuting with some choice of complex scalar multiplication such as $(\alpha\cdot Id_U\pm\beta\cdot J_U)(\vec{u}) = \alpha\cdot\vec{u}\pm\beta\cdot J_U(\vec{u})$, but we are intentionally avoiding the introduction of the field of complex numbers as scalars or for any other use. In particular, we will not attempt to consider any scalar valued trace for morphisms $(V,J_V)\to(V,J_V)$;

this without-complex-numbers approach will only apply to the generalized trace, where the output is another morphism.

In this Section, we will review just enough of the ideas and notation for linear complex structures to propose a definition of the generalized trace for the category $\mathscr{C}$. See [6] for notes giving a more detailed development of complex structure operators.

When the vector space V has a CSO J_V, it is sometimes convenient to abbreviate the pair (V,J_V) by one letter, $\mathbf{V}$. However, a real vector space V may have several complex structures (and this itself is a situation where doing linear algebra with CSOs can be more clear than with complex scalars). Two CSOs can (but do not necessarily) commute, as in the following Lemma (left as an exercise).

LEMMA 4.1. *Given V and two CSOs J_1, J_2, the following are equivalent:*

1. *J_1 and J_2 commute, i.e., $J_1 \circ J_2 = J_2 \circ J_1$;*
2. *The composite $J_1 \circ J_2$ is an involution, i.e., $(J_1 \circ J_2) \circ (J_1 \circ J_2) = Id_V$.*

Any involution B on a real vector space produces a direct sum $V = V_1 \oplus V_2$, where V_1 is the $+1$ eigenspace (the fixed point set) and V_2 is the -1 eigenspace. The projection onto V_2 with kernel V_1 is $P = \frac{1}{2} \cdot (Id_V - B)$.

NOTATION 4.2. In the case of commuting CSOs on V, and the involution $J_1 \circ J_2$ as in Lemma 4.1, let V_c denote the -1 eigenspace, so that $\vec{v} \in V_c \iff J_1(J_2(\vec{v})) = -\vec{v} \iff J_1(\vec{v}) = J_2(\vec{v})$. So, V_c is exactly the real subspace of V where $J_1 = J_2$, and $J_1|_{V_c} = J_2|_{V_c}$ is a canonically induced CSO on V_c. The projection from V to V_c, where the kernel is the $+1$ eigenspace, is $P_c = \frac{1}{2} \cdot (Id_V - J_1 \circ J_2)$. P_c is c-linear from both (V,J_1) and (V,J_2) to $(V_c, J_1|_{V_c})$. If V_a denotes the $+1$ eigenspace of $J_1 \circ J_2$, then $V = V_c \oplus V_a$ and V_a is the subspace of V where the two CSOs are opposite, $J_1|_{V_a} = -J_2|_{V_a}$.

EXAMPLE 4.3. Given $\mathbf{U} = (U,J_U)$ and $\mathbf{V} = (V,J_V)$, the maps $\mathrm{Hom}(Id_U, J_V)$ and $\mathrm{Hom}(J_U, Id_V)$ are commuting CSOs on $\mathrm{Hom}(U,V)$, so Lemma 4.1 applies. The real subspace of $\mathrm{Hom}(U,V)$ where the two CSOs agree, as in Notation 4.2, is the vector space of c-linear maps, and also the set of morphisms in $\mathscr{C}$ from $\mathbf{U}$ to $\mathbf{V}$:

$$\mathrm{Hom}_c((U,J_U),(V,J_V)) = \{A \in \mathrm{Hom}(U,V) : A \circ J_U = J_V \circ A\}.$$

The projection onto the subspace is

$$\begin{aligned} P_c &= \frac{1}{2} \cdot (Id_{\mathrm{Hom}(U,V)} - \mathrm{Hom}(Id_U, J_V) \circ \mathrm{Hom}(J_U, Id_V)) \\ &= \frac{1}{2} \cdot (Id_{\mathrm{Hom}(U,V)} - \mathrm{Hom}(J_U, J_V)) : A \mapsto \frac{1}{2} \cdot (A - J_V \circ A \circ J_U). \end{aligned}$$

The subspace has a canonical CSO, so as an object in $\mathscr{C}$, the pair can be denoted

$$\mathrm{Hom}_c(\mathbf{U},\mathbf{V}) = (\mathrm{Hom}_c((U,J_U),(V,J_V)), \mathrm{Hom}(Id_U, J_V)|_{\mathrm{Hom}_c((U,J_U),(V,J_V))}).$$

EXAMPLE 4.4. Given $\mathbf{U} = (U, J_U)$ and $\mathbf{V} = (V, J_V)$, the two maps $[Id_U \otimes J_V]$, $[J_U \otimes Id_V] \in \mathrm{Hom}(U \otimes V, U \otimes V)$ are commuting CSOs on $U \otimes V$, so Lemma 4.1 applies. The direct sum from Notation 4.2 is denoted:

$$U \otimes V = (U \otimes_c V) \oplus (U \otimes_a V), \tag{4.1}$$

so that the subspace of $U \otimes V$ where the two CSOs agree is

$$U \otimes_c V = \{\vec{w} \in U \otimes V : [Id_U \otimes J_V](\vec{w}) = [J_U \otimes Id_V](\vec{w})\}, \tag{4.2}$$

and it has a canonical CSO, $J_{U \otimes_c V} = [Id_U \otimes J_V]|_{U \otimes_c V} = [J_U \otimes Id_V]|_{U \otimes_c V}$. The symbol $\mathbf{U} \otimes_c \mathbf{V}$ will be used to denote the object in $\mathscr{C}$ given by this subspace paired with the CSO.

So, the category $\mathscr{C}$ has a tensor product $\otimes_c$; it is associative as described in the Proof of Lemma 4.12. (In terms of complex linear algebra, the idea is that the real subspace $U \otimes_c V$ corresponds to the tensor product "over $\mathbb{C}$" where complex scalars can move from $\mathbf{U}$ to $\mathbf{V}$. The elements of the complementary subspace $U \otimes_a V$, corresponding to V_a in Notation 4.2, are the "antilinear" tensors where moving a complex scalar introduces a conjugation.)

EXAMPLE 4.5. The vector space $\mathbb{R}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \right\}$ admits distinct CSOs; for example, the following matrices all satisfy the definition:

$$J_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ J_3 = \begin{bmatrix} 47 & -34 \\ 65 & -47 \end{bmatrix}, \ J_4 = \begin{bmatrix} -5 & 26 \\ -1 & 5 \end{bmatrix}, \ldots$$

None of these is any more canonical than the others, although the first one could be called the "standard" CSO for $\mathbb{R}^2$ with the (x, y) coordinate system, due to its resemblance to a $+90^\circ$ rotation matrix. For any object $\mathbf{V}$ in $\mathscr{C}$, there exists some (not necessarily unique or canonical) invertible c-linear map from $\mathbf{V} \otimes_c (\mathbb{R}^2, J_1)$ to $\mathbf{V}$ (more details appear in Example 5.12). So the tensor product $\otimes_c$ does have at least one unit object in $\mathscr{C}$, and it is unique only up to isomorphism. The interesting difference between $\mathscr{C}$ and the category of all vector spaces is not whether there exists a tensor unit, but that $\mathscr{C}$ does not have a distinguished unit object and scalar multiplication isomorphisms in the same way that the category of all vector spaces has the canonical object $\mathbb{K}$ and the canonical ℓ maps.

LEMMA 4.6. *For V with commuting CSOs J_1, J_2, and another space V' with commuting CSOs J_1', J_2', if a map $H : V \to V'$ satisfies $H \circ J_1 = J_1' \circ H$ and $H \circ J_2 = J_2' \circ H$, then H respects the direct sums of ± 1 eigenspaces, and H restricts to a c-linear map $V_c \to V_c'$, which is invertible if H is.*

Proof. The statement about respecting the direct sum means that if $\vec{v}$ is a -1 eigenvector of $J_1 \circ J_2$ (so $(J_1 \circ J_2)(\vec{v}) = -\vec{v} \iff J_1(\vec{v}) = J_2(\vec{v}) \iff \vec{v} \in V_c$), then $H(\vec{v})$ is a -1 eigenvector of $J_1' \circ J_2'$. This is easily checked (using only the weaker property $H\circ J_1\circ J_2 = J_1'\circ J_2'\circ H$), and also holds for the $+1$ eigenspace. The c-linear property refers to the canonical CSOs on V_c and V_c': if $\vec{v}\in V_c$, then $J_1|_{V_c}(\vec{v})\in V_c$, and

$$\begin{aligned} H|_{V_c}(J_1|_{V_c}(\vec{v})) &= H(J_1(\vec{v})) = H(J_2(\vec{v})) \\ &= J_2'(H(\vec{v})) = J_1'(H(\vec{v})) = J_1'|_{V_c'}(H|_{V_c}(\vec{v})). \end{aligned}$$

If H has an inverse, then it also respects the direct sum and the restriction of the inverse to V_c' is the inverse of $H|_{V_c}: V_c\to V_c'$. $\square$

EXAMPLE 4.7. If $\mathbf{U}=(U,J_U)$ and $\mathbf{V}=(V,J_V)$, then $U\otimes V$ and $V\otimes U$ both have commuting pairs of CSOs as in Example 4.4. The switching map $s:U\otimes V\to V\otimes U$ (as in Notation 2.4) satisfies $s\circ[J_U\otimes Id_V]=[Id_V\otimes J_U]\circ s$ and $s\circ[Id_V\otimes J_V]=[J_V\otimes Id_V]\circ s$, so Lemma 4.6 applies and s restricts to a c-linear map $\mathbf{s}:\mathbf{U}\otimes_c\mathbf{V}\to\mathbf{V}\otimes_c\mathbf{U}$.

EXAMPLE 4.8. For c-linear maps $A:\mathbf{U}\to\mathbf{U}'$ and $B:\mathbf{V}\to\mathbf{V}'$, the map

$$[A\otimes B]: U\otimes V\to U'\otimes V'$$

satisfies both $[A\otimes B]\circ[J_U\otimes Id_V] = [J_{U'}\otimes Id_{V'}]\circ[A\otimes B]$ and $[A\otimes B]\circ[Id_U\otimes J_V] = [Id_{U'}\otimes J_{V'}]\circ[A\otimes B]$, so Lemma 4.6 applies and $[A\otimes B]$ restricts to a c-linear map, denoted $[A\otimes_c B]:\mathbf{U}\otimes_c\mathbf{V}\to\mathbf{U}'\otimes_c\mathbf{V}'$.

EXAMPLE 4.9. For c-linear maps $A:\mathbf{U}'\to\mathbf{U}$ and $B:\mathbf{V}\to\mathbf{V}'$, Lemma 4.6 applies to the map

$$\mathrm{Hom}(A,B):\mathrm{Hom}(U,V)\to\mathrm{Hom}(U',V')$$

and the corresponding pairs of CSOs from Example 4.3. The restricted c-linear map can be denoted

$$\mathrm{Hom}_c(A,B):\mathrm{Hom}_c(\mathbf{U},\mathbf{V})\to\mathrm{Hom}_c(\mathbf{U}',\mathbf{V}'): F\mapsto B\circ F\circ A.$$

We will need to work with some spaces with three mutually commuting CSOs, as in the following Example 4.10. Lemma 4.6 can be generalized for maps between such spaces, but we will just sketch the following special case that will be needed later.

EXAMPLE 4.10. For any objects $\mathbf{U}=(U,J_U)$, $\mathbf{V}=(V,J_V)$, $\mathbf{X}=(X,J_X)$, a canonical map from Notation 3.3, such as (3.1),

$$n: V\otimes\mathrm{Hom}(X,U)\to\mathrm{Hom}(X,V\otimes U),$$

is c-linear with respect to each of the three corresponding pairs of CSOs on the domain and target:

$$\begin{aligned} n\circ[J_V\otimes Id_{\mathrm{Hom}(X,U)}] &= \mathrm{Hom}(Id_X,[J_V\otimes Id_U])\circ n, \\ n\circ[Id_V\otimes\mathrm{Hom}(J_X,Id_U)] &= \mathrm{Hom}(J_X,Id_{V\otimes U})\circ n, \\ n\circ[Id_V\otimes\mathrm{Hom}(Id_X,J_U)] &= \mathrm{Hom}(Id_X,[Id_V\otimes J_U])\circ n. \end{aligned}$$

Each of these three equations follows from Lemma 3.10. The three CSOs on the domain commute pairwise, and similarly for the target. Lemma 4.6 applies to any two out of the three CSO pairs; an example we will need later is these two CSOs on the domain: $[J_V \otimes Id_{\mathrm{Hom}(X,U)}]$ and $[Id_V \otimes \mathrm{Hom}(J_X, Id_U)]$. The subspace of $V \otimes \mathrm{Hom}(X,U)$ where these two complex structures agree can be denoted $\mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{X},U)$, as in Example 4.4, where the bold letters indicate the tensor product in $\mathscr{C}$ of the objects $\mathbf{V}$ and

$$\mathrm{Hom}(\mathbf{X},U) = (\mathrm{Hom}(X,U), \mathrm{Hom}(J_X, Id_U)).$$

Similarly in the target, the subspace of $\mathrm{Hom}(X, V \otimes U)$ where the two CSOs $\mathrm{Hom}(Id_X, [J_V \otimes Id_U])$ and $\mathrm{Hom}(J_X, Id_{V\otimes U})$ agree can be denoted $\mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes U)$ as in Example 4.3. The map n respects these direct sums and restricts to a c-linear map $n : \mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{X},U) \to \mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes U)$. The third CSO in the domain, $[Id_V \otimes \mathrm{Hom}(Id_X, J_U)]$, also respects the direct sum and restricts to a CSO on the subspace $\mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{X},U)$ that commutes with the CSO induced by the first two. The subspace of $\mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{X},U)$ where these two restricted CSOs agree is exactly the subspace of $V \otimes \mathrm{Hom}(X,U)$ where all three CSOs agree, and can be denoted $\mathbf{V} \otimes_c \mathrm{Hom}_c(\mathbf{X},\mathbf{U})$. (This subspace does not depend on which two out of three CSOs start the construction.) The projection from Notation 4.2 will be denoted

$$\mathbf{P}_c : \mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{X},U) \to \mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{X},\mathbf{U}). \tag{4.3}$$

Similarly in the target, the third CSO $[Id_V \otimes \mathrm{Hom}(J_X, Id_U)]$ restricts to the subspace $\mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes U)$ and commutes with the induced CSO, so the subspace where all three CSOs agree is $\mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes_c \mathbf{U})$. If we denote $P_c : V \otimes U \to \mathbf{V} \otimes_c \mathbf{U}$, then this projection map can be denoted:

$$\mathrm{Hom}_c(Id_X, P_c) : \mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes U) \to \mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes_c \mathbf{U}).$$

The map n restricts to a c-linear map between the subspaces where all three commuting CSOs agree, denoted $\mathbf{n}$, as in the following commutative diagram.

$$\begin{array}{ccc}
\mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{X},U) & \xrightarrow{\ \mathbf{P}_c\ } & \mathbf{V} \otimes_c \mathrm{Hom}_c(\mathbf{X},\mathbf{U}) \\
\Big\downarrow n & & \Big\downarrow \mathbf{n} \\
\mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes U) & \xrightarrow[\mathrm{Hom}_c(Id_X, P_c)]{} & \mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes_c \mathbf{U})
\end{array}$$

If $n : V \otimes \mathrm{Hom}(X,U) \to \mathrm{Hom}(X, V \otimes U)$ is invertible, then so are the restrictions n and $\mathbf{n}$ in the diagram. There is a similar construction for other versions of n maps such as (3.2).

LEMMA 4.11. *If V is finite dimensional with CSO J_V, then there exists an ordered basis for V of the form*

$$(\vec{v}_1, J_V(\vec{v}_1), \vec{v}_2, J_V(\vec{v}_2), \ldots, \vec{v}_N, J_V(\vec{v}_N)). \tag{4.4}$$

For any U with CSO J_U and basis $\{\vec{u}_l\}$, the set

$$\{\vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l)) : i = 1,\dots,N, l \in \mathbf{L}\} \tag{4.5}$$

is a basis of $\mathbf{V} \otimes_c \mathbf{U}$.

Proof. The existence of such a basis for V is elementary (although it uses $\mathbb{K} = \mathbb{R}$ and may not work for other fields of scalars; we refer to [6]), but note that in this Section, V has real dimension $2N$, which is a change from the notation in Sections 2 and 3.

Every element in $U \otimes V$ is a finite sum of the form

$$\left(\sum_{i=1}^{N}\sum_{l\in\mathbf{L}} a_{il}\vec{v}_i \otimes \vec{u}_l\right) + \left(\sum_{i'=1}^{N}\sum_{l'\in\mathbf{L}} b_{i'l'}(J_V(\vec{v}_{i'})) \otimes \vec{u}_{l'}\right). \tag{4.6}$$

If this element is in $\mathbf{U} \otimes_c \mathbf{V}$, then

$$\left(\sum_{i=1}^{N}\sum_{l\in\mathbf{L}} a_{il}(J_V(\vec{v}_i)) \otimes \vec{u}_l\right) - \left(\sum_{i'=1}^{N}\sum_{l'\in\mathbf{L}} b_{i'l'}\vec{v}_{i'} \otimes \vec{u}_{l'}\right)$$
$$= \left(\sum_{i=1}^{N}\sum_{l\in\mathbf{L}} a_{il}\vec{v}_i \otimes (J_U(\vec{u}_l))\right) + \left(\sum_{i'=1}^{N}\sum_{l'\in\mathbf{L}} b_{i'l'}(J_V(\vec{v}_{i'})) \otimes (J_U(\vec{u}_{l'}))\right).$$

By the independence (over $\mathbb{R}$) of the set $\{\vec{v}_i \otimes \vec{u}_l\} \cup \{(J_V(\vec{v}_{i'})) \otimes \vec{u}_{l'}\}$ in $U \otimes V$,

$$-\sum_{i'=1}^{N}\sum_{l'\in\mathbf{L}} b_{i'l'}\vec{v}_{i'} \otimes \vec{u}_{l'} = \sum_{i=1}^{N}\sum_{l\in\mathbf{L}} a_{il}\vec{v}_i \otimes (J_U(\vec{u}_l)),$$

and applying $[J_V \otimes Id_U]$ to both sides gives

$$-\sum_{i'=1}^{N}\sum_{l'\in\mathbf{L}} b_{i'l'}(J_V(\vec{v}_{i'})) \otimes \vec{u}_{l'} = \sum_{i=1}^{N}\sum_{l\in\mathbf{L}} a_{il}(J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l));$$

we can conclude that if any element (4.6) is in $\mathbf{U} \otimes_c \mathbf{V}$, then it is of the form

$$\sum_{i=1}^{N}\sum_{l\in\mathbf{L}} a_{il}\left(\vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_V(\vec{u}_l))\right),$$

so the set (4.5) spans $\mathbf{U} \otimes_c \mathbf{V}$. The independence of (4.5) also follows from the independence of the set $\{\vec{v}_i \otimes \vec{u}_l\} \cup \{(J_V(\vec{v}_{i'})) \otimes \vec{u}_{l'}\}$ in $U \otimes V$. $\square$

LEMMA 4.12. *Given vector spaces U, V, X, if V is finite dimensional with CSO J_V and basis*

$$(\vec{v}_1, J_V(\vec{v}_1), \dots, \vec{v}_N, J_V(\vec{v}_N)),$$

and X is finite dimensional with CSO J_X and basis

$$(\vec{x}_1, J_X(\vec{x}_1), \dots, \vec{x}_Q, J_X(\vec{x}_Q)),$$

then for any CSO J_U and basis $\{\vec{u}_l\}$ for U, the set

$$\begin{aligned}\{&\vec{v}_i \otimes \vec{u}_l \otimes \vec{x}_q - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l)) \otimes \vec{x}_q \\ &-(J_V(\vec{v}_i)) \otimes \vec{u}_l \otimes (J_X(\vec{x}_q)) - \vec{v}_i \otimes (J_U(\vec{u}_l)) \otimes (J_X(\vec{x}_q))\}\end{aligned} \tag{4.7}$$

is a basis of $\mathbf{V} \otimes_c \mathbf{U} \otimes_c \mathbf{X}$.

Proof. Recall from Example 4.4 that $\mathbf{V} \otimes_c \mathbf{U}$ is the subspace of $V \otimes U$ where the CSOs $[Id_V \otimes J_U]$ and $[J_V \otimes Id_U]$ agree, and that the CSO on $\mathbf{V} \otimes_c \mathbf{U}$ is their common restriction $[Id_V \otimes J_U]|_{\mathbf{V}\otimes_c\mathbf{U}} = [J_V \otimes Id_U]|_{\mathbf{V}\otimes_c\mathbf{U}}$. In the same way, $(\mathbf{V} \otimes_c \mathbf{U}) \otimes_c \mathbf{X}$ is the subspace of $(\mathbf{V} \otimes_c \mathbf{U}) \otimes X$ where the CSOs $[[J_V \otimes Id_U]|_{\mathbf{V}\otimes_c\mathbf{U}} \otimes Id_X]$ and $[Id_{\mathbf{V}\otimes_c\mathbf{U}} \otimes J_X]$ agree. By Lemma 4.11, $(\mathbf{V} \otimes_c \mathbf{U}) \otimes_c \mathbf{X}$ is spanned by basis elements of the form

$$\begin{aligned}&(\vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l))) \otimes \vec{x}_q \\ &\quad - ([J_V \otimes Id_U]|_{\mathbf{V}\otimes_c\mathbf{U}}(\vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l)))) \otimes (J_X(\vec{x}_q)) \\ &= (\vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l))) \otimes \vec{x}_q \\ &\quad - ((J_V(\vec{v}_i)) \otimes \vec{u}_l + \vec{v}_i \otimes (J_U(\vec{u}_l))) \otimes (J_X(\vec{x}_q)).\end{aligned} \tag{4.8}$$

Similarly using Lemma 4.11 again, $\mathbf{V} \otimes_c (\mathbf{U} \otimes_c \mathbf{X})$ is spanned by basis elements of the form

$$\begin{aligned}&\vec{v}_i \otimes (\vec{u}_l \otimes \vec{x}_q - (J_U(\vec{u}_l)) \otimes (J_X(\vec{x}_q))) \\ &-(J_V(\vec{v}_i)) \otimes ((J_U(\vec{u}_l)) \otimes \vec{x}_q + \vec{u}_l \otimes (J_X(\vec{x}_q))).\end{aligned} \tag{4.9}$$

Under the identification of the real tensor products $(V \otimes U) \otimes X$ and $V \otimes (U \otimes X)$ with the triple product $V \otimes U \otimes X$ as mentioned in Notation 2.2, both (4.8) and (4.9) can be expanded out and parentheses removed, so that they are equal to each other and to the expression in (4.7). The subspace of $V \otimes U \otimes X$ spanned by these elements can be unambiguously denoted $\mathbf{V} \otimes_c \mathbf{U} \otimes_c \mathbf{X}$; it is the subspace where all three of these commuting CSOs on $V \otimes U \otimes X$ are equal:

$$\begin{aligned}[][J_V \otimes Id_U] \otimes Id_X] &= [J_V \otimes Id_{U\otimes X}] \\ [[Id_V \otimes J_U] \otimes Id_X] &= [Id_V \otimes [J_U \otimes Id_X]] \\ [Id_V \otimes [Id_U \otimes J_X]] &= [Id_{V\otimes U} \otimes J_X].\end{aligned}$$

As in Example 4.4, the notation $\mathbf{V} \otimes_c \mathbf{U} \otimes_c \mathbf{X}$ will also be used to denote the object in $\mathscr{C}$ given by pairing this real subspace with the CSO equal to the restriction of any of the above three. □

LEMMA 4.13. *If X is finite dimensional with CSO J_X and ordered basis of the form*

$$(\vec{x}_1, J_X(\vec{x}_1), \dots, \vec{x}_Q, J_X(\vec{x}_Q)),$$

then for any U with CSO J_U and basis $\{\vec{u}_l\}$, the set of maps

$$\{\Phi^c_{ql} : q = 1, \dots, Q, l \in \mathbf{L}\}, \tag{4.10}$$

with each Φ^c_{ql} *defined on basis elements of* X *by the formula:*

$$\vec{x}_{q'} \mapsto \delta_{qq'}\vec{u}_l$$
$$J_X(\vec{x}_{q'}) \mapsto \delta_{qq'}J_U(\vec{u}_l)$$

is a basis of $\mathrm{Hom}_c(\mathbf{X},\mathbf{U})$.

Proof. It is straightforward to check that each $\Phi^c_{ql} \in \mathrm{Hom}_c(\mathbf{X},\mathbf{U})$. Any element $A \in \mathrm{Hom}_c(\mathbf{X},\mathbf{U})$ is determined by its values on the $\vec{x}_1,\dots,\vec{x}_Q$ basis elements of X:

$$A(\vec{x}_q) = \sum_{l\in\mathbf{L}} A_{lq}\vec{u}_l \implies A(J_X(\vec{x}_q)) = J_U(A(\vec{x}_q)) = \sum_{l\in\mathbf{L}} A_{lq}J_U(\vec{u}_l). \tag{4.11}$$

$\mathrm{Hom}_c(\mathbf{X},\mathbf{U})$ is spanned by the set (4.10): corresponding to the finite list of coefficients from (4.11) for any $A \in \mathrm{Hom}_c(\mathbf{X},\mathbf{U})$,

$$\sum_{q=1}^{Q}\sum_{l\in\mathbf{L}} A_{lq}\Phi^c_{ql} : \vec{x}_{q'} \mapsto \sum_{q=1}^{Q}\sum_{l\in\mathbf{L}} A_{lq}\delta_{qq'}\vec{u}_l = \sum_{l\in\mathbf{L}} A_{lq'}\vec{u}_l = A(\vec{x}_{q'}).$$

To show that (4.10) is an independent set, suppose $\sum_{q=1}^{Q}\sum_{l\in\mathbf{L}} b_{ql}\Phi^c_{ql} = 0_{\mathrm{Hom}_c(U,V)}$. Then, for any q',

$$\vec{0}_U = \sum_{q=1}^{Q}\sum_{l\in\mathbf{L}} b_{ql}\Phi^c_{ql}(\vec{x}_{q'}) = \sum_{q=1}^{Q}\sum_{l\in\mathbf{L}} b_{ql}\delta_{qq'}\vec{u}_l = \sum_{l\in\mathbf{L}} b_{q'l}\vec{u}_l,$$

so every coefficient is zero by the independence of the basis for U. □

The next Theorem is the main result of this Section; there is enough structure in the category $\mathscr{C}$ to use an analogue of formula (3.6) to define a c-linear trace. However, $\mathscr{C}$ is different enough from the category of all vector spaces, so that there is a need to give another Proof showing that the formula is independent of the choices of evaluation and coevaluation.

THEOREM 4.14. *Given objects in* $\mathscr{C}$, $\mathbf{U} = (U,J_U)$, $\mathbf{V} = (V,J_V)$, $\mathbf{W} = (W,J_W)$, *suppose there exist an object* $\mathbf{X} = (X,J_X)$ *and morphisms*

$$\eta^{\mathbf{U}} : \mathbf{U} \to \mathbf{V} \otimes_c \mathrm{Hom}_c(\mathbf{X},\mathbf{U})$$
$$\varepsilon^{\mathbf{U}} : \mathrm{Hom}_c(\mathbf{X},\mathbf{U}) \otimes_c \mathbf{V} \to \mathbf{U}$$
$$\varepsilon^{\mathbf{W}} : \mathrm{Hom}_c(\mathbf{X},\mathbf{W}) \otimes_c \mathbf{V} \to \mathbf{W}$$

such that this composite is equal to a switching morphism:

$$[Id_V \otimes_c \varepsilon^{\mathbf{U}}] \circ [\eta^{\mathbf{U}} \otimes_c Id_V] = \mathbf{s} : \mathbf{U} \otimes_c \mathbf{V} \to \mathbf{V} \otimes_c \mathbf{U}. \tag{4.12}$$

Suppose further that V and X are both finite dimensional, so that the $\mathbf{n}$ *morphisms in the following diagram are invertible, and that the diagram is commutative for any morphism* $F:\mathbf{V}\otimes_c\mathbf{U}\to\mathbf{V}\otimes_c\mathbf{W}$,

$$\begin{array}{ccc} \mathbf{V}\otimes_c\mathbf{U} & \xrightarrow{\quad F\quad} & \mathbf{V}\otimes_c\mathbf{W} \\ \uparrow{\scriptstyle [Id_V\otimes_c\varepsilon^{\mathbf{U}}]} & & \uparrow{\scriptstyle [Id_V\otimes_c\varepsilon^{\mathbf{W}}]} \\ \mathbf{V}\otimes_c\mathrm{Hom}_c(\mathbf{X},\mathbf{U})\otimes_c\mathbf{V} & & \mathbf{V}\otimes_c\mathrm{Hom}_c(\mathbf{X},\mathbf{W})\otimes_c\mathbf{V} \\ \downarrow{\scriptstyle [\mathbf{n}_{\mathbf{U}}\otimes_c Id_V]} & & \downarrow{\scriptstyle [\mathbf{n}_4\otimes_c Id_V]} \\ \mathrm{Hom}_c(\mathbf{X},\mathbf{V}\otimes_c\mathbf{U})\otimes_c\mathbf{V} & \xrightarrow[{[\mathrm{Hom}_c(Id_X,F)\otimes_c Id_V]}]{} & \mathrm{Hom}_c(\mathbf{X},\mathbf{V}\otimes_c\mathbf{W})\otimes_c\mathbf{V} \end{array} \tag{4.13}$$

in the sense that

$$\begin{aligned} & F\circ[Id_V\otimes_c\varepsilon^{\mathbf{U}}]\circ[\mathbf{n}_{\mathbf{U}}\otimes_c Id_V]^{-1} \\ =\ & [Id_V\otimes_c\varepsilon^{\mathbf{W}}]\circ[\mathbf{n}_4\otimes_c Id_V]^{-1}\circ[\mathrm{Hom}_c(Id_X,F)\otimes_c Id_V]. \end{aligned}$$

Then the canonical map

$$\mathbf{n}_{\mathbf{W}}:\mathrm{Hom}_c(\mathbf{X},\mathbf{W})\otimes_c\mathbf{V}\to\mathrm{Hom}_c(\mathbf{X},\mathbf{V}\otimes_c\mathbf{W})$$

is also invertible, and for any morphism $A:\mathbf{V}\otimes_c\mathbf{U}\to\mathbf{V}\otimes_c\mathbf{W}$*, the composite map from* $\mathbf{U}$ *to* $\mathbf{W}$ *in the following diagram depends only on* A *and not on* $(X,J_X,\eta^{\mathbf{U}},\varepsilon^{\mathbf{U}},\varepsilon^{\mathbf{W}})$.

$$\begin{array}{ccc} \mathrm{Hom}_c(\mathbf{X},\mathbf{V}\otimes_c\mathbf{U}) & \xrightarrow{\mathrm{Hom}_c(Id_X,A)} & \mathrm{Hom}_c(\mathbf{X},\mathbf{V}\otimes_c\mathbf{W}) \\ \uparrow{\scriptstyle \mathbf{n}_{\mathbf{U}}} & & \downarrow{\scriptstyle \mathbf{n}_{\mathbf{W}}^{-1}} \\ \mathbf{V}\otimes_c\mathrm{Hom}_c(\mathbf{X},\mathbf{U}) & & \mathrm{Hom}_c(\mathbf{X},\mathbf{W})\otimes_c\mathbf{V} \\ \uparrow{\scriptstyle \eta^{\mathbf{U}}} & & \downarrow{\scriptstyle \varepsilon^{\mathbf{W}}} \\ \mathbf{U} & & \mathbf{W} \end{array} \tag{4.14}$$

Proof. The proof proceeds in the same way as the Proof of Theorem 3.7; the difference is in choosing basis sets, using Lemma 4.11, Lemma 4.12, and Lemma 4.13.

Using the basis (4.10) for $\mathrm{Hom}_c(\mathbf{X},\mathbf{U})$, the basis (4.4) for V, and the basis (4.5) for the tensor product, the set

$$\{\vec{v}_i\otimes\Phi^c_{ql}-(J_V(\vec{v}_i))\otimes(J_U\circ\Phi^c_{ql})\}$$

is a basis for $\mathbf{V}\otimes_c\mathrm{Hom}_c(\mathbf{X},\mathbf{U})$. For each basis element $\vec{u}_{l'}$ of U, there are real coefficients $\eta^{\mathbf{U}}_{iqll'}$ (finitely many non-zero for each l') so that

$$\eta^{\mathbf{U}}:\vec{u}_{l'}\mapsto\sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{l\in\mathbf{L}}\eta^{\mathbf{U}}_{iqll'}\left(\vec{v}_i\otimes\Phi^c_{ql}-(J_V(\vec{v}_i))\otimes(J_U\circ\Phi^c_{ql})\right).$$

Similarly, there are real coefficients $\varepsilon^{\mathbf{U}}_{l'qli}$ (finitely many non-zero for each l) so that

$$\varepsilon^{\mathbf{U}}:\Phi^c_{ql}\otimes\vec{v}_i-(J_U\circ\Phi^c_{ql})\otimes(J_V(\vec{v}_i))\mapsto\sum_{l'\in\mathbf{L}}\varepsilon^{\mathbf{U}}_{l'qli}\vec{u}_{l'}. \tag{4.15}$$

By the c-linearity of $\varepsilon^{\mathbf{U}}$,

$$\varepsilon^{\mathbf{U}}:(J_U\circ\Phi^c_{ql})\otimes\vec{v}_i+\Phi^c_{ql}\otimes(J_V(\vec{v}_i))\mapsto\sum_{l'\in\mathbf{L}}\varepsilon^{\mathbf{U}}_{l'qli}J_U(\vec{u}_{l'}).$$

The switching morphism from (4.12) acts on basis elements of $\mathbf{U}\otimes_c\mathbf{V}$ by:

$$\mathbf{s}:\vec{u}_{l'}\otimes\vec{v}_{i'}-(J_U(\vec{u}_{l'}))\otimes(J_V(\vec{v}_{i'}))\mapsto\vec{v}_{i'}\otimes\vec{u}_{l'}-(J_V(\vec{v}_{i'}))\otimes(J_U(\vec{u}_{l'})).$$

The first hypothesis on $\eta^{\mathbf{U}}$ and $\varepsilon^{\mathbf{U}}$ is that this gives the same output:

$$\begin{aligned}
&[Id_V\otimes_c\varepsilon^{\mathbf{U}}]\circ[\eta^{\mathbf{U}}\otimes_c Id_V]:\vec{u}_{l'}\otimes\vec{v}_{i'}-(J_U(\vec{u}_{l'}))\otimes(J_V(\vec{v}_{i'}))\\
\mapsto\ &[Id_V\otimes_c\varepsilon^{\mathbf{U}}]((\eta^{\mathbf{U}}(\vec{u}_{l'}))\otimes\vec{v}_{i'}-([J_V\otimes Id_{\mathrm{Hom}_c(\mathbf{X},\mathbf{U})}](\eta^{\mathbf{U}}(\vec{u}_{l'})))\otimes(J_V(\vec{v}_{i'})))\\
=\ &[Id_V\otimes_c\varepsilon^{\mathbf{U}}]\left(\left(\sum_{i=1}^N\sum_{q=1}^Q\sum_{l\in\mathbf{L}}\eta^{\mathbf{U}}_{iqll'}(\vec{v}_i\otimes\Phi^c_{ql}-(J_V(\vec{v}_i))\otimes(J_U\circ\Phi^c_{ql}))\right)\otimes\vec{v}_{i'}\right.\\
&\left.-\left(\sum_{i=1}^N\sum_{q=1}^Q\sum_{l\in\mathbf{L}}\eta^{\mathbf{U}}_{iqll'}((J_V(\vec{v}_i))\otimes\Phi^c_{ql}+\vec{v}_i\otimes(J_U\circ\Phi^c_{ql}))\right)\otimes(J_V(\vec{v}_{i'}))\right)\\
=\ &\sum_{i=1}^N\sum_{q=1}^Q\sum_{l\in\mathbf{L}}\eta^{\mathbf{U}}_{iqll'}\Big(\vec{v}_i\otimes(\varepsilon^{\mathbf{U}}(\Phi^c_{ql}\otimes\vec{v}_{i'}-(J_U\circ\Phi^c_{ql})\otimes(J_V(\vec{v}_{i'}))))\\
&-(J_V(\vec{v}_i))\otimes(\varepsilon^{\mathbf{U}}((J_U\circ\Phi^c_{ql})\otimes\vec{v}_{i'}+\Phi^c_{ql}\otimes(J_V(\vec{v}_{i'}))))\Big)\\
=\ &\sum_{i=1}^N\sum_{q=1}^Q\sum_{l\in\mathbf{L}}\eta^{\mathbf{U}}_{iqll'}\left(\vec{v}_i\otimes\left(\sum_{l''\in\mathbf{L}}\varepsilon^{\mathbf{U}}_{l''qli'}\vec{u}_{l''}\right)-(J_V(\vec{v}_i))\otimes\left(\sum_{l''\in\mathbf{L}}\varepsilon^{\mathbf{U}}_{l''qli'}J_U(\vec{u}_{l''})\right)\right)\\
=\ &\sum_{i=1}^N\sum_{q=1}^Q\sum_{l\in\mathbf{L}}\sum_{l''\in\mathbf{L}}\eta^{\mathbf{U}}_{iqll'}\varepsilon^{\mathbf{U}}_{l''qli'}(\vec{v}_i\otimes\vec{u}_{l''}-(J_V(\vec{v}_i))\otimes(J_U(\vec{u}_{l''}))).
\end{aligned}$$

The step from the above first line to the second uses the c-linearity of $\eta^{\mathbf{U}}$. The last sum matches the output of the switching map when:

$$\sum_{q=1}^Q\sum_{l\in\mathbf{L}}\eta^{\mathbf{U}}_{iqll'}\varepsilon^{\mathbf{U}}_{l''qli'}=\delta_{ii'}\delta_{l'l''}, \tag{4.16}$$

which is analogous to (3.8).

As in (4.10), choose the basis set for $\mathrm{Hom}_c(\mathbf{X},\mathbf{W})$:

$$\{\Psi^c_{qk}:q=1,\dots,n,k\in\mathbf{K}\},$$

with each Ψ^c_{qk} defined on basis elements of X by the formula:

$$\begin{aligned}
\vec{x}_{q'}&\mapsto\delta_{qq'}\vec{w}_k\\
J_X(\vec{x}_{q'})&\mapsto\delta_{qq'}J_W(\vec{w}_k).
\end{aligned}$$

Then as in (4.15), there are real coefficients $\varepsilon^{\mathbf{W}}_{k'qki}$ (finitely many non-zero for each k) so that

$$\varepsilon^{\mathbf{W}} : \Psi^c_{qk} \otimes \vec{v}_i - (J_W \circ \Psi^c_{qk}) \otimes (J_V(\vec{v}_i)) \mapsto \sum_{k' \in \mathbf{K}} \varepsilon^{\mathbf{W}}_{k'qki} \vec{w}_{k'}.$$

For $F : \mathbf{V} \otimes_c \mathbf{U} \to \mathbf{V} \otimes_c \mathbf{W}$, and each basis element $\vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l))$, there are real coefficients $F_{i'kil}$ (finitely many non-zero for each l) so that

$$\begin{aligned} F \ &: \ \vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l)) \\ &\mapsto \sum_{i'=1}^{N} \sum_{k \in \mathbf{K}} F_{i'kil} \left(\vec{v}_{i'} \otimes \vec{w}_k - (J_V(\vec{v}_{i'})) \otimes (J_W(\vec{w}_k)) \right). \end{aligned} \tag{4.17}$$

A basis for $\mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes_c \mathbf{U})$ can be chosen in the same way as (4.10), with

$$\begin{aligned} \vec{x}_{q'} &\mapsto \delta_{qq'}(\vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l))), \\ J_X(\vec{x}_{q'}) &\mapsto \delta_{qq'}[Id_V \otimes J_U]|_{\mathbf{V} \otimes_c \mathbf{U}}(\vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l))), \end{aligned}$$

but this map is exactly the same as

$$\begin{aligned} &\mathbf{n}_{\mathbf{U}}(\vec{v}_i \otimes \Phi^c_{ql} - (J_V(\vec{v}_i)) \otimes (J_U \circ \Phi^c_{ql})) : \\ \vec{x}_{q'} &\mapsto \vec{v}_i \otimes (\Phi^c_{ql}(\vec{x}_{q'})) - (J_V(\vec{v}_i)) \otimes (J_U(\Phi^c_{ql}(\vec{x}_{q'}))) \\ &= \vec{v}_i \otimes (\delta_{qq'}\vec{u}_l) - (J_V(\vec{v}_i)) \otimes (\delta_{qq'} J_U(\vec{u}_l)), \\ J_X(\vec{x}_{q'}) &\mapsto \vec{v}_i \otimes (\Phi^c_{ql}(J_X(\vec{x}_{q'}))) - (J_V(\vec{v}_i)) \otimes (J_U(\Phi^c_{ql}(J_X(\vec{x}_{q'})))) \\ &= \vec{v}_i \otimes (\delta_{qq'} J_U(\vec{u}_l)) + (J_V(\vec{v}_i)) \otimes (\delta_{qq'}\vec{u}_l). \end{aligned}$$

Similarly, the maps

$$\begin{aligned} &\mathbf{n}_4(\vec{v}_i \otimes \Psi^c_{qk} - (J_V(\vec{v}_i)) \otimes (J_W \circ \Psi^c_{qk})) : \\ \vec{x}_{q'} &\mapsto \delta_{qq'}(\vec{v}_i \otimes \vec{w}_k - (J_V(\vec{v}_i)) \otimes (J_W(\vec{w}_k))), \\ J_X(\vec{x}_{q'}) &\mapsto \delta_{qq'}[Id_V \otimes J_W]|_{\mathbf{V} \otimes_c \mathbf{W}}(\vec{v}_i \otimes \vec{w}_k - (J_V(\vec{v}_i)) \otimes (J_W(\vec{w}_k))) \end{aligned}$$

form a basis for $\mathrm{Hom}_c(\mathbf{X}, \mathbf{V} \otimes_c \mathbf{W})$.

To calculate the composites in the diagram (4.13), start with:

$$\begin{aligned} &(\mathrm{Hom}_c(Id_X, F) \circ \mathbf{n}_{\mathbf{U}})(\vec{v}_i \otimes \Phi^c_{ql} - (J_V(\vec{v}_i)) \otimes (J_U \circ \Phi^c_{ql})) : \\ \vec{x}_{q'} &\mapsto (F \circ (\mathbf{n}_{\mathbf{U}}(\vec{v}_i \otimes \Phi^c_{ql} - (J_V(\vec{v}_i)) \otimes (J_U \circ \Phi^c_{ql}))))(\vec{x}_{q'}) \\ &= F(\vec{v}_i \otimes (\delta_{qq'}\vec{u}_l) - (J_V(\vec{v}_i)) \otimes (\delta_{qq'} J_U(\vec{u}_l))) \\ &= \delta_{qq'} \sum_{i''=1}^{N} \sum_{k \in \mathbf{K}} F_{i''kil}(\vec{v}_{i''} \otimes \vec{w}_k - (J_V(\vec{v}_{i''})) \otimes (J_W(\vec{w}_k))) \\ &= \sum_{i''=1}^{N} \sum_{k \in \mathbf{K}} F_{i''kil}(\mathbf{n}_4(\vec{v}_{i''} \otimes \Psi^c_{qk} - (J_V(\vec{v}_{i''})) \otimes (J_W \circ \Psi^c_{qk})))(\vec{x}_{q'}). \end{aligned}$$

Since c-linear maps from X are determined by their values on $\vec{x}_{q'}$, it follows that

$$\begin{aligned} &\mathbf{n}_4^{-1} \circ \mathrm{Hom}_c(Id_X, F) \circ \mathbf{n}_{\mathbf{U}} : \vec{v}_i \otimes \Phi^c_{ql} - (J_V(\vec{v}_i)) \otimes (J_U \circ \Phi^c_{ql}) \\ &\mapsto \sum_{i''=1}^{N} \sum_{k \in \mathbf{K}} F_{i''kil}(\vec{v}_{i''} \otimes \Psi^c_{qk} - (J_V(\vec{v}_{i''})) \otimes (J_W \circ \Psi^c_{qk})), \end{aligned} \tag{4.18}$$

which is analogous to (3.9).

For basis elements of $\mathbf{V}\otimes_c \mathrm{Hom}_c(\mathbf{X},\mathbf{U})\otimes_c \mathbf{V}$ as in Lemma 4.12,

$$\begin{aligned}
&[(\mathbf{n}_4^{-1}\circ \mathrm{Hom}_c(Id_X,F)\circ \mathbf{n}_\mathbf{U})\otimes_c Id_V]:\\
&\vec{v}_i\otimes\Phi_{ql}\otimes\vec{v}_{i'}-(J_V(\vec{v}_i))\otimes(J_U\circ\Phi^c_{ql})\otimes\vec{v}_{i'}\\
&-(J_V(\vec{v}_i))\otimes\Phi^c_{ql}\otimes(J_V(\vec{v}_{i'}))-\vec{v}_i\otimes(J_U\circ\Phi^c_{ql})\otimes(J_V(\vec{v}_{i'}))\\
\mapsto\ &((\mathbf{n}_4^{-1}\circ \mathrm{Hom}_c(Id_X,F)\circ \mathbf{n}_\mathbf{U})(\vec{v}_i\otimes\Phi^c_{ql}-(J_V(\vec{v}_i))\otimes(J_U\circ\Phi^c_{ql})))\otimes\vec{v}_{i'}\\
&-((\mathbf{n}_4^{-1}\circ \mathrm{Hom}_c(Id_X,F)\circ \mathbf{n}_\mathbf{U}\circ[J_V\otimes Id_{\mathrm{Hom}_c(X,U)}])(\vec{v}_i\otimes\Phi^c_{ql}\\
&-(J_V(\vec{v}_i))\otimes(J_U\circ\Phi^c_{ql})))\otimes(J_V(\vec{v}_{i'}))\\
=\ &\left(\sum_{i''=1}^N\sum_{k\in\mathbf{K}}F_{i''kil}(\vec{v}_{i''}\otimes\Psi^c_{qk}-(J_V(\vec{v}_{i''}))\otimes(J_W\circ\Psi^c_{qk}))\right)\otimes\vec{v}_{i'}\\
&-\left([J_V\otimes Id_{\mathrm{Hom}_c(X,W)}]\left(\sum_{i''=1}^N\sum_{k\in\mathbf{K}}F_{i''kil}(\vec{v}_{i''}\otimes\Psi^c_{qk}\right.\right.\\
&\qquad\left.\left.-(J_V(\vec{v}_{i''}))\otimes(J_W\circ\Psi^c_{qk}))\right)\right)\otimes(J_V(\vec{v}_{i'}))\\
=\ &\sum_{i''=1}^N\sum_{k\in\mathbf{K}}F_{i''kil}(\vec{v}_{i''}\otimes(\Psi^c_{qk}\otimes\vec{v}_{i'}-(J_W\circ\Psi^c_{qk})\otimes(J_V(\vec{v}_{i'})))\\
&-(J_V(\vec{v}_{i''}))\otimes((J_W\circ\Psi^c_{qk})\otimes\vec{v}_{i'}+\Psi^c_{qk}\otimes(J_V(\vec{v}_{i'})))).
\end{aligned}$$

The c-linear map $[Id_V\otimes_c \varepsilon^\mathbf{W}]$ takes the above output to:

$$\sum_{i''=1}^N\sum_{k\in\mathbf{K}}\sum_{k'\in\mathbf{K}}F_{i''kil}\varepsilon^\mathbf{W}_{k'qki'}(\vec{v}_{i''}\otimes\vec{w}_{k'}-(J_V(\vec{v}_{i''}))\otimes(J_W(\vec{w}_{k'}))),\tag{4.19}$$

which is analogous to (3.10).

The hypothesis (4.13) is that for any c-linear F, the expressions (4.19) and (4.20) are equal:

$$\begin{aligned}
&F\circ[Id_V\otimes\varepsilon^\mathbf{U}]:\\
&\vec{v}_i\otimes\Phi_{ql}\otimes\vec{v}_{i'}-\vec{v}_i\otimes(J_U\circ\Phi^c_{ql})\otimes(J_V(\vec{v}_{i'}))\\
&-(J_V(\vec{v}_i))\otimes(J_U\circ\Phi^c_{ql})\otimes\vec{v}_{i'}-(J_V(\vec{v}_i))\otimes\Phi^c_{ql}\otimes(J_V(\vec{v}_{i'}))\\
\mapsto\ &F\left(\vec{v}_i\otimes\left(\sum_{l'\in\mathbf{L}}\varepsilon^\mathbf{U}_{l'qli'}\vec{u}_{l'}\right)-(J_V(\vec{v}_i))\otimes\left(\sum_{l'\in\mathbf{L}}\varepsilon^\mathbf{U}_{l'qli'}J_U(\vec{u}_{l'})\right)\right)\\
=\ &\sum_{l'\in\mathbf{L}}\varepsilon^\mathbf{U}_{l'qli'}\sum_{i''=1}^N\sum_{k'\in\mathbf{K}}F_{i''k'il'}(\vec{v}_{i''}\otimes\vec{w}_{k'}-(J_V(\vec{v}_{i''}))\otimes(J_W(\vec{w}_{k'}))),
\end{aligned}\tag{4.20}$$

so, in analogy with (3.11), for any i, q, l, i', i'', k', these finite sums are equal:

$$\sum_{k\in\mathbf{K}}F_{i''kil}\varepsilon^\mathbf{W}_{k'qki'}=\sum_{l'\in\mathbf{L}}\varepsilon^\mathbf{U}_{l'qli'}F_{i''k'il'},\tag{4.21}$$

which is the same relation as (3.12).

For the composite from the diagram (4.14) in the conclusion of the Theorem, a calculation analogous to (4.18) gives, for any A as in (4.17):

$$\begin{aligned} &\mathbf{n}_{\mathbf{W}}^{-1}\circ \mathrm{Hom}_c(Id_X,A)\circ \mathbf{n}_{\mathbf{U}}:\vec{v}_i\otimes \Phi^c_{ql}-(J_V(\vec{v}_i))\otimes (J_U\circ \Phi^c_{ql}) \\ &\mapsto \sum_{i'=1}^{N}\sum_{k\in\mathbf{K}} A_{i'kil}(\Psi^c_{qk}\otimes \vec{v}_{i'}-(J_W\circ \Psi^c_{qk})\otimes (J_V(\vec{v}_{i'}))). \end{aligned} \tag{4.22}$$

The next steps use (4.16), (4.21), and (4.22).

$$\begin{aligned} &\varepsilon^{\mathbf{W}}\circ \mathbf{n}_{\mathbf{W}}^{-1}\circ \mathrm{Hom}_c(Id_X,A)\circ \mathbf{n}_{\mathbf{U}}\circ \eta^{\mathbf{U}}: \\ \vec{u}_l &\mapsto \varepsilon^{\mathbf{W}}\left((\mathbf{n}_{\mathbf{W}}^{-1}\circ \mathrm{Hom}_c(Id_X,A)\circ \mathbf{n}_{\mathbf{U}})\left(\sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{l'\in\mathbf{L}}\eta^{\mathbf{U}}_{iql'l}(\vec{v}_i\otimes \Phi_{ql'}\right.\right. \\ &\qquad \left.\left. -(J_V(\vec{v}_i))\otimes (J_U\circ \Phi^c_{ql'}))\right)\right) \\ &= \varepsilon^{\mathbf{W}}\left(\sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{l'\in\mathbf{L}}\eta^{\mathbf{U}}_{iql'l}\sum_{i'=1}^{N}\sum_{k\in\mathbf{K}}A_{i'kil'}(\Psi^c_{qk}\otimes \vec{v}_{i'}-(J_W\circ \Psi^c_{qk})\otimes (J_V(\vec{v}_{i'})))\right) \\ &= \sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{l'\in\mathbf{L}}\sum_{i'=1}^{N}\sum_{k\in\mathbf{K}}\sum_{k'\in\mathbf{K}}\eta^{\mathbf{U}}_{iql'l}A_{i'kil'}\varepsilon^{\mathbf{W}}_{k'qki'}\vec{w}_{k'} \\ &= \sum_{i=1}^{N}\sum_{q=1}^{Q}\sum_{l'\in\mathbf{L}}\sum_{i'=1}^{N}\sum_{k'\in\mathbf{K}}\sum_{l''\in\mathbf{L}}\eta^{\mathbf{U}}_{iql'l}\varepsilon^{\mathbf{U}}_{l''ql'i'}A_{i'k'il''}\vec{w}_{k'} \\ &= \sum_{i=1}^{N}\sum_{i'=1}^{N}\sum_{k'\in\mathbf{K}}\sum_{l''\in\mathbf{L}}\delta_{ii'}\delta_{ll''}A_{i'k'il''}\vec{w}_{k'} \\ &= \sum_{i=1}^{N}\sum_{k'\in\mathbf{K}}A_{ik'il}\vec{w}_{k'}. \end{aligned} \tag{4.23}$$

The conclusion is that (4.23) does not depend on the choices of X, J_X, $\eta^{\mathbf{U}}$, $\varepsilon^{\mathbf{U}}$, or $\varepsilon^{\mathbf{W}}$. □

The output of the composite (4.14), the last sum (4.23), ends up looking a lot like (2.6) and (3.15), as a result of the choices made for a basis. One difference is that in (4.23), the sum from 1 to N is only over half the real dimension of V. The composite map (4.14) from the conclusion of Theorem 4.14 can be used as a definition of the generalized trace for morphisms $A:\mathbf{V}\otimes_c\mathbf{U}\to\mathbf{V}\otimes_c\mathbf{W}$ in the category $\mathscr{C}$:

$$Tr_{\mathbf{V};\mathbf{U},\mathbf{W}}(A)=\varepsilon^{\mathbf{W}}\circ \mathbf{n}_{\mathbf{W}}^{-1}\circ \mathrm{Hom}_c(Id_X,A)\circ \mathbf{n}_{\mathbf{U}}\circ \eta^{\mathbf{U}}:\mathbf{U}\to\mathbf{W}. \tag{4.24}$$

Another conclusion from Theorem 4.14 is that the generalized trace does not depend on any choice of unit object for $\otimes_c$ in $\mathscr{C}$ as discussed in Example 4.4, and formula (4.24) does not even require that such a choice be made.

Theorem 4.14 can only be used to find the generalized trace of a map between tensor products in $\mathscr{C}$ defined as in Example 4.4; if you have N^2 complex numbers arranged into a square and you want a complex number for the trace of the corresponding linear transformation as in (1.1), this approach won't help and you should add up the diagonal entries. However, in the interest of giving a concrete, real matrix calculation to illustrate formulas (4.23) and (4.24), Example 4.17 makes some simple choices for objects (V,J_V) and (U,J_U) in $\mathscr{C}$. We first need to find some specific evaluation and coevaluation morphisms in $\mathscr{C}$ — so that the hypothesis of Theorem 4.14 is non-vacuous.

EXAMPLE 4.15. Recall, for any real vector spaces V, U, the canonical evaluation map from Notation 3.1:

$$Ev_{VU} : \mathrm{Hom}(V,U) \otimes V \to U : A \otimes \vec{v} \mapsto A(\vec{v}).$$

If V and U have CSOs J_V and J_U, the subspace of $\mathrm{Hom}(V,U) \otimes V$ where all three induced CSOs agree,

$$[\mathrm{Hom}(J_V, Id_U) \otimes Id_V] = [\mathrm{Hom}(Id_V, J_U) \otimes Id_V] = [Id_{\mathrm{Hom}(V,U)} \otimes J_V],$$

is spanned by elements of the form

$$A \otimes \vec{v} - (J_U \circ A) \otimes (J_V(\vec{v})),$$

for c-linear maps A. The restriction of Ev_{VU} to this subspace, denoted

$$Ev^c_{\mathbf{VU}} : \mathrm{Hom}_c(\mathbf{V},\mathbf{U}) \otimes_c \mathbf{V} \to \mathbf{U},$$

acts on such elements:

$$Ev^c_{\mathbf{VU}} : A \otimes \vec{v} - (J_U \circ A) \otimes (J_V(\vec{v})) \mapsto A(\vec{v}) - (J_U \circ A)(J_V(\vec{v})) = 2A(\vec{v}),$$

the last step using the c-linearity of A. $Ev^c_{\mathbf{VU}}$ is itself c-linear:

$$\begin{aligned} & Ev^c_{\mathbf{VU}} \circ [\mathrm{Hom}(Id_V, J_U) \otimes Id_V] : \\ A \otimes \vec{v} - (J_U \circ A) \otimes (J_V(\vec{v})) \mapsto\ & Ev^c_{\mathbf{VU}}((J_U \circ A) \otimes \vec{v} + A \otimes (J_V(\vec{v}))) \\ =\ & J_U(A(\vec{v})) + A(J_V(\vec{v})) = 2J_U(A(\vec{v})) \\ =\ & (J_U \circ Ev^c_{\mathbf{VU}})(A \otimes \vec{v} - (J_U \circ A) \otimes (J_V(\vec{v}))). \end{aligned}$$

For $\mathbf{W} = (W, J_W)$, $Ev^c_{\mathbf{VU}}$ and $Ev^c_{\mathbf{VW}}$ satisfy c-linear versions of Lemma 3.2, Lemma 3.5, and Lemma 3.6 (the details are omitted here), so the compatibility condition (4.13) is satisfied for $\mathbf{X} = \mathbf{V}$, $\varepsilon^{\mathbf{U}} = Ev^c_{\mathbf{VU}}$, $\varepsilon^{\mathbf{W}} = Ev^c_{\mathbf{VW}}$, and any c-linear F.

EXAMPLE 4.16. To find a coevaluation in $\mathscr{C}$ corresponding to the evaluation from Example 4.15, recall the result of Lemma 3.12,

$$\eta_{VU} : U \to V \otimes \mathrm{Hom}(V,U) : \vec{u} \mapsto n_U^{-1}(n_5(Id_V \otimes \vec{u})).$$

This gives a formula for this real coevaluation that does not depend on a choice of basis. Both n_U and n_5 are c-linear on corresponding pairs of the three commuting CSOs on each domain and target, as in Example 4.10. η_{VU} is c-linear with respect to J_U and $[Id_V \otimes \mathrm{Hom}(Id_V, J_U)]$:

$$\begin{aligned}\eta_{VU} : J_U(\vec{u}) \mapsto\ & n_U^{-1}(n_5(Id_V \otimes (J_U(\vec{u})))) \\ =\ & n_U^{-1}(n_5([Id_{\mathrm{Hom}(V,V)} \otimes J_U](Id_V \otimes \vec{u}))) \\ =\ & [Id_V \otimes \mathrm{Hom}(Id_V, J_U)](n_U^{-1}(n_5(Id_V \otimes \vec{u}))) \\ =\ & [Id_V \otimes \mathrm{Hom}(Id_V, J_U)](\eta_{VU}(\vec{u})).\end{aligned}$$

Also by the c-linearity of the n maps, because Id_V is c-linear, the image of η_{VU} is contained in the subspace of $V \otimes \mathrm{Hom}(V,U)$ where two of the three CSOs agree, $[J_V \otimes Id_{\mathrm{Hom}(V,U)}]$ and $[Id_V \otimes \mathrm{Hom}(J_V, Id_U)]$ — this is the subspace $\mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{V}, U)$ from Example 4.10 with $\mathbf{X} = \mathbf{V}$. So, to get a c-linear map from U to the subspace of $V \otimes \mathrm{Hom}(V,U)$ where all three CSOs agree, compose η_{VU} with the c-linear projection map $\mathbf{P}_c$ from (4.3), which equals the restriction of the following map to the subspace $\mathbf{V} \otimes_c \mathrm{Hom}(\mathbf{V}, U)$:

$$\frac{1}{2}\left(Id_{V \otimes \mathrm{Hom}(V,U)} - [J_V \otimes Id_{\mathrm{Hom}(V,U)}] \circ [Id_V \otimes \mathrm{Hom}(Id_V, J_U)]\right). \tag{4.25}$$

Define the c-linear map

$$\eta^c_{\mathbf{VU}} = \mathbf{P}_c \circ \eta_{VU} : \mathbf{U} \to \mathbf{V} \otimes_c \mathrm{Hom}_c(\mathbf{V}, \mathbf{U}). \tag{4.26}$$

To get an expression for η_{VU} in terms of the basis $\{\vec{u}_l\}$ for U, we need to adapt the expression (3.18) to the $2N$-element basis (4.4) for V:

$$\eta_{VU} : \vec{u}_l \mapsto \sum_{i=1}^{N} \left(\vec{v}_i \otimes \Phi_{il} + (J_V(\vec{v}_i)) \otimes \Phi'_{il}\right),$$

where

$$\begin{aligned}\Phi_{il} : \vec{v}_{i'} &\mapsto \delta_{ii'}\vec{u}_l, \\ \Phi_{il} : J_V(\vec{v}_{i'}) &\mapsto \vec{0}_U, \\ \Phi'_{il} : \vec{v}_{i'} &\mapsto \vec{0}_U, \\ \Phi'_{il} : J_V(\vec{v}_{i'}) &\mapsto \delta_{ii'}\vec{u}_l.\end{aligned}$$

With this notation, the maps (4.10) from Lemma 4.13 in the case $X = V$ satisfy:

$$\Phi^c_{il} = \Phi_{il} + J_U \circ \Phi'_{il}.$$

Then the composite $\eta^c_{\mathbf{VU}}$ is given by the formula

$$\begin{aligned}\mathbf{P}_c \circ \eta_{VU} : \vec{u}_l \mapsto & \frac{1}{2}\sum_{i=1}^{N}\left(\vec{v}_i \otimes \Phi_{il} + (J_V(\vec{v}_i)) \otimes \Phi'_{il}\right) \\ & -\frac{1}{2}\sum_{i=1}^{N}\left((J_V(\vec{v}_i)) \otimes (J_U \circ \Phi_{il}) - \vec{v}_i \otimes (J_U \circ \Phi'_{il})\right) \\ = & \frac{1}{2}\sum_{i=1}^{N}(\vec{v}_i \otimes \Phi^c_{il} - (J_V(\vec{v}_i)) \otimes (J_U \circ \Phi^c_{il}))\,.\end{aligned}$$

This choice for $\eta^{\mathbf{U}} = \eta^c_{\mathbf{VU}}$, together with $\varepsilon^{\mathbf{U}} = Ev^c_{\mathbf{VU}}$ from Example 4.15, satisfies the hypothesis (4.12) from Theorem 4.14. The following composite is equal to the switching map on $\mathbf{U} \otimes_c \mathbf{V}$.

$$\begin{aligned} & \vec{u}_l \otimes \vec{v}_i - (J_U(\vec{u}_l)) \otimes (J_V(\vec{v}_i)) \\ \mapsto\ & ([Id_V \otimes_c Ev^c_{\mathbf{VU}}] \circ [\eta^c_{\mathbf{VU}} \otimes_c Id_V])(\vec{u}_l \otimes \vec{v}_i - (J_U(\vec{u}_l)) \otimes (J_V(\vec{v}_i))) \\ =\ & [Id_V \otimes_c Ev^c_{\mathbf{VU}}]((\eta^c_{\mathbf{VU}}(\vec{u}_l)) \otimes \vec{v}_i \\ & -([Id_V \otimes \mathrm{Hom}(Id_V, J_U)](\eta^c_{\mathbf{VU}}(\vec{u}_l))) \otimes (J_V(\vec{v}_i))) \\ =\ & [Id_V \otimes_c Ev^c_{\mathbf{VU}}]\left(\frac{1}{2}\left(\sum_{i'=1}^{N}(\vec{v}_{i'} \otimes \Phi^c_{i'l} \otimes \vec{v}_i - (J_V(\vec{v}_{i'})) \otimes (J_U \circ \Phi^c_{i'l}) \otimes \vec{v}_i)\right)\right. \\ & \left. -\frac{1}{2}\left(\sum_{i'=1}^{N}(\vec{v}_{i'} \otimes (J_U \circ \Phi^c_{i'l}) \otimes (J_V(\vec{v}_i)) + (J_V(\vec{v}_{i'})) \otimes \Phi^c_{i'l} \otimes (J_V(\vec{v}_i)))\right)\right) \\ =\ & \frac{1}{2}\sum_{i'=1}^{N}(\vec{v}_{i'} \otimes (Ev^c_{\mathbf{VU}}(\Phi^c_{i'l} \otimes \vec{v}_i - (J_U \circ \Phi^c_{i'l}) \otimes (J_V(\vec{v}_i)))) \\ & - (J_V(\vec{v}_{i'})) \otimes (J_U \circ Ev^c_{\mathbf{VU}})(\Phi^c_{i'l} \otimes \vec{v}_i - (J_U \circ \Phi^c_{i'l}) \otimes (J_V(\vec{v}_i)))) \\ =\ & \frac{1}{2}\sum_{i'=1}^{N}(\vec{v}_{i'} \otimes (2\Phi^c_{i'l}(\vec{v}_i)) - (J_V(\vec{v}_{i'})) \otimes (J_U(2\Phi^c_{i'l}(\vec{v}_i)))) \\ =\ & \frac{1}{2}\sum_{i'=1}^{N}(\vec{v}_{i'} \otimes (2\delta_{ii'}\vec{u}_l) - (J_V(\vec{v}_{i'})) \otimes (J_U(2\delta_{ii'}\vec{u}_l))) \\ =\ & \vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l)).\end{aligned}$$

EXAMPLE 4.17. Let $V = \mathbb{R}^4$, with a CSO given by the matrix

$$J_V = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

so that this list is an ordered basis for V as in (4.4):

$$(\vec{v}_1, J_V(\vec{v}_1), \vec{v}_2, J_V(\vec{v}_2)) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right).$$

Let $U = W = \mathbb{R}^2$, with CSO $J_U = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and the ordered basis

$$(\vec{u}_1, \vec{u}_2) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

A basis for $\mathbf{V} \otimes_c \mathbf{U}$, as in Lemma 4.11, has four elements:

$$\{\vec{v}_1 \otimes \vec{u}_1 - (J_V(\vec{v}_1)) \otimes (J_U(\vec{u}_1)), \dots, \vec{v}_2 \otimes \vec{u}_2 - (J_V(\vec{v}_2)) \otimes (J_U(\vec{u}_2))\}.$$

For real constants $a, \dots, h$, the following matrix defines a c-linear transformation $(V, J_V) \to (V, J_V)$:

$$B = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ e & f & g & h \\ -f & e & -h & g \end{bmatrix}.$$

The c-linear map

$$A = [B \otimes_c Id_U] : \mathbf{V} \otimes_c \mathbf{U} \to \mathbf{V} \otimes_c \mathbf{U}$$

has trace as in (4.24):

$$Tr_{\mathbf{V};\mathbf{U},\mathbf{U}}(A) = \varepsilon^{\mathbf{U}} \circ \mathbf{n}_{\mathbf{U}}^{-1} \circ \mathrm{Hom}_c(Id_X, [B \otimes_c Id_U]) \circ \mathbf{n}_{\mathbf{U}} \circ \eta^{\mathbf{U}} : \mathbf{U} \to \mathbf{U}.$$

To get a 2×2 matrix representation for this c-linear map, we need the coefficients $A_{ili'l'}$ as in (4.17). For example,

$$\begin{aligned} & A(\vec{v}_1 \otimes \vec{u}_1 - (J_V(\vec{v}_1)) \otimes (J_U(\vec{u}_1))) \\ &= [B \otimes_c Id_U] \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} a \\ -b \\ e \\ -f \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} b \\ a \\ f \\ e \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \sum_{i=1}^{2} \sum_{l=1}^{2} A_{il11}(\vec{v}_i \otimes \vec{u}_l - (J_V(\vec{v}_i)) \otimes (J_U(\vec{u}_l))), \end{aligned}$$

with

$$A_{1111} = a,\ A_{1211} = -b,\ A_{2111} = e,\ A_{2211} = -f.$$

Similarly,

$$\begin{aligned} A_{1112} &= b,\ A_{1212} = a,\ A_{2112} = f,\ A_{2212} = e, \\ A_{1121} &= c,\ A_{1221} = -d,\ A_{2121} = g,\ A_{2221} = -h, \\ A_{1122} &= d,\ A_{1222} = c,\ A_{2122} = h,\ A_{2222} = g. \end{aligned}$$

Then, formula (4.23) gives:

$$\begin{aligned} \vec{u}_1 &\mapsto \sum_{i=1}^{2}\sum_{l=1}^{2} A_{ili1}\vec{u}_l \\ &= (A_{1111} + A_{2121})\vec{u}_1 + (A_{1211} + A_{2221})\vec{u}_2 = \begin{bmatrix} a+g \\ -b-h \end{bmatrix}, \\ \vec{u}_2 &\mapsto \sum_{i=1}^{2}\sum_{l=1}^{2} A_{ili2}\vec{u}_l \\ &= (A_{1112} + A_{2122})\vec{u}_1 + (A_{1212} + A_{2222})\vec{u}_2 = \begin{bmatrix} b+h \\ a+g \end{bmatrix} \\ \implies Tr_{\mathbf{V};\mathbf{U},\mathbf{U}}(A) &= \begin{bmatrix} a+g & b+h \\ -(b+h) & a+g \end{bmatrix}. \end{aligned} \tag{4.27}$$

5. Relating the complex trace to the real trace

Given V with finite dimension and a CSO J, a c-linear map $(V,J) \to (V,J)$ has a real matrix representation $A_{2N\times 2N}$ with respect to some real basis. In this Section, we continue the without-complex-numbers approach in order to find an analogue of (1.1) for such matrices $A_{2N\times 2N}$, and to see how (1.1) and the generalized trace expression (4.27) from Example 4.17 are related.

When comparing the trace in the category $\mathscr{C}$ to the trace in the category of real vector spaces, there is some risk of confusing them, and as seen in Example 4.15 and Example 4.16, there are some factors of 2 and $\frac{1}{2}$ that merit close attention. Our first step is to be more precise about direct sums (which have already appeared in Section 4) by introducing some notation. The following Definition is an "external direct sum" construction (as in [1] §6).

DEFINITION 5.1. Given real vector spaces U, U_1, U_2, and ordered pairs of maps (P_1,P_2) and (Q_1,Q_2), where $P_i : U \to U_i$, $Q_i : U_i \to U$ for $i = 1,2$, U is a *direct sum* of U_1 and U_2 means:

$$\begin{aligned} Q_1 \circ P_1 + Q_2 \circ P_2 &= Id_U \\ P_1 \circ Q_1 &= Id_{U_1} \\ P_2 \circ Q_2 &= Id_{U_2}. \end{aligned}$$

NOTATION 5.2. The data from Definition 5.1 will be abbreviated $U = U_1 \oplus U_2$, when the maps P_i (called *projections*) and Q_i (*inclusions*) are understood.

It follows from Definition 5.1 that $P_i \circ Q_I = 0_{\mathrm{Hom}(U_I, U_i)}$ for $i \neq I$.

EXAMPLE 5.3. For an involution $K : U \to U$, consider the ± 1 eigenspaces, $U_1 = \{\vec{u} \in U : K(\vec{u}) = \vec{u}\}$, and $U_2 = \{\vec{u} \in U : K(\vec{u}) = -\vec{u}\}$. Then $U = U_1 \oplus U_2$, where Q_i are the subspace inclusion maps, and the projections are:

$$P_1 = \frac{1}{2} \cdot (Id_U + K), \ \ P_2 = \frac{1}{2} \cdot (Id_U - K).$$

A special case of Example 5.3 already appeared in Notation 4.2.

LEMMA 5.4. *Given U, the following are equivalent:*

1. *$U = U_1 \oplus U_2$ and there exists an invertible map $R : U_1 \to U_2$;*
2. *U admits a CSO J and an involution C which anticommutes with J (i.e., $C \circ C = Id_U$ and $C \circ J = -J \circ C$).*

Proof. The significance of the Lemma is not whether there exist such structures, but how one can be constructed from the other.

To show *1.* $\Longrightarrow$ *2.*, let U_1, U_2, Q_1, Q_2, P_1, P_2 be as in Definition 5.1. Then for R as in *1.*,

$$J = Q_2 \circ R \circ P_1 - Q_1 \circ R^{-1} \circ P_2 \tag{5.1}$$

is a CSO (easily checked), and

$$C = Q_1 \circ P_1 - Q_2 \circ P_2 \tag{5.2}$$

is an involution anticommuting with J.

Conversely, to show *2.* $\Longrightarrow$ *1.*, the involution C produces, as in Example 5.3, a direct sum $U = U_1 \oplus U_2$ with projections

$$P_1' = \frac{1}{2}(Id_U + C), \ P_2' = \frac{1}{2}(Id_U - C) \tag{5.3}$$

and corresponding subspace inclusions Q_1', Q_2'. In particular, $Q_1' \circ P_1' : U \to U$ is also given by the formula $\frac{1}{2}(Id_U + C)$, and similarly for $Q_2' \circ P_2'$. The composite

$$P_2' \circ J \circ Q_1' : U_1 \to U_2 \tag{5.4}$$

is invertible, with inverse

$$-P_1' \circ J \circ Q_2' : U_2 \to U_1, \tag{5.5}$$

so *1.* holds. □

In the above Proof, neither implication is given by a canonical construction; some signs in (5.1), (5.2), (5.4), (5.5) could have been chosen differently. The above choices are, however, consistent with each other, in the sense that *2.* $\Longrightarrow$ *1.* $\Longrightarrow$ *2.* returns the same data J, C.

DEFINITION 5.5. Given a real vector space U with a CSO J, a real linear map $C:U\to U$ is a *real structure operator* means: C is an involution that anticommutes with J.

NOTATION 5.6. More briefly, a real structure operator is called a *RSO* with respect to the given J, and (U,J) is said to have a *real structure*. There is a canonical (unordered) pair of subspaces, where U_1 is the fixed point set of C and U_2 is the -1 eigenspace, so the notational convention will be to order them with U_1 first, and to refer to the direct sum produced by C as in Lemma 5.4 as $U_1\oplus U_2$. (Having chosen C and this ordering, the maps P_1', P_2', Q_1', Q_2' are canonical even if the map R from (5.4) in Lemma 5.4 is not.)

LEMMA 5.7. *Given U with CSO J_U and RSO C_U, and another space W with CSO J_W, any c-linear map $A:(U,J_U)\to(W,J_W)$ is determined by its values on the fixed point subspace U_1.*

Proof. The meaning of the Lemma is that if $B:\mathbf{U}\to\mathbf{W}$ is another c-linear map, and $A\circ Q_1'=B\circ Q_1':U_1\to W$, then $A=B$. An analogous idea, but depending on a choice of basis, was used in Section 4. Here, the result follows from only the properties of the direct sum from Lemma 5.4.

$$\begin{aligned}
A &= A\circ(Q_1'\circ P_1'+Q_2'\circ P_2')\\
&= A\circ Q_1'\circ P_1' - A\circ J_U\circ J_U\circ\left(\frac{1}{2}\cdot(Id_U-C)\right)\\
&= A\circ Q_1'\circ P_1' - J_W\circ A\circ\left(\frac{1}{2}\cdot(Id_U+C)\right)\circ J_U\\
&= (A\circ Q_1')\circ P_1' - J_W\circ(A\circ Q_1')\circ P_1'\circ J_U. \quad \square
\end{aligned} \tag{5.6}$$

EXAMPLE 5.8. Given U with any CSO J_U and RSO C_U, and the direct sum structure $U=U_1\oplus U_2$ produced by C_U as in Notation 5.6, and another space V with CSO J_V, there are two commuting CSOs on $V\otimes U$. An involution on $V\otimes U$ is $[Id_V\otimes C_U]$, which commutes with the CSO $[J_V\otimes Id_U]$ and anticommutes with the other CSO $[Id_V\otimes J_U]$. The involution $[Id_V\otimes C_U]$ produces a direct sum $(V\otimes U_1)\oplus(V\otimes U_2)$, as in Example 5.3. This notation is justified by the equality of the projection maps

$$P_1=\frac{1}{2}(Id_{V\otimes U}+[Id_V\otimes C_U])=[Id_V\otimes P_1']=[Id_V\otimes(\frac{1}{2}(Id_U+C_U))]$$

and similarly for $P_2=[Id_V\otimes P_2']$, so $Q_1=[Id_V\otimes Q_1']$ and $Q_2=[Id_V\otimes Q_2']$ are the subspace inclusion maps for the images of the projections. Because $[J_V\otimes Id_U]$ commutes with the involution $[Id_V\otimes C_U]$, it respects the direct sum and restricts to a CSO on each subspace, $V\otimes U_1$ and $V\otimes U_2$. Another involution on $V\otimes U$ is $[J_V\otimes Id_U]\circ[Id_V\otimes J_U]$, as in Example 4.4; this produces a different direct sum structure for $V\otimes U$. Let

$$P_c=\frac{1}{2}(Id_{V\otimes U}-[J_V\otimes J_U]):V\otimes U\to\mathbf{V}\otimes_c\mathbf{U} \tag{5.7}$$

denote the projection onto the -1 eigenspace as in Notation 4.2. Let $P_a = \frac{1}{2}(Id_{V\otimes U} + [J_V \otimes J_U])$ denote the projection on the $+1$ eigenspace $V \otimes_a U$, with corresponding subspace inclusions Q_c and Q_a. The composite $Q_c \circ P_c$ is also given by the formula (5.7). The composite

$$P_c \circ Q_1 : (V \otimes U_1, [J_V \otimes Id_U]|_{V\otimes U_1}) \to \mathbf{V} \otimes_c \mathbf{U} \tag{5.8}$$

is c-linear:

$$P_c \circ Q_1 \circ [J_V \otimes Id_U]|_{V\otimes U_1} = P_c \circ [J_V \otimes Id_U] \circ Q_1 = [J_V \otimes Id_U]|_{\mathbf{V}\otimes_c \mathbf{U}} \circ P_c \circ Q_1,$$

and similarly for $P_c \circ Q_2$. The map $P_c \circ Q_1$ is also invertible; an inverse is given by

$$(P_c \circ Q_1)^{-1} = 2P_1 \circ Q_c : \mathbf{V} \otimes_c \mathbf{U} \to V \otimes U_1. \tag{5.9}$$

First, note that the involution $[Id_V \otimes C_U]$ satisfies:

$$\begin{aligned}
Q_c \circ P_c \circ [Id_V \otimes C_U] &= \frac{1}{2}(Id_{V\otimes U} - [J_V \otimes J_U]) \circ [Id_V \otimes C_U] \\
&= [Id_V \otimes C_U] \circ \frac{1}{2}(Id_{V\otimes U} + [J_V \otimes J_U]) \\
&= [Id_V \otimes C_U] \circ Q_a \circ P_a.
\end{aligned}$$

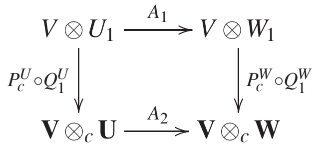
Then

$$\begin{aligned}
P_c \circ Q_1 \circ (2P_1 \circ Q_c) &= 2P_c \circ Q_c \circ P_c \circ (\frac{1}{2}(Id_{V\otimes U} + [Id_V \otimes C_U])) \circ Q_c \\
&= P_c \circ Q_c \circ P_c \circ Q_c + P_c \circ [Id_V \otimes C_U] \circ Q_a \circ P_a \circ Q_c \\
&= Id_{\mathbf{V}\otimes_c \mathbf{U}} + 0_{\mathrm{Hom}(\mathbf{V}\otimes_c \mathbf{U}, \mathbf{V}\otimes_c \mathbf{U})}.
\end{aligned}$$

Similarly, the composite in the other order is $(2P_1 \circ Q_c) \circ (P_c \circ Q_1) = Id_{V\otimes U_1}$.

REMARK 5.9. The two maps in (5.9) could be re-scaled to $(\sqrt{2}P_c \circ Q_1)^{-1} = \sqrt{2}P_1 \circ Q_c$ to have a more symmetric appearance, and such a re-scaling would not affect the results in the rest of this Section.

The following Theorem 5.10 is the main result of this Section; it finds an expression for the trace in $\mathscr{C}$ in terms of real traces. The maps A_1 and A_2 are analogous to, respectively, A and $\mathbf{A}$ from (1.1). The object $\mathbf{V} = (V, J_V)$ and the object $\mathbf{U} = (U, J_U)$ with RSO C_U and direct sum structure $U = U_1 \oplus U_2$ are as in Example 5.8. The map from (5.7) is re-labeled P_c^U, with corresponding inclusion $Q_c^U = Q_c$. The inclusion $Q_1 : V \otimes U_1 \to V \otimes U$ is re-labeled Q_1^U. The object $\mathbf{W} = (W, J_W)$ has RSO C_W and direct sum structure $W = W_1 \oplus W_2$ as in Notation 5.6, with projections $P_1'' : W \to W_1$, $P_2'' : W \to W_2$ and corresponding inclusions Q_1'', Q_2''. Denote the map corresponding to (5.7) by $P_c^W : V \otimes W \to \mathbf{V} \otimes_c \mathbf{W}$, and denote the inclusion $Q_1^W : V \otimes W_1 \to V \otimes W$. The hypothesis (5.10) with the invertible composites $P_c^U \circ Q_1^U$ and $P_c^W \circ Q_1^W$ states the commutativity of the following diagram,



which can be thought of as showing that A_1 and A_2 are real and complex versions of each other.

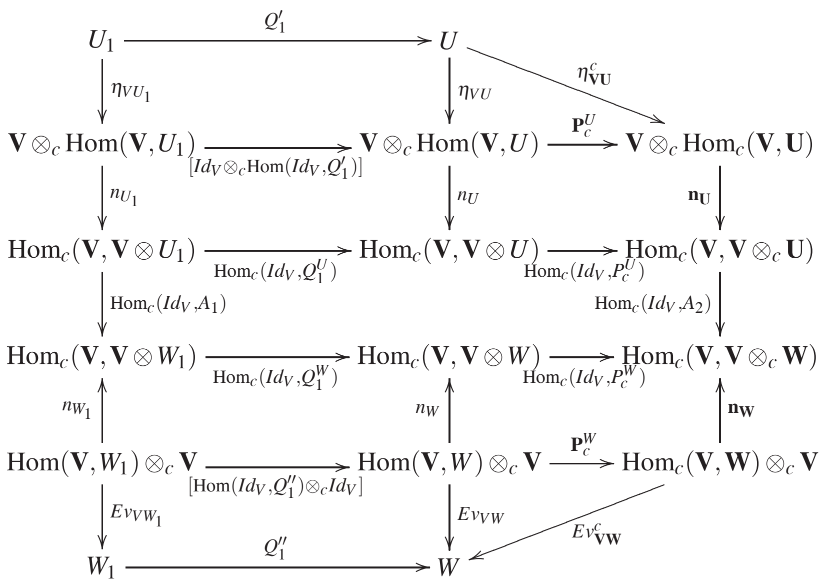
THEOREM 5.10. *Given* $\mathbf{V} = (V, J_V)$ *with finite dimension,* $\mathbf{U} = (U, J_U)$ *with RSO* C_U*, and* $\mathbf{W} = (W, J_W)$ *with RSO* C_W*, if* $A_1 : V \otimes U_1 \to V \otimes W_1$ *is c-linear with respect to* $[J_V \otimes Id_{U_1}]$ *and* $[J_V \otimes Id_{W_1}]$*, and* $A_2 : \mathbf{V} \otimes_c \mathbf{U} \to \mathbf{V} \otimes_c \mathbf{W}$ *is c-linear with respect to the induced CSOs, and*

$$A_2 \circ P_c^U \circ Q_1^U = P_c^W \circ Q_1^W \circ A_1 : V \otimes U_1 \to \mathbf{V} \otimes_c \mathbf{W}, \tag{5.10}$$

then

$$\begin{aligned} Tr_{\mathbf{V};\mathbf{U},\mathbf{W}}(A_2) = \; & \frac{1}{2} Q_1'' \circ (Tr_{V;U_1,W_1}(A_1)) \circ P_1' \\ & - \frac{1}{2} J_W \circ Q_1'' \circ (Tr_{V;U_1,W_1}([J_V \otimes Id_{W_1}] \circ A_1)) \circ P_1' \\ & - \frac{1}{2} J_W \circ Q_1'' \circ (Tr_{V;U_1,W_1}(A_1)) \circ P_1' \circ J_U \\ & - \frac{1}{2} Q_1'' \circ (Tr_{V;U_1,W_1}([J_V \otimes Id_{W_1}] \circ A_1)) \circ P_1' \circ J_U. \end{aligned}$$

Proof. The following diagram shows composites that define a real trace on the left and a trace in $\mathscr{C}$ on the right. The notation will be explained below.



By the main results from the previous Sections, Theorem 3.7 and Theorem 4.14, we can choose any evaluation and coevaluation maps to compute the traces. On the right side, the downward composite from U to W is $Tr_{\mathbf{V};\mathbf{U},\mathbf{W}}(A_2)$ as in (4.24), using the canonical evaluation from Example 4.15, and the coevaluation constructed in Example 4.16. In particular, the upper right triangle in the diagram is commutative by the definition (4.26), where the map $\mathbf{P}_c$ from Example 4.16 has been re-labeled $\mathbf{P}_c^U$. The square below that triangle, with n_U and $\mathbf{n_U}$ is commutative, as in Example 4.10. The lower square with n_W, $\mathbf{n_W}$ is also commutative as in Example 4.10, with $\mathbf{P}_c^W$ analogous to (4.3), but with some re-ordering of spaces and $\mathbf{X}=\mathbf{V}$, so that, in analogy with (4.25), the projection $\mathbf{P}_c^W$ is a restriction of this projection:

$$\frac{1}{2}\left(Id_{\mathrm{Hom}(V,W)\otimes V}-[Id_{\mathrm{Hom}(V,W)}\otimes J_V]\circ[\mathrm{Hom}(Id_V,J_W)\otimes Id_V]\right). \tag{5.11}$$

The center block with A_1 and A_2 is commutative by the hypothesis (5.10).

The left column starts with the coevaluation map η_{VU_1} from Example 3.11, and takes advantage of the observation from Example 4.16 that its image is contained in the subspace $\mathbf{V}\otimes_c\mathrm{Hom}(\mathbf{V},U_1)$, as in Example 4.10 (but without a CSO on U_1). Every step in the composite stays in the subspaces where the two CSOs induced by J_V are equal, using the c-linearity of n_{U_1} and n_{W_1} (as in Example 4.10), and A_1 (by hypothesis). The last map is the restriction of the canonical evaluation Ev_{VW_1} from Notation 3.1 to the subspace $\mathrm{Hom}(\mathbf{V},W_1)\otimes_c\mathbf{V}$.

The commutativity of the block on the left with n_{U_1} and n_U follows from Lemma 3.10, and so does the commutativity of the block with n_{W_1} and n_W. The upper left block with the coevaluations is then easily seen to be commutative using Lemma 3.12. The lower left block with the evaluations is commutative by Lemma 3.2.

The only block in the diagram that is not commutative, and this is the key step for the Theorem, is the lower right triangle. An element of $\mathrm{Hom}(\mathbf{V},W)\otimes_c\mathbf{V}$ of the form

$$B\otimes\vec{v}-(B\circ J_V)\otimes(J_V(\vec{v})), \tag{5.12}$$

for a (real linear) $B\in\mathrm{Hom}(V,W)$ as in Lemma 4.11, is mapped by Ev_{VW} to

$$B(\vec{v})-B(J_V(J_V(\vec{v})))=2B(\vec{v}).$$

The quantity (5.12) is mapped by $\mathbf{P}_c^W$, as in (5.11), to:

$$\frac{1}{2}(B\otimes\vec{v}-(B\circ J_V)\otimes(J_V(\vec{v}))-(J_W\circ B)\otimes(J_V(\vec{v}))-(J_W\circ B\circ J_V)\otimes\vec{v}),$$

which is then mapped by $Ev^c_{\mathbf{VW}}$ to

$$B(\vec{v})-J_W(B(J_V(\vec{v}))).$$

The conclusion is that

$$Ev^c_{\mathbf{VW}}\circ\mathbf{P}_c^W=\frac{1}{2}Ev_{VW}-\frac{1}{2}J_W\circ Ev_{VW}\circ[Id_{\mathrm{Hom}(V,W)}\otimes_c J_V].$$

The composite in the lowest two blocks of the diagram is then:

$$\begin{aligned}
&Ev^c_{\mathbf{VW}} \circ \mathbf{P}^W_c \circ [\mathrm{Hom}(Id_V, Q''_1) \otimes_c Id_V] \\
&= \frac{1}{2} Ev_{VW} \circ [\mathrm{Hom}(Id_V, Q''_1) \otimes_c Id_V] \\
&\quad - \frac{1}{2} J_W \circ Ev_{VW} \circ [Id_{\mathrm{Hom}(V,W)} \otimes_c J_V] \circ [\mathrm{Hom}(Id_V, Q''_1) \otimes_c Id_V] \\
&= \frac{1}{2} Ev_{VW} \circ [\mathrm{Hom}(Id_V, Q''_1) \otimes_c Id_V] \\
&\quad - \frac{1}{2} J_W \circ Ev_{VW} \circ [\mathrm{Hom}(Id_V, Q''_1) \otimes_c Id_V] \circ [Id_{\mathrm{Hom}(V,W_1)} \otimes_c J_V] \\
&= \frac{1}{2} Q''_1 \circ Ev_{VW_1} - \frac{1}{2} J_W \circ Q''_1 \circ Ev_{VW_1} \circ [Id_{\mathrm{Hom}(V,W_1)} \otimes_c J_V]. \qquad (5.13)
\end{aligned}$$

Starting with the clockwise composite around the outside of the diagram from U_1 to W, and then using the commutativity of the upper part of the diagram, step (5.13), and the c-linearity of n_{W_1}, gives:

$$\begin{aligned}
&Tr_{\mathbf{V};\mathbf{U},\mathbf{W}}(A_2) \circ Q'_1 \\
&= Ev^c_{\mathbf{VW}} \circ \mathbf{n}^{-1}_{\mathbf{W}} \circ \mathrm{Hom}_c(Id_V, A_2) \circ \mathbf{n}_{\mathbf{U}} \circ \eta^c_{\mathbf{VW}} \circ Q'_1 \\
&= Ev^c_{\mathbf{VW}} \circ \mathbf{P}^W_c \circ [\mathrm{Hom}(Id_V, Q''_1) \otimes_c Id_V] \circ \\
&\quad n^{-1}_{W_1} \circ \mathrm{Hom}_c(Id_V, A_1) \circ n_{U_1} \circ \eta_{VU_1} \\
&= \left(\frac{1}{2} Q''_1 \circ Ev_{VW_1} - \frac{1}{2} J_W \circ Q''_1 \circ Ev_{VW_1} \circ [Id_{\mathrm{Hom}(V,W_1)} \otimes_c J_V] \right) \circ \\
&\quad n^{-1}_{W_1} \circ \mathrm{Hom}_c(Id_V, A_1) \circ n_{U_1} \circ \eta_{VU_1} \\
&= \frac{1}{2} Q''_1 \circ Ev_{VW_1} \circ n^{-1}_{W_1} \circ \mathrm{Hom}_c(Id_V, A_1) \circ n_{U_1} \circ \eta_{VU_1} \\
&\quad - \frac{1}{2} J_W \circ Q''_1 \circ Ev_{VW_1} \circ n^{-1}_{W_1} \circ \mathrm{Hom}_c(Id_V, [J_V \otimes Id_{W_1}] \circ A_1) \circ n_{U_1} \circ \eta_{VU_1} \\
&= \frac{1}{2} Q''_1 \circ (Tr_{V;U_1,W_1}(A_1)) - \frac{1}{2} J_W \circ Q''_1 \circ (Tr_{V;U_1,W_1}([J_V \otimes Id_{W_1}] \circ A_1)).
\end{aligned}$$

Because $Tr_{\mathbf{V};\mathbf{U},\mathbf{W}}(A_2)$ is c-linear, by Lemma 5.7 it is uniquely determined on U by the above formula showing its restriction to U_1, and the claimed result follows from (5.6):

$$\begin{aligned}
&Tr_{\mathbf{V};\mathbf{U},\mathbf{W}}(A_2) \qquad (5.14) \\
&= ((Tr_{\mathbf{V};\mathbf{U},\mathbf{W}}(A_2)) \circ Q'_1) \circ P'_1 - J_W \circ ((Tr_{\mathbf{V};\mathbf{U},\mathbf{W}}(A_2)) \circ Q'_1) \circ P'_1 \circ J_U \\
&= \frac{1}{2} Q''_1 \circ (Tr_{V;U_1,W_1}(A_1)) \circ P'_1 \\
&\quad - \frac{1}{2} J_W \circ Q''_1 \circ (Tr_{V;U_1,W_1}([J_V \otimes Id_{W_1}] \circ A_1)) \circ P'_1
\end{aligned}$$

$$-\frac{1}{2}J_W\circ Q_1''\circ(Tr_{V;U_1,W_1}(A_1))\circ P_1'\circ J_U$$
$$-\frac{1}{2}Q_1''\circ(Tr_{V;U_1,W_1}([J_V\otimes Id_{W_1}]\circ A_1))\circ P_1'\circ J_U. \quad \square \tag{5.15}$$

REMARK 5.11. Both quantities, (5.14), and the sum of four terms (5.15), depend on the CSOs J_V, J_U, J_W. The first term out of the four depends only on the direct sum structures $U=U_1\oplus U_2$ and $W=W_1\oplus W_2$, but not on either complex structure J_W or J_U, nor on J_V except for the hypothesis that A_1 is c-linear. Because the Proof of Theorem 5.10 used formula (4.24) to define the trace in $\mathscr{C}$, (5.14) does not depend on any choice of unit object for $\otimes_c$ in $\mathscr{C}$. We remark further that the Proof of Theorem 5.10 is not entirely basis-free; the properties of the coevaluation $\eta^c_{\mathbf{VU}}$ from Example 4.16 were developed using a choice of basis.

EXAMPLE 5.12. Our goal in this example is to construct an object $\mathbf{U}$ in $\mathscr{C}$ and an invertible morphism (5.18) from any object $\mathbf{V}=(V,J_V)$ in $\mathscr{C}$ to $\mathbf{V}\otimes_c\mathbf{U}$, in terms of the construction of Example 5.8, and to elaborate on the statement from Example 4.5 that such an object is not unique by showing what sort of choices are involved in the construction. The objects $\mathbf{U}=\mathbf{U}_\lambda$ chosen for this Example depend on a real parameter $\lambda\neq 0$ and have a real structure, so that Theorem 5.10 can be applied in Example 5.14.

We make the initial assumption that there exists a real vector space U that admits a direct sum structure of the form $U=\mathbb{R}\oplus\mathbb{R}$, and fix a choice of such a direct sum, with projections (P_1,P_2) and inclusions (Q_1,Q_2) as in Definition 5.1.

The involution $C_U=Q_1\circ P_1-Q_2\circ P_2$ from (5.2) in Lemma 5.4 respects this direct sum (in the sense that it commutes with $Q_1\circ P_1$ and $Q_2\circ P_2$; the involution $-C_U$ also does this). The *2.* $\implies$ *1.* construction from Lemma 5.4 applied to the involution C_U defines another direct sum $U=U_1\oplus U_2$, which is distinct from the initial direct sum; it is an internal direct sum where U_1 and U_2 are both subspaces of U with inclusions Q_i' and projections P_i' depending on C_U as in (5.3). For $i=1,2$, the composite $P_i'\circ Q_i:\mathbb{R}\to U_i$ is invertible, with inverse $P_i\circ Q_i'$.

The construction from Example 5.8 (and the notation from Theorem 5.10) applies to any $\mathbf{V}=(V,J_V)$ and any vector space U with an involution producing a direct sum $U=U_1\oplus U_2$ to define the direct sum $V\otimes U=V\otimes U_1\oplus V\otimes U_2$, with an inclusion map $Q_1^U=[Id_V\otimes Q_1']:V\otimes U_1\to V\otimes U$. This direct sum depends only on the involution C_U, and the CSO $[J_V\otimes Id_U]|_{V\otimes U_1}$ on $V\otimes U_1$ depends only on J_V, it does not assume U has a complex structure.

From the *1.* $\implies$ *2.* step of Lemma 5.4, any linear isomorphism $R:\mathbb{R}\to\mathbb{R}$ is of the form $R=\lambda\cdot Id_{\mathbb{R}}$ for some $\lambda\neq 0$, and a CSO for U from (5.1) is:

$$J_{U,\lambda}=\lambda\cdot Q_2\circ P_1-\lambda^{-1}\cdot Q_1\circ P_2. \tag{5.16}$$

Recall that in general, even after choosing R, the choices made in Lemma 5.4 are not the only choices for a CSO and RSO, but in this case, any CSO anticommuting with $C_U=Q_1\circ P_1-Q_2\circ P_2$ must be of the form (5.16). Denote this object $\mathbf{U}_\lambda=(U,J_{U,\lambda})$.

Example 5.8 also considered the other internal direct sum

$$V \otimes U = (V \otimes_c U) \oplus (V \otimes_a U) \tag{5.17}$$

defined in terms of both complex structures J_V and $J_{U,\lambda}$, as in (4.1) from Example 4.4; denote the object from (4.2) by $\mathbf{V} \otimes_c \mathbf{U}_\lambda = (V \otimes_c U, J_{V \otimes_c U})$. Denote by $P_c^{U,\lambda}$: $V \otimes U \to \mathbf{V} \otimes_c \mathbf{U}_\lambda$ the projection from (5.7), and similarly denote the inclusion $Q_c^{U,\lambda}$: $\mathbf{V} \otimes_c \mathbf{U}_\lambda \to V \otimes U$. From (5.8), there is an invertible, c-linear map

$$P_c^{U,\lambda} \circ Q_1^U : V \otimes U_1 \to \mathbf{V} \otimes_c \mathbf{U}_\lambda.$$

Using the above choices for C_U and $J_{U,\lambda}$, and the scalar multiplication map $\ell : V \otimes \mathbb{R} \to V$, gives the following sequence of maps,

$$V \xrightarrow{\ell^{-1}} V \otimes \mathbb{R} \xrightarrow{[Id_V \otimes (P_1' \circ Q_1)]} V \otimes U_1 \xrightarrow{[Id_V \otimes Q_1']} V \otimes U \xrightarrow{P_c^{U,\lambda}} \mathbf{V} \otimes_c \mathbf{U}_\lambda. \tag{5.18}$$

The first three steps are c-linear with respect to the CSOs induced by J_V, the last step is c-linear with respect to the CSOs induced by J_V and $J_{U,\lambda}$, and the overall composite is invertible and c-linear. Only the last step depends on the choice made for $J_{U,\lambda}$. Simplifying the middle steps using $Q_1' \circ P_1' \circ Q_1 = Q_1$ gives the invertible, c-linear map:

$$P_c^{U,\lambda} \circ [Id_V \otimes Q_1] \circ \ell^{-1} : \mathbf{V} \to \mathbf{V} \otimes_c \mathbf{U}_\lambda.$$

REMARK 5.13. There is still some choice of scalar multiple for $P_c^{U,\lambda} \circ Q_1^U$ as in Remark 5.9. For $\lambda = 1$, the direct sum (5.17) corresponds to the construction usually denoted $V^{1,0} \oplus V^{0,1}$ in complex geometry, so (5.18) is a c-linear invertible map $\mathbf{V} \to V^{1,0}$. In situations involving a metric or symplectic form on V, the scale factor $\sqrt{2}$ from Remark 5.9 is sometimes preferred.

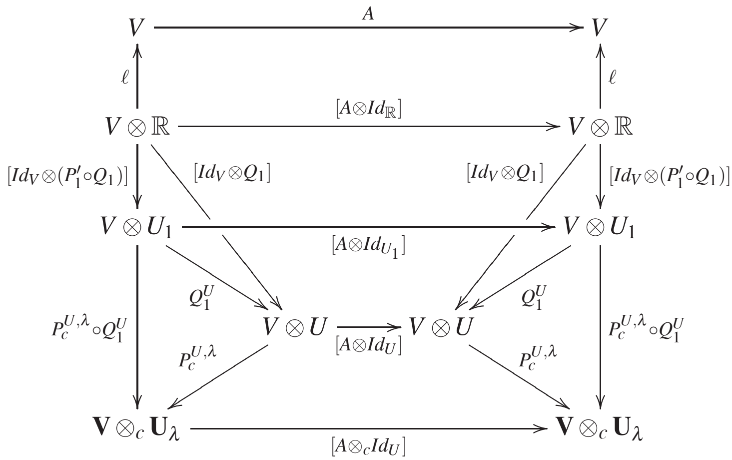
It was remarked after formula (4.24) for the trace in $\mathscr{C}$ that the trace $Tr_{\mathbf{V};\mathbf{U},\mathbf{W}}(A)$ does not depend on any choice of unit object for $\otimes_c$ in $\mathscr{C}$. An exception to this remark occurs when such a choice has been made and it appears in (4.24) as either $\mathbf{V}$, $\mathbf{U}$, or $\mathbf{W}$. In the following Example 5.14, the unit object for $\otimes_c$ constructed in Example 5.12 is used for both $\mathbf{U}$ and $\mathbf{W}$ in (4.24).

EXAMPLE 5.14. In this example, we show how an analogue of (1.1) can be expressed in terms of the above development of the generalized trace in $\mathscr{C}$.

Let $\mathbf{V} = (V, J_V)$ be finite dimensional and let $\mathbf{U}_\lambda = (U, J_{U,\lambda})$ be the unit object for $\otimes_c$ constructed in Example 5.12, with the same notation for the internal direct sum $U = U_1 \oplus U_2$ and the external direct sum $U = \mathbb{R} \oplus \mathbb{R}$. Let $A : \mathbf{V} \to \mathbf{V}$ be c-linear.

The following diagram is commutative. The downward composite on the left side is the invertible c-linear map from (5.18), and is equal to the right side. The commutativity of every block in the diagram is easily checked. For the lowest block, recall that lowest arrow $[A \otimes_c Id_U]$ is defined as the restriction of $[A \otimes Id_U]$ to the subspace

$\mathbf{V}\otimes_c\mathbf{U}_\lambda$ of $V\otimes U$, so $[A\otimes_c Id_U]=P_c^{U,\lambda}\circ[A\otimes Id_U]\circ Q_c^{U,\lambda}$. Using (5.7) and the c-linearity of A and Id_U (for any $J_{U,\lambda}$), $[A\otimes_c Id_U]\circ P_c^{U,\lambda}=P_c^{U,\lambda}\circ[A\otimes Id_U]\circ Q_c^{U,\lambda}\circ P_c^{U,\lambda}=P_c^{U,\lambda}\circ Q_c^{U,\lambda}\circ P_c^{U,\lambda}\circ[A\otimes Id_U]=P_c^{U,\lambda}\circ[A\otimes Id_U]$.



Theorem 5.10 applies, with $\mathbf{W}=\mathbf{U}=\mathbf{U}_\lambda$, $A_1=[A\otimes Id_{U_1}]$, and $A_2=[A\otimes_c Id_U]$. The hypothesis (5.10) is satisfied by the commutativity of the above diagram. The conclusion from Theorem 5.10 is this equality of maps $U\to U$:

$$\begin{aligned}
&Tr_{\mathbf{V};\mathbf{U}_\lambda,\mathbf{U}_\lambda}([A\otimes_c Id_U])\\
=&\frac{1}{2}Q_1'\circ(Tr_{V;U_1,U_1}([A\otimes Id_{U_1}]))\circ P_1'\\
&-\frac{1}{2}J_{U,\lambda}\circ Q_1'\circ(Tr_{V;U_1,U_1}([J_V\otimes Id_{U_1}]\circ[A\otimes Id_{U_1}]))\circ P_1'\\
&-\frac{1}{2}J_{U,\lambda}\circ Q_1'\circ(Tr_{V;U_1,U_1}([A\otimes Id_{U_1}]))\circ P_1'\circ J_{U,\lambda}\\
&-\frac{1}{2}Q_1'\circ(Tr_{V;U_1,U_1}([J_V\otimes Id_{U_1}]\circ[A\otimes Id_{U_1}]))\circ P_1'\circ J_{U,\lambda}. \qquad (5.19)
\end{aligned}$$

From the diagram,

$$[A\otimes Id_{U_1}]=[Id_V\otimes(P_1'\circ Q_1)]\circ[A\otimes Id_{\mathbb{R}}]\circ[Id_V\otimes(P_1\circ Q_1')],$$

and similarly for $[J_V\otimes Id_{U_1}]\circ[A\otimes Id_{U_1}]=[(J_V\circ A)\otimes Id_{U_1}]$. Then Theorem 3.14 applies, so that (5.19) is equal to:

$$\begin{aligned}
=&\frac{1}{2}Q_1'\circ P_1'\circ Q_1\circ(Tr_{V;\mathbb{R},\mathbb{R}}([A\otimes Id_{\mathbb{R}}]))\circ P_1\circ Q_1'\circ P_1'\\
&-\frac{1}{2}J_{U,\lambda}\circ Q_1'\circ P_1'\circ Q_1\circ(Tr_{V;\mathbb{R},\mathbb{R}}([(J_V\circ A)\otimes Id_{\mathbb{R}}]))\circ P_1\circ Q_1'\circ P_1'\\
&-\frac{1}{2}J_{U,\lambda}\circ Q_1'\circ P_1'\circ Q_1\circ(Tr_{V;\mathbb{R},\mathbb{R}}([A\otimes Id_{\mathbb{R}}]))\circ P_1\circ Q_1'\circ P_1'\circ J_{U,\lambda}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}Q_1'\circ P_1'\circ Q_1\circ(Tr_{V;\mathbb{R},\mathbb{R}}([(J_V\circ A)\otimes Id_{\mathbb{R}}]))\circ P_1\circ Q_1'\circ P_1'\circ J_{U,\lambda}\\
=&\ \frac{1}{2}Q_1\circ(Tr_{V;\mathbb{R},\mathbb{R}}(\ell^{-1}\circ A\circ\ell))\circ P_1\\
&-\frac{1}{2}J_{U,\lambda}\circ Q_1\circ(Tr_{V;\mathbb{R},\mathbb{R}}(\ell^{-1}\circ J_V\circ A\circ\ell))\circ P_1\\
&-\frac{1}{2}J_{U,\lambda}\circ Q_1\circ(Tr_{V;\mathbb{R},\mathbb{R}}(\ell^{-1}\circ A\circ\ell))\circ P_1\circ J_{U,\lambda}\\
&-\frac{1}{2}Q_1\circ(Tr_{V;\mathbb{R},\mathbb{R}}(\ell^{-1}\circ J_V\circ A\circ\ell))\circ P_1\circ J_{U,\lambda}. \qquad (5.20)
\end{aligned}
$$

As previously remarked, the first of the four terms in (5.20) depends only on the direct sum $U=\mathbb{R}\oplus\mathbb{R}$ from the beginning of Example 5.12, and not on the subsequent choices for C_U and $J_{U,\lambda}$.

The next step uses the formula (2.7) from Example 2.13, so that (5.20) is equal to this expression involving the real scalar trace $Tr_V(A)$:

$$
\begin{aligned}
=&\ \frac{1}{2}Q_1\circ(Tr_V(A)\cdot Id_{\mathbb{R}})\circ P_1 \qquad (5.21)\\
&-\frac{1}{2}J_{U,\lambda}\circ Q_1\circ(Tr_V(J_V\circ A)\cdot Id_{\mathbb{R}})\circ P_1\\
&-\frac{1}{2}J_{U,\lambda}\circ Q_1\circ(Tr_V(A)\cdot Id_{\mathbb{R}})\circ P_1\circ J_{U,\lambda}\\
&-\frac{1}{2}Q_1\circ(Tr_V(J_V\circ A)\cdot Id_{\mathbb{R}})\circ P_1\circ J_{U,\lambda}\\
=&\ \frac{1}{2}Tr_V(A)\cdot(Q_1\circ P_1-J_{U,\lambda}\circ Q_1\circ P_1\circ J_{U,\lambda})\\
&-\frac{1}{2}Tr_V(J_V\circ A)\cdot(J_{U,\lambda}\circ Q_1\circ P_1+Q_1\circ P_1\circ J_{U,\lambda}). \qquad (5.22)
\end{aligned}
$$

Finally, using the formula (5.16), $J_{U,\lambda}=\lambda\cdot Q_2\circ P_1-\lambda^{-1}\cdot Q_1\circ P_2$, (5.22) simplifies to:

$$
Tr_{\mathbf{V};\mathbf{U}_\lambda,\mathbf{U}_\lambda}([A\otimes_c Id_U])=\frac{1}{2}Tr_V(A)\cdot Id_U-\frac{1}{2}Tr_V(J_V\circ A)\cdot J_{U,\lambda}. \qquad (5.23)
$$

The concluding observations are that (5.23) is the claimed generalization of (1.1), and that the first term $\frac{1}{2}Tr_V(A)\cdot Id_U$ does not depend on any of the extra structure on U (the direct sum, RSO, or CSO from Example 5.12). The 2×2 matrix representation of (5.23) in the case $\lambda=1$ is consistent with the calculation (4.27) from Example 4.17.

6. Conclusion

The construction of Sections 3–5 could be adapted to other categories of vector spaces with supplemental structures, for example, the category of vector spaces with real structures (V,J,C) and morphisms that respect both operators. The category $\mathscr{C}$, where only some of the objects have a real structure, would also be a natural framework for a basis-free approach to vector valued Hermitian forms on vector spaces (or vector bundles) and using the generalized trace to compute tensor contraction with respect to a Hermitian metric.

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