Operators and Matrices
Volume 15, Number 2 (2021), 581–587
doi:10.7153/oam-2021-15-39

INEQUALITIES RELATED TO THE GEOMETRIC MEAN OF ACCRETIVE MATRICES

JUNTONG LIU, JIN-JIN MEI AND DENGPENG ZHANG*

(*Communicated by F. Hansen*)

Abstract. We present some inequalities related to the recently defined geometric mean of two accretive matrices. Firstly, we show that if the block matrix $\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}$ is accretive, then the singular values of $(X+Y)/2$ are weakly log majorized by the singular values of the geometric mean of A and B. This extends a result of M. Lin.

1. Introduction

The set of all $n\times n$ complex matrices is denoted by $\mathbb{M}_n$. We say that $A\in\mathbb{M}_n$ is *accretive* if its real (or Hermitian) part $\Re A := (A+A^*)/2$ is positive definite, where A^* means the conjugate transpose of A. For two positive definite matrices $A,B\in\mathbb{M}_n$, their geometric mean is defined by

$$A\sharp B := B^{1/2}(B^{-1/2}AB^{-1/2})^{1/2}B^{1/2}.$$

It is easy to prove that the geometric mean $A\sharp B$ is the unique positive definite solution to the Ricatti equation $XB^{-1}X = A$. This observation enables one to see that the role of A,B in the geometric mean is symmetric, that is $A\sharp B = B\sharp A$. By a limit process, the definition could be extended for positive semidefinite matrices. For more information about matrix geometric mean, we refer to [4, Chapter 4].

Extending the geometric mean of two positive definite matrices, Drury [5] recently defined the geometric mean for two accretive matrices $A,B\in\mathbb{M}_n$ via the formula

$$A\sharp B := \left(\frac{2}{\pi}\int_0^\infty (tA+t^{-1}B)^{-1}\frac{dt}{t}\right)^{-1},$$

in which we continue to use the standard notation $A\sharp B$ for the geometric mean. The geometric mean for accretive matrices enjoys several appealing properties; see [5]. A weighted version was subsequently proposed by Raissouli, Moslehian and Furuichi [12]. It is clear from the formula that if A,B are accretive, then so is $A\sharp B$.

Mathematics subject classification (2020): 15A42.
Keywords and phrases: Geometric mean, accretive matrix.
* Corresponding author.

For two Hermitian $A, B \in \mathbb{M}_n$, we write $A \geqslant B$ (resp. $A > B$) if $A - B$ is positive semidefinite (resp. positive definite). It is well known that if $A, B \in \mathbb{M}_n$ are positive semidefinite, then we have the noncommutative AM-GM inequality

$$\frac{A+B}{2} \geqslant A \sharp B. \tag{1}$$

It was pointed out in [10, Eq. (9)] that a direct analogue of (1)

$$\Re \frac{A+B}{2} \geqslant \Re(A \sharp B)$$

for accretive $A, B \in \mathbb{M}_n$ fails.

A remarkable property about the geometric mean is the following inequality due to Lin and Sun [9]: Let $A, B \in \mathbb{M}_n$ be accretive. Then

$$\Re(A \sharp B) \geqslant (\Re A) \sharp (\Re B). \tag{2}$$

This inequality would play an important role in our derivations. Again, we mention that the corresponding weighted version was given in [12].

In this paper, we consider several results related to the geometric mean of accretive matrices. The remaining of this section is some notation used in the article. The eigenvalues, singular values of $A \in \mathbb{M}_n$ are denoted by $\lambda_j(A), \sigma_j(A),\ j = 1, \ldots, n$, respectively such that $\lambda_1(A) \geqslant \cdots \geqslant \lambda_n(A)$, $\sigma_1(A) \geqslant \cdots \geqslant \sigma_n(A)$ (whenever the eigenvalues are all real). For $A, B \in \mathbb{M}_n$, if

$$\prod_{j=1}^{k} \sigma_j(A) \leqslant \prod_{j=1}^{k} \sigma_j(B)$$

for all $k = 1, \ldots, n$, then we say that the singular values of A are weakly log majorized by the singular values of B and we denote the relation by

$$\sigma(A) \prec_{wlog} \sigma(B).$$

For more information about majorization, we refer to [13, Chapter 3] or [14, Chapter 10].

2. A weak log majorization

Let $A, B, X, Y \in \mathbb{M}_n$. If

$$M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \quad \text{and} \quad M^\tau = \begin{pmatrix} A & X^* \\ X & B \end{pmatrix}$$

are both positive semidefinite, then we say that M is PPT (i.e., positive partial transpose). In [11], Lin proved that if M is PPT, then

$$\sigma(X) \prec_{wlog} \sigma(A \sharp B). \tag{3}$$

For an alternative proof of (3), see [7]. We could extend the notion to accretive matrices. If

$$M = \begin{pmatrix} A & X \\ Y^* & B \end{pmatrix} \quad \text{and} \quad M^\tau = \begin{pmatrix} A & Y^* \\ X & B \end{pmatrix}$$

are both accretive, then we say that M is APT (i.e., accretive partial transpose). Clearly, the class of APT matrices include the class of PPT matrices. A relevant notion SPT (i.e., sectorial partial transpose) has appeared in [6].

We extend Lin's result to the case of APT matrices as follows.

THEOREM 2.1. *Let $A,B,X,Y \in \mathbb{M}_n$. If $M = \begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}$ is APT, then*

$$\sigma\left(\frac{X+Y}{2}\right) \prec_{wlog} \sigma(A\sharp B). \tag{4}$$

Proof. By the Fan-Hoffman inqeuality [3, p. 73],

$$\lambda_j(\Re(A\sharp B) \leqslant \sigma_j(A\sharp B)$$

for all $j = 1,\dots,n$. Moreover, since

$$\Re(A\sharp B) \geqslant (\Re A)\sharp(\Re B)$$

and by the Weyl's monontonicity theorem for the eigenvalues [3, p. 63], we have

$$\lambda_j((\Re A)\sharp(\Re B)) \leqslant \lambda_j(\Re(A\sharp B))$$

for all $j = 1,\dots,n$. These enable us to conclude

$$\sigma((\Re A)\sharp(\Re B)) \prec_{wlog} \sigma(A\sharp B). \tag{5}$$

As M is APT, we see that

$$\Re M = \begin{pmatrix} \Re A & (X+Y)/2 \\ (X+Y)^*/2 & \Re B \end{pmatrix} \quad \text{and} \quad \Re(M^\tau) = \begin{pmatrix} \Re A & (X+Y)^*/2 \\ (X+Y)/2 & \Re B \end{pmatrix} = (\Re M)^\tau$$

are both positive definite. In other words, $\Re M$ is PPT. Therefore, applying (3) to $\Re M$ gives

$$\sigma\left(\frac{X+Y}{2}\right) \prec_{wlog} \sigma((\Re A)\sharp(\Re B)). \tag{6}$$

The desired result now follows from (5) and (6). $\square$

It is apparent that if M is PPT (in this case, $X = Y$), then (4) becomes Lin's result (3). An immediate corollary of the previous theorem is the following.

COROLLARY 2.2. *Let $A,B,X \in \mathbb{M}_n$. If $M = \begin{pmatrix} A & X \\ X & B \end{pmatrix}$ is accretive, then*

$$\sigma(\Re X) \prec_{wlog} \sigma(A\sharp B).$$

3. A matrix inequality

In [1], Ando proved the following interesting result.

PROPOSITION 3.1. *Let* $A_j, B_j, X, Y \in \mathbb{M}_n$. *If* $\begin{pmatrix} A_j & X \\ X^* & B_j \end{pmatrix}$, $j = 1,2$, *are positive semidefinite, then so is* $\begin{pmatrix} A_1 \sharp A_2 & X \\ X^* & B_1 \sharp B_2 \end{pmatrix}$.

The next result is an extention of this.

PROPOSITION 3.2. *Let* $A_j, B_j, X, Y \in \mathbb{M}_n$. *If* $\begin{pmatrix} A_j & X \\ Y^* & B_j \end{pmatrix}$, $j = 1,2$, *are accretive, then so is* $\begin{pmatrix} A_1 \sharp A_2 & X \\ Y^* & B_1 \sharp B_2 \end{pmatrix}$.

Proof. The condition says $\Re \begin{pmatrix} A_j & X \\ Y & B_j \end{pmatrix} = \begin{pmatrix} \Re A_j & (X+Y)/2 \\ (X+Y)^*/2 & \Re B_j \end{pmatrix}$, $j = 1,2$, are positive definite. Then by the positivity of the Schur complement,

$$\Re A_j > \left(\frac{X+Y}{2}\right) (\Re B_j)^{-1} \left(\frac{X+Y}{2}\right)^*, \qquad j = 1,2.$$

On the other hand, the key inequality (2) implies

$$\Big(\Re(B_1 \sharp B_2)\Big)^{-1} \leqslant \Big((\Re B_1) \sharp (\Re B_2)\Big)^{-1}.$$

Therefore,

$$\begin{aligned}
&\left(\frac{X+Y}{2}\right) \Big(\Re(B_1 \sharp B_2)\Big)^{-1} \left(\frac{X+Y}{2}\right)^* \\
\leqslant &\left(\frac{X+Y}{2}\right) \Big((\Re B_1) \sharp (\Re B_2)\Big)^{-1} \left(\frac{X+Y}{2}\right)^* \\
= &\left(\frac{X+Y}{2}\right) \Big((\Re B_1)^{-1} \sharp (\Re B_2)^{-1}\Big) \left(\frac{X+Y}{2}\right)^* \\
\leqslant &\left(\left(\frac{X+Y}{2}\right) (\Re B_1)^{-1} \left(\frac{X+Y}{2}\right)^*\right) \sharp \left(\left(\frac{X+Y}{2}\right) (\Re B_2)^{-1} \left(\frac{X+Y}{2}\right)^*\right) \\
< &\ (\Re A_1) \sharp (\Re A_2) \leqslant \Re(A_1 \sharp A_2),
\end{aligned}$$

in which the second inequality is due to [4, Theorem 4.1.5 (ii)]. This implies the block matrix $\begin{pmatrix} \Re(A_1 \sharp A_2) & (X+Y)/2 \\ (X+Y)^*/2 & \Re(B_1 \sharp B_2) \end{pmatrix}$ is positive definite. In other words, $\begin{pmatrix} A_1 \sharp A_2 & X \\ Y^* & B_1 \sharp B_2 \end{pmatrix}$ is accretive. $\square$

In [7], Lee proved the following matrix inequality.

THEOREM 3.3. *Let $A,B,X \in \mathbb{M}_n$. If $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ is PPT, then for some unitary matrix $V \in \mathbb{M}_n$*

$$2|X| \leqslant A\sharp B + V^*(A\sharp B)V.$$

We make use of Proposition 3.2 to extend Theorem 3.3.

THEOREM 3.4. *Let $A,B,X,Y \in \mathbb{M}_n$. If $\begin{pmatrix} A & X \\ Y^* & B \end{pmatrix}$ is APT, then for some unitary matrix $V \in \mathbb{M}_n$*

$$|X+Y| \leqslant \Re\Big(A\sharp B + V^*(A\sharp B)V\Big).$$

Proof. Since $\begin{pmatrix} A & Y^* \\ X & B \end{pmatrix}$ is accretive, so is $\begin{pmatrix} B & X \\ Y^* & A \end{pmatrix}$ by a congruence with $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. It follows from Proposition 3.2 that $\begin{pmatrix} A\sharp B & X \\ Y^* & A\sharp B \end{pmatrix}$ is accretive, that is,

$$\begin{pmatrix} \Re(A\sharp B) & (X+Y)/2 \\ (X+Y)^*/2 & \Re(A\sharp B) \end{pmatrix}$$

is positive definite. Consider the polar decomposition $X+Y = V|X+Y|$, where $V \in \mathbb{M}_n$ is unitary. Then

$$\begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix}^* \begin{pmatrix} \Re(A\sharp B) & (X+Y)/2 \\ (X+Y)^*/2 & \Re(A\sharp B) \end{pmatrix} \begin{pmatrix} V & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} V^*(\Re(A\sharp B))V & |X+Y|/2 \\ |X+Y|/2 & \Re(A\sharp B) \end{pmatrix}$$

is positive definite. Therefore by a simple congruence with $\begin{pmatrix} I & -I \end{pmatrix}$, we have the desired inequality. $\square$

4. The geometric mean of A and A^*

In this section, we present some inequalities about $A\sharp A^*$.

PROPOSITION 4.1. *If $A \in \mathbb{M}_n$ is accretive, then $A\sharp A^* \geqslant \Re A$.*

Proof. Clearly $A\sharp A^*$ is Hermitian and accretive, so $A\sharp A^*$ is positive definite. Then we observe that the block matrix $\begin{pmatrix} A\sharp A^* & A \\ A^* & A\sharp A^* \end{pmatrix}$ is positive semidefinite by using the Schur complement, for

$$A\sharp A^* - A(A\sharp A^*)^{-1}A^* = A\sharp A^* - A\Big(A^{-1}\sharp (A^*)^{-1}\Big)A^* = 0.$$

Therefore,

$$\left\langle v\oplus -v, \begin{pmatrix} A\sharp A^* & A \\ A^* & A\sharp A^* \end{pmatrix}(v\oplus -v)\right\rangle \geqslant 0, \quad \forall v \in \mathbb{C}^n.$$

Expanding this gives

$$2\langle v, (A\sharp A^*)v\rangle \geqslant \langle v, (A+A^*)v\rangle, \quad \forall v \in \mathbb{C}^n,$$

as desired. $\square$

We say that $A \in \mathbb{M}_n$ is a contraction if $I \geqslant A^*A$. Using the obvious fact that $\det A\sharp B = \det A^{1/2} \det B^{1/2}$, we see that the following corollary is stronger than the Hua's determinantal inequality [14, p. 231]: For $A, B \in \mathbb{M}_n$ contractive,

$$|\det(I - A^*B)|^2 \geqslant \det(I - A^*A)\det(I - B^*B).$$

For other strengthenings of the Hua's determinantal inequality in the level of eigenvalues or singular values, we refer to [8].

COROLLARY 4.2. *If $A, B \in \mathbb{M}_n$ are contractions, then*

$$(I - A^*B)\sharp(I - B^*A) \geqslant (I - A^*A)\sharp(I - B^*B).$$

Proof. We need the following observation of Ando [2]: $(A-B)^*(A-B) \geqslant 0$ gives $A^*A + B^*B \geqslant A^*B + B^*A$, and so

$$\Re(I - A^*B) \geqslant \frac{(I - A^*A) + (I - B^*B)}{2}$$

Now by Proposition 4.1,

$$(I - A^*B)\sharp(I - B^*A) \geqslant \Re(I - A^*B).$$

And the easy fact

$$\frac{(I - A^*A) + (I - B^*B)}{2} \geqslant (I - A^*A)\sharp(I - B^*B).$$

Hence the conclusion. $\square$

The positivity of the block matrix in the proof of previous proposition also implies the following inequality about the usual operator norm.

COROLLARY 4.3. *If $A \in \mathbb{M}_n$ is accretive, then*

$$\|A\sharp A^*\| \geqslant \|A\|.$$

Disclosure statement

No potential conflict of interest was reported by the authors.

Acknowledgement. The work is supported by a grant from National Natural Science Foundation Project of China (No. 11601314); Key Project of Anhui Provincial Department of Education (No. KJ2019A0534); Anhui Natural Science Foundation (No. 1908085qa08, 2008085MA12); The Building of Brand Speciality Projects of Fuyang Normal University (No. 2019PPZY01); Young Talents Program of Fuyang Normal University (No. rcxm202002).

REFERENCES

[1] T. ANDO, *Geometric mean and norm Schwarz inequality*, Ann. Funct. Anal. 7 (2016) 1–8.
[2] T. ANDO, *Hua-Marcus inequalities*, Linear Multilinear Algebra 8 (1980) 347–352.
[3] R. BHATIA, *Matrix Analysis*, GTM 169, Springer-Verlag, New York, 1997.
[4] R. BHATIA, *Positive Definite Matrices*, Princeton University Press, Princeton, 2007.
[5] S. DRURY, *Principal powers of matrices with positive definite real part*, Linear Multilinear Algebra 63 (2) (2015) 296–301.
[6] L. KUAI, *An extension of the Fiedler-Markham determinant inequality*, Linear Multilinear Algebra 66 (2018) 547–553.
[7] E.-Y. LEE, *The off-diagonal block of a PPT matrix*, Linear Algebra Appl. 486 (2015) 449–453.
[8] M. LIN, *The Hua matrix and inequalities related to contractive matrices*, Linear Algebra Appl. 511 (2016) 22–30.
[9] M. LIN, F. SUN, *A property of the geometric mean of accretive operator*, Linear Multilinear Algebra 65 (2017) 433–437.
[10] M. LIN, *Some inequalities for sector matrices*, Oper. Matrices 10 (2016) 915–921.
[11] M. LIN, *Inequalities related to 2×2 block PPT matrices*, Oper. Matrices 9 (2015) 917–924.
[12] M. RAISSOULI, M. S. MOSLEHIAN, S. FURUICHI, *Relative entropy and Tsallis entropy of two accretive operators*, C. R. Acad. Sci. Paris, Ser. I 355 (2017) 687–693.
[13] X. ZHAN, *Matrix Theory*, GSM 147, American Mathematical Society, Providence, RI, 2013.
[14] F. ZHANG, *Matrix Theory: Basic Results and Techniques*, second edition, Springer, New York, 2011.

(Received January 7, 2020)

Juntong Liu
School of Mathematics and Statistics
Fuyang Normal university
Fuyang, 236041, China
e-mail: juntongliu82@163.com

Jin-Jin Mei
School of Mathematics and Statistics
Fuyang Normal university
Fuyang, 236041, China
e-mail: meijinjin666@126.com

Dengpeng Zhang
School of statistics and mathematics
Guangdong University of Finance and Economics
Guangzhou 510320, China
e-mail: zhangdengpeng@sina.cn