Operators and Matrices
Volume 15, Number 2 (2021), 589–614
doi:10.7153/oam-2021-15-40

GRAPH COMPLEMENT CONJECTURE FOR CLASSES OF SHADOW GRAPHS

MONSIKARN JANSRANG* AND SIVARAM K. NARAYAN

(*Communicated by H. J. Woerdeman*)

Abstract. The real minimum semidefinite rank of a graph G, denoted $\mathrm{mr}_+^{\mathbb{R}}(G)$, is defined to be the minimum rank among all real symmetric positive semidefinite matrices whose zero/nonzero pattern corresponds to the graph G. The inequality $\mathrm{mr}_+^{\mathbb{R}}(G)+\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant |G|+2$ is called the graph complement conjecture, denoted GCC_+, where $\overline{G}$ is the complement of G and $|G|$ is the number of vertices in G. A known definition of shadow graph $S(G)$ and a variant of this definition denoted $\mathrm{Shad}(G)$ are given. It is shown that $S(G)$ satisfies GCC_+ when G is a tree or a unicyclic graph or a complete graph. Under additional conditions on $\overline{G}$, it is shown that $S(G)$ satisfies GCC_+ when G is a k-tree or a chordal graph. Moreover, whenever G satisfies GCC_+ and $\overline{G}$ does not contain any isolated vertices, it is shown that $\mathrm{Shad}(G)$ satisfies GCC_+.

1. Introduction

A *graph* G consists of a set of vertices $V(G)=\{v_1,v_2,\ldots,v_n\}$ and a set of edges $E(G)$, where an edge is defined to be an unordered pair of vertices. The *order* of G, denoted $|G|$, is the cardinality of $V(G)$. A graph is said to be *simple* if it has no multiple edges or loops. A *multigraph* G consists of possible multiple edges but has no loops. The *complement* of a graph $G(V,E)$ is the graph $\overline{G}(V,\overline{E})$, where $\overline{E}$ consists of all the unordered pairs of vertices that are not in $E(G)$.

An $n\times n$ matrix $A=[a_{ij}]$ is said to be combinatorially symmetric when $a_{ij}=0$ if and only if $a_{ji}=0$. We say that $\mathcal{G}(A)$ is the graph of an $n\times n$ combinatorially symmetric matrix $A=[a_{ij}]$ if $V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{\{v_i,v_j\}: a_{ij}\neq 0, i\neq j\}$. The main diagonal entries of A play no role in determining $\mathcal{G}(A)$. Define $\mathcal{S}(G,\mathbb{F})$ to be the set of all $n\times n$ matrices A that are *real symmetric* if $\mathbb{F}=\mathbb{R}$ and *complex Hermitian* if $\mathbb{F}=\mathbb{C}$ whose graph is G. The sets $\mathcal{S}_+(G,\mathbb{F})$ are the corresponding subsets of positive semidefinite (psd) matrices. The smallest possible rank of any matrix A in $\mathcal{S}(G,\mathbb{F})$ is called the *minimum rank* of G, denoted by $\mathrm{mr}(G,\mathbb{F})$, and the smallest possible rank of any matrix A in $\mathcal{S}_+(G,\mathbb{F})$ is called the *minimum semidefinite rank* of G, denoted either $\mathrm{mr}_+^{\mathbb{R}}(G)$ or $\mathrm{mr}_+^{\mathbb{C}}(G)$. Many results on this topic are mentioned in ([16], Topics in Combinatorial Matrix Theory 46).

An interesting conjecture was presented at the 2006 AIM workshop at Palo Alto, CA, called the graph complement conjecture or GCC for short [13]. The conjecture

Mathematics subject classification (2020): 05C50, 15A03, 15A18, 15B57.
Keywords and phrases: Shadow graphs, minimum semidefinite rank, graph complement conjecture.
* Corresponding author.

is the following inequality $\mathrm{mr}(G)+\mathrm{mr}(\overline{G}) \leqslant |G|+2$. A variant of GCC known as GCC_+, is the following inequality:

$$\mathrm{mr}_+^{\mathbb{R}}(G)+\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant |G|+2.$$

Since $\mathcal{S}_+(G,\mathbb{R}) \subseteq \mathcal{S}(G,\mathbb{R})$, it follows that $\mathrm{mr}(G) \leqslant \mathrm{mr}_+^{\mathbb{R}}(G)$ and whenever GCC_+ holds so does GCC.

The study of GCC and GCC_+ are part of the questions in graph theory called the Nordhaus–Gaddum type problems, which involve bounding the sum of a graph parameter evaluated at G and its complement $\overline{G}$. This question has been considered for graph parameters such as the chromatic number, the independence number and the domination number ([16], section 46.7).

The graph complement conjecture GCC_+ has been shown to hold true for some graph classes. In [13] it was shown that trees satisfy GCC_+. Later, GCC_+ was shown to hold for unicyclic graphs [15], chordal graphs [19], graphs with $\delta(G) \geqslant |G|-3$ [2], partial 3-trees [21] and k-connected partial k-trees [21]. In this paper we prove that certain new classes of graphs satisfy GCC_+.

The paper is organized as follows: In section 2 we present graph theory preliminaries and some known results on $\mathrm{mr}_+^{\mathbb{R}}(G)$ that will be used in the paper. In section 3 we define the shadow graph $S(G)$ and give upper bounds for the minimum semidefinite rank of $S(G)$ and the minimum semidefinite rank of its complement $\overline{S(G)}$. We also show that when $\overline{G}$ is either a tree or a unicyclic graph $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) = \mathrm{mr}_+^{\mathbb{R}}(\overline{G})+1$. In section 4 we prove that $S(G)$ satisfies GCC_+ when G belongs to certain graph classes. The complete result is stated in Theorem 1.

THEOREM 1. *If G belongs to any of the following graph classes, then $S(G)$ satisfies GCC_+. The graph classes are*

1. *G is a tree.*
2. *G is a unicyclic graph.*
3. *G is a complete graph.*
4. *G is a k-tree such that $\overline{G}$ does not contain any isolated vertices.*
5. *G is a partial k-tree with $k \geqslant 2$ where G has a complete subgraph K_{k+1} and $\overline{G}$ does not contain any isolated vertices.*
6. *G is a chordal graph such that $\overline{G}$ does not contain any isolated vertices.*

In section 5, we give a different definition of a shadow graph and denote it $\mathrm{Shad}(G)$. The result we obtained for $\mathrm{Shad}(G)$ is as follows:

THEOREM 2. *If G satisfies GCC_+ and $\overline{G}$ does not contain any isolated vertices, then* $\mathrm{Shad}(G)$ *satisfies GCC_+.*

Moreover, in section 6 we show that the shadow graphs $S(G)$ discussed in section 4 also satisfy the “delta conjecture” which states $\mathrm{mr}_+^{\mathbb{R}}(G) \leqslant |G|-\delta(G)$ where $\delta(G)$ is the minimum degree of the vertices in G.

2. Preliminaries

In this section, we present some graph theory preliminaries and some known results concerning the minimum semidefinite rank.

2.1. Graph theory preliminaries

Given a simple graph G, let $V(G)$ be the set of vertices and $E(G)$ be the set of edges, where the elements of $E(G)$ are unordered pairs of vertices. An edge joining vertices x and y will be written either as xy or $\{x,y\}$. If $e = xy$, then we say vertices x and y are adjacent vertices. Moreover, $e = xy$ is said to be incident to both x and y or x (or y) is incident with the edge e.

Given a vertex $v \in V(G)$, the *neighborhood* $N(v)$ of v is the set of vertices that are adjacent to v and the *closed neighborhood* $N[v]$ is $N(v) \cup \{v\}$. The *degree* of a vertex v in G, denoted by $d_G(v)$, is the cardinality of $N(v)$. We will use $d(v)$ instead of $d_G(v)$ when G is clear in the context. If $d_G(v) = 1$, then v is called a *pendant* vertex of G. We denote $\delta(G)$ to be the minimum degree of the vertices in G. Two vertices u and v in a graph G are said to be *duplicate vertices* if u is adjacent to v and $N(u) = N(v)$, or equivalently $N[u] = N[v]$.

A *path* is a simple graph whose vertices $\{v_1, v_2, \ldots, v_n\}$ can be ordered so that two vertices are adjacent if and only if they are consecutive in the list ([23], p. 5). A path on n vertices is denoted by P_n. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle ([23], p. 5). A cycle on n vertices is denoted by C_n. A graph G is said to be *connected* if there is a path between any two vertices of G. A *tree* is a connected graph without any cycles.

A *subgraph* $H = (V(H), E(H))$ of $G = (V(G), E(G))$ is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and we say G is a *supergraph* of H. An *induced subgraph* H of G is a subgraph with $V(H) \subseteq V(G)$ and $E(H) = \{\{v_i, v_j\} \in E(G) : v_i, v_j \in V(H)\}$. We use $G[R]$ to denote the subgraph of G induced by the set of vertices $R \subseteq V(G)$. A *spanning subgraph* of a graph G is a subgraph whose vertex set is $V(G)$.

An *independent set* in a graph G is a set of pairwise non-adjacent vertices in G. The cardinality of a largest independent set in G is called the *independence number* of G, denoted by $\alpha(G)$. A *star graph* S_n on n vertices is a tree with an independent set of $n-1$ pendant vertices and a center vertex x, such that x is adjacent to all the $n-1$ vertices.

A *complete graph* is a simple graph in which the vertices are pairwise adjacent. A *clique* is a subgraph of pairwise adjacent vertices. A vertex v is said to be a *simplicial* vertex in a graph G if the induced subgraph $G[N[v]]$ is a clique. The size of a maximum clique in a graph G is called the *clique number* of G, denoted by $\omega(G)$. A *chordal graph* is a graph in which there are no induced cycles on four or more vertices.

Let $G_1, G_2, \ldots, G_k$ be simple subgraphs of a connected graph G on two or more vertices. We say that $G_1, G_2, \ldots, G_k$ cover a graph G if each vertex of G is a vertex of at least one G_i, and for every pair of vertices u and v that are adjacent in G, there is at least one G_i in which u and v are adjacent. If each G_i is a clique, then it is a *clique* *cover* of G. The minimum number of cliques needed to cover all the edges of G is called the *clique cover number* of G, denoted by $\mathrm{cc}(G)$.

The *join* of two graphs G and H, denoted $G \vee H$, is the graph with the vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{\{u,v\} : u \in V(G), v \in V(H)\}$.

Suppose G is decomposable into two graphs, G_1 and G_2, sharing only one vertex v such that if $u \in V(G_1)$ and $w \in V(G_2)$, then $\{u,w\} \in E(G)$ only if $u = v$ or $w = v$. Then G_1 and G_2 are joined at a cut vertex v, and we write $G = G_1 \circ G_2$ and call it a vertex sum of G_1 and G_2.

The *contraction* of an edge $e = \{u,v\} \in E(G)$ involves the deletion of e and merging the vertices u and v into a new vertex w and keeping all the edges in G incident to either u or v. A *minor* of a graph G is any graph obtainable from G by means of a sequence of vertex and edge deletions and edge contractions ([4], p. 268). Alternatively, consider a partition $(V_0, V_1, \ldots, V_k)$ of V such that $G[V_i]$ is connected, $1 \leqslant i \leqslant k$, and let H be the graph obtained from G by deleting V_0 and contracting each induced subgraph $G[V_i]$, $1 \leqslant i \leqslant k$, to a single vertex. Then any spanning subgraph F of H is a minor of G. Note that in the definition of a minor any multiple edge can be replaced by a single edge.

2.2. The minimum semidefinite rank of graphs

Let $M_n(\mathbb{C})$ be the set of complex square matrices. A matrix $A \in M_n(\mathbb{C})$ is said to be *Hermitian* if $A = A^*$ where A^* is the conjugate transpose of A. A Hermitian matrix $A \in M_n(\mathbb{C})$ is said to be *positive semidefinite* (psd) if $x^*Ax \geqslant 0$ for all nonzero $x \in \mathbb{C}^n$. Since a principal submatrix of a psd matrix is psd ([17], p. 430), it follows that the main diagonal entries $a_{ii} \geqslant 0$. Moreover, a positive semidefinite matrix has a zero entry on its main diagonal if and only if the entire row and column to which that entry belongs is zero ([17], p. 432, Observation 7.1.10). As a consequence, if $A \in \mathcal{S}_+(G,\mathbb{F})$ where G is connected and $|G| \geqslant 2$, the main diagonal entries of A are strictly positive. For a given graph G, the *complex minimum semidefinite rank* of G is defined to be

$$\mathrm{mr}_+^{\mathbb{C}}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}_+(G,\mathbb{C})\}$$

and the *real minimum semidefinite rank* of G is defined to be

$$\mathrm{mr}_+^{\mathbb{R}}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}_+(G,\mathbb{R})\}.$$

Since $\mathcal{S}_+(G,\mathbb{R}) \subseteq \mathcal{S}_+(G,\mathbb{C})$, we have $\mathrm{mr}_+^{\mathbb{C}}(G) \leqslant \mathrm{mr}_+^{\mathbb{R}}(G)$. An example of a graph G where $\mathrm{mr}_+^{\mathbb{C}}(G) < \mathrm{mr}_+^{\mathbb{R}}(G)$ is given in [1]. It is clear that if GCC_+ holds for $\mathrm{mr}_+^{\mathbb{R}}(G)$, then it also holds for $\mathrm{mr}_+^{\mathbb{C}}(G)$.

We denote $\mathrm{M}(G)$ to be the *maximum nullity* among matrices in $\mathcal{S}(G,\mathbb{R})$, $\mathrm{M}_+^{\mathbb{R}}(G)$ to be the maximum nullity among matrices in $\mathcal{S}_+(G,\mathbb{R})$ and $\mathrm{M}_+^{\mathbb{C}}(G)$ to be the maximum nullity among matrices in $\mathcal{S}_+(G,\mathbb{C})$. Using the rank-nullity theorem, we have $\mathrm{mr}(G) + \mathrm{M}(G) = \mathrm{mr}_+^{\mathbb{R}}(G) + \mathrm{M}_+^{\mathbb{R}}(G) = \mathrm{mr}_+^{\mathbb{C}}(G) + \mathrm{M}_+^{\mathbb{C}}(G) = |G|$.

When the result does not depend on the real or complex entries of the psd matrices corresponding to a given graph G we will denote the minimum semidefinite rank and the maximum nullity as $\mathrm{mr}_+(G)$ and $\mathrm{M}_+(G)$, respectively. When discussing GCC_+ we will only consider real minimum semidefinite rank $\mathrm{mr}_+^{\mathbb{R}}(G)$.

If a graph G is disconnected, then the direct sum of psd matrices for the connected components G_i, $i=1,2,\dots,k$ of G yields a psd matrix for the graph G. In this case, $\mathrm{mr}_+(G)=\sum_{i=1}^{k}\mathrm{mr}_+(G_i)$. Therefore, it suffices to find $\mathrm{mr}_+(G)$ for a connected graph G.

The adjacency matrix $A=[a_{ij}]$ of a simple graph G on n vertices $\{v_1,v_2,\dots,v_n\}$ consists of entries $a_{ij}=1$ when v_i and v_j are adjacent and $a_{ij}=0$ otherwise. Let the matrix $D=\mathrm{diag}\{d(v_1),\dots,d(v_n)\}$. Then, $L(G)=D(G)-A(G)$ is called the *(classical) Laplacian matrix* of G.

Let $\overrightarrow{u}=(u_1,u_2,\dots,u_n)$ and $\overrightarrow{v}=(v_1,v_2,\dots,v_n)$ be two vectors in $\mathbb{C}^n$. The *Euclidean inner product* of $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined as $\langle\overrightarrow{u},\overrightarrow{v}\rangle=\sum_{i=1}^{n}u_i\overline{v_i}$. Any two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{C}^n$ are said to be *orthogonal* if $\langle\overrightarrow{u},\overrightarrow{v}\rangle=0$.

Suppose $v_1,v_2,\dots,v_n$ are the vertices of a simple graph G. We associate the vectors $\overrightarrow{v_1},\overrightarrow{v_2},\dots,\overrightarrow{v_n}$ in $\mathbb{C}^m$ to the vertices $v_1,v_2,\dots,v_n$, such that, for $i\neq j$, $\langle\overrightarrow{v_i},\overrightarrow{v_j}\rangle\neq 0$ if and only if $\{v_i,v_j\}\in E(G)$ for $1\leqslant i,j\leqslant n$. We say that $\overrightarrow{V}=\{\overrightarrow{v_1},\overrightarrow{v_2},\dots,\overrightarrow{v_n}\}$ is a *vector representation* of G. Let X be a matrix given by $X=[\overrightarrow{v_1}\cdots\overrightarrow{v_n}]$. Then X^*X is a psd matrix called the *Gram matrix* of $\overrightarrow{V}$ with respect to the Euclidean inner product where $\mathrm{rank}(\overrightarrow{V}):=\dim(\mathrm{Span}(\overrightarrow{V}))=\mathrm{rank}(X^*X)$. Since any psd matrix A can be written as X^*X for some $X\in M_{m,n}(\mathbb{C})$ with $\mathrm{rank}(A)=\mathrm{rank}(X)$ ([17], p. 440), each psd matrix is the Gram matrix of a set of vectors $\overrightarrow{V}$. Thus, finding a psd matrix representing G with rank k and finding a vector representation of G in $\mathbb{R}^k$ are equivalent problems.

A real symmetric matrix A is said to satisfy the *Strong Arnold Property* if there does not exist an $n\times n$ symmetric matrix $X\neq 0$ such that

- $AX=0$
- $A\circ X=0$
- $I\circ X=0,$

where $\circ$ denotes the entrywise (Hadamard) product and I is the identity matrix. The parameter $\nu(G)$ is defined to be the maximum nullity among matrices $A\in\mathcal{S}_+(G,\mathbb{R})$ that satisfy the Strong Arnold Property [17].

2.3. Some prior results on the minimum semidefinite rank

For any connected graph G on n vertices, the Laplacian matrix $L(G)$ of G is a psd matrix with rank $n-1$ [18] and it follows that $\mathrm{mr}_+(G)\leqslant n-1$. Further, $\mathrm{mr}_+(G)=n-1$ if and only if G is a tree on n vertices ([22], Theorem 4.1). For a complete graph K_n where $n\geqslant 2$, the $n\times n$ matrix J of all ones is in $\mathcal{S}_+(K_n,\mathbb{C})$ and it follows that $\mathrm{mr}_+(K_n)=1$. Further, $\mathrm{mr}_+(G)=1$ if and only if $G=K_n$ for $n\geqslant 2$. Thus, for any connected graph G with $|G|\geqslant 2$, if G is neither a tree nor a complete graph, then $2\leqslant\mathrm{mr}_+(G)\leqslant|G|-2$. Note that $\mathrm{mr}_+(K_1)=0$.

Since a principal submatrix of a psd matrix is psd ([17], p. 430) and the rank of a submatrix can never be greater than that of the matrix ([17], p. 430, Observation 7.1.2), the minimum semidefinite rank of any induced subgraph H of a given graph G gives a lower bound for the minimum semidefinite rank of G. For a cycle C_n, since a path P on $n-1$ vertices is an induced subgraph of C_n, we have $\text{mr}_+(C_n) \geqslant \text{mr}_+(P_{n-1}) = n-2$. Since C_n is not a tree, it follows that $\text{mr}_+(C_n) = n-2$.

DEFINITION 1. [20] Let G be a multigraph. If $v \in V(G)$, the *orthogonal vertex removal of* v from G, denoted $G \ominus v$, is a multigraph modified from $G[V(G) - \{v\}]$ by adding $P(u,w)$ additional edges between each pair $u, w \in N(v)$, where $P(u,w)$ is the product of the number of edges from v to u and from v to w.

DEFINITION 2. Let G be a connected multigraph with $|G| = n$. Define an $n \times n$ symmetric or Hermitian *psd* matrix $A = [a_{ij}]$ corresponding to G as follows:

- $a_{ij} \neq 0$ if v_i and v_j are joined by exactly one edge.
- $a_{ij} = 0$ if $v_i \neq v_j$ and v_i and v_j are not adjacent.
- a_{ij} is any real number if v_i and v_j are joined by multiple edges.

Let $\mathcal{S}_+(G,\mathbb{F})$ denote the set of all $n \times n$ psd matrices which satisfy the above properties where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then $\text{mr}_+^{\mathbb{F}}(G) = \min\{\text{rank}(A) | A \in \mathcal{S}_+(G,\mathbb{F})\}$.

RESULT 1. ([6], Corollary 3.5) If G is a simple connected graph and v is a pendant vertex, then $\text{mr}_+(G) = \text{mr}_+(G-v)+1 = \text{mr}_+(G \ominus v)+1$.

RESULT 2. ([3], Lemma 2.5) If G is a connected graph and v is a vertex of degree two, then $\text{mr}_+(G) = \text{mr}_+(G \ominus v)+1$.

DEFINITION 3. [6] A *simplicial vertex* of a multigraph G is a vertex v such that the induced subgraph $G[N[v]]$ is a clique in G.

RESULT 3. ([6], Lemma 3.4) If v is a simplicial vertex of a connected multigraph G that is joined to at least one neighbor by exactly one edge, then $\text{mr}_+(G) = \text{mr}_+(G \ominus v)+1$.

RESULT 4. ([6], Proposition 3.1 and Theorem 3.6) For a connected graph G, $\text{mr}_+(G) \leqslant \text{cc}(G)$. In particular, $\text{mr}_+(G) = \text{cc}(G)$ if G is a chordal graph.

RESULT 5. [5] For a connected graph G, we have $\text{mr}_+^{\mathbb{R}}(G) \geqslant \alpha(G)$.

3. Shadow graph $S(G)$ and its complement $\overline{S(G)}$

In this section we give the definition of shadow graph $S(G)$ found in [9]. We give upper bounds for the minimum semidefinite rank of $S(G)$ and the minimum semidefinite rank of its complement $\overline{S(G)}$. We show that when $\overline{G}$ is a tree or when $\overline{G}$ is a unicyclic graph $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) = \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$.

DEFINITION 4. ([9], p. 276) Given a graph G, the *shadow graph* $S(G)$ is obtained from G by adding for each vertex u of G, a new vertex v, called the shadow vertex of u, and joining v to the neighbors of u in G.

EXAMPLE 1. The following are the shadow graphs $S(G)$ of the path P_5 and the cycle C_4. The shadow vertices are represented as black vertices.

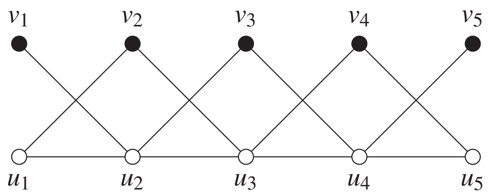


Figure 1: $S(P_5)$

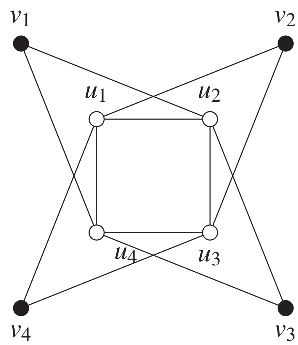


Figure 2: $S(C_4)$

OBSERVATION 1. In the definition of $S(G)$ note that the vertex u of G and its shadow vertex v are not adjacent in $S(G)$ and the shadow vertices are pairwise nonadjacent in $S(G)$.

THEOREM 3. *If G is a connected graph with $|G| \geqslant 3$, then $|G| \leqslant \mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - 2$.*

Proof. Since G is connected and $|G| \geqslant 3$, there is a vertex u in G such that $d_G(u) \geqslant 2$. Let u_1, u_2 be the neighbors of u and v be the shadow vertex of u. Then the set of vertices $\{u, v, u_1, u_2\}$ induces a cycle in $S(G)$. Hence $S(G)$ is not a tree and $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - 2$ when $|G| \geqslant 3$. It is clear from Observation 1 and the definition of $S(G)$ that the shadow vertices of $S(G)$ form a largest independent set of size $|G|$. Since the independence number is a lower bound for the minimum semidefinite rank ([6], Corollary 2.7), we have $|G| \leqslant \mathrm{mr}_+^{\mathbb{R}}(S(G))$. $\square$

Next, we give an example of a class of G such that $\mathrm{mr}_+^{\mathbb{R}}(S(G)) = |S(G)| - 2$.

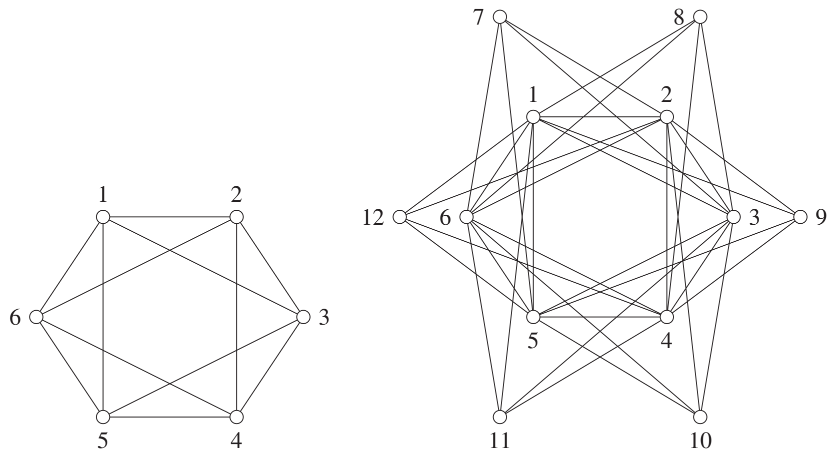
PROPOSITION 1. *Let P_n be a path on $n \geqslant 3$ vertices. Then* $\mathrm{mr}_+^{\mathbb{R}}(S(P_n)) = |S(P_n)| - 2$.

Proof. For $n = 3$ or $n = 4$, it is easy to verify that $\mathrm{mr}_+^{\mathbb{R}}(S(P_n))$ is equal to $|S(P_n)| - 2$. Assume $n \geqslant 5$. Let $V(S(P_n)) = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ where v_i is the shadow vertex of u_i, $1 \leqslant i \leqslant n$. Note that $d_{S(G)}(v_1) = d_{S(G)}(v_n) = 1$ and $d_{S(G)}(v_i) = 2$ for $2 \leqslant i \leqslant n-1$. By orthogonally removing the vertices v_i $(1 \leqslant i \leqslant n)$ and using Results 1 and 2 we have $\mathrm{mr}_+^{\mathbb{R}}(S(P_n)) = |P_n| + \mathrm{mr}_+^{\mathbb{R}}(H)$ where H is the graph such that $N(u_1) = \{u_2, u_3\}$, $N(u_n) = \{u_{n-2}, u_{n-1}\}$, $N(u_2) = \{u_1, u_3, u_4\}$, $N(u_{n-1}) = \{u_{n-3}, u_{n-2}, u_n\}$ and $N(u_i) = \{u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}\}$ for $3 \leqslant i \leqslant n-2$. Since H is a chordal graph with $\mathrm{cc}(H) = |P_n| - 2$, by Result 4 $\mathrm{mr}_+^{\mathbb{R}}(H) = |P_n| - 2$. Therefore, $\mathrm{mr}_+^{\mathbb{R}}(S(P_n)) = |P_n| + \mathrm{mr}_+^{\mathbb{R}}(H) = |P_n| + |P_n| - 2 = |S(P_n)| - 2$. □

REMARK 1. There are other classes of graphs such as the star graph S_n (in Example 2) that show the upper bound in Theorem 3 is sharp. We also know a class of circulant graphs (in Example 3) for which the lower bound in Theorem 3 is attained.

EXAMPLE 2. Let $S(S_{n+1})$ be the shadow graph of a star on $n+1$ vertices where $n \geqslant 2$. Then $\mathrm{mr}_+^{\mathbb{R}}(S(S_{n+1})) = |S(S_{n+1})| - 2$.

Proof. Let $V(S(S_{n+1})) = \{u_1, \ldots, u_n, x, v_1, \ldots, v_n, \tilde{x}\}$ where v_i is the shadow vertex of u_i, x is the center vertex of S_{n+1} and $\tilde{x}$ is the shadow vertex of x. Since u_i is a pendant vertex in S_{n+1}, v_i is a pendant vertex in $S(S_{n+1})$. Applying Result 1 inductively to vertices v_i, we have $\mathrm{mr}_+^{\mathbb{R}}(S(S_{n+1})) = \mathrm{mr}_+^{\mathbb{R}}(K_{2,n}) + n$. So, $\mathrm{mr}_+^{\mathbb{R}}(S(S_{n+1})) = n + n = 2n = (2n+2) - 2 = |S(S_{n+1})| - 2$. □



(a) $\mathrm{Circ}(6, \{1,2\})$ (b) $S(\mathrm{Circ}(6, \{1,2\}))$

Figure 3: *The circulant graph* $\mathrm{Circ}(6, \{1,2\})$ *and its shadow graph*

DEFINITION 5. A circulant graph $\mathrm{Circ}(n,S)$ is a graph with n vertices in which every vertex i (where $i \in \{1,2,\ldots,n\}$) is adjacent to vertices $i+j \pmod n$ and $i-j \pmod n$ for each j in S where $S \subseteq \{1,2,\ldots,n\}$.

The example of the circulant graph $\mathrm{Circ}(6,\{1,2\})$ and its shadow graph is shown in Figure 3.

EXAMPLE 3. Consider $G = \mathrm{Circ}(6,\{1,2\})$ in Figure 3. We give a matrix $M \in \mathcal{S}_+(S(G))$ with $\mathrm{rank}(M) = 6 = |G|$.

Proof. Let
$$M = \begin{bmatrix} A & B \\ B^T & I_{6\times 6} \end{bmatrix} = \left[\begin{array}{cccccc|cccccc} 4 & 2 & 2 & 0 & -2 & -2 & 0 & 1 & 1 & 0 & 1 & 1 \\ 2 & 4 & 2 & -2 & 0 & 2 & 1 & 0 & 1 & 1 & 0 & 1 \\ 2 & 2 & 4 & 2 & 2 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & -2 & 2 & 4 & 2 & -2 & 0 & 1 & -1 & 0 & 1 & -1 \\ -2 & 0 & 2 & 2 & 4 & 2 & 1 & 0 & -1 & 1 & 0 & -1 \\ -2 & 2 & 0 & -2 & 2 & 4 & 1 & -1 & 0 & 1 & -1 & 0 \\ \hline 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

where A is a matrix corresponding to G. Since the set of shadow vertices $\{7,8,9,10,11,12\}$ is an independent set, we choose the matrix corresponding to the shadow vertices to be a $I_{6\times 6}$. For the matrix B, we have that $B = \begin{bmatrix} J - I_{3\times 3} & J - I_{3\times 3} \\ D & D \end{bmatrix}$ where J is the 3×3 matrix of all ones and $D = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$. Notice that $A = BB^T$. Since $I_{6\times 6}$ is a positive definite and $A - BI^{-1}B^T = A - BB^T = A - A = 0$, by Schur complement for positive semidefiniteness we have that M is psd and

$$\mathrm{rank}(M) = \mathrm{rank}(I_{6\times 6}) + \mathrm{rank}(A - BI_{6\times 6}^{-1}B^T) = 6 + \mathrm{rank}(0) = 6. \quad \square$$

We use the same idea as above to generalize as below.

COROLLARY 1. *Let*

$$G = \mathrm{Circ}\left(n, \{1,2,\ldots,\frac{n-2}{2}\}\right)$$

where n is even and $n \geqslant 6$. Then $\mathrm{mr}_+^{\mathbb{R}}(S(G)) = |G| = n$.

Proof. Denote $V(G) = \{1,2,\ldots,n\}$ and the set of shadow vertices by

$$\{n+1, n+2, \ldots, 2n\}$$

where $\forall j \in \{n+1, n+2, \dots, 2n\}$, j is the shadow vertex of $j-n$. By the definition of G, every vertex $i \in \{1,2,\dots,n\}$, i is adjacent to all vertices except the vertex $\frac{n}{2}+i$ (mod n). Note that every vertex i in G, i is not adjacent to its shadow vertex in $S(G)$. Moreover, the set of shadow vertices forms an independent set in $S(G)$. Define M to be a 2×2 block matrix where

$$M = \begin{bmatrix} A & B \\ B^T & I_{n\times n} \end{bmatrix},$$

where each entry a_{ij} in A corresponds to the adjacency between vertices in G, each entry b_{ij} in B corresponds to the adjacency between the vertices in G and their shadow vertices and the identity matrix $I_{n\times n}$ corresponds to the adjacency between vertices in $\{n+1, n+2, \dots, 2n\}$. Define the 2×2 block matrix B as

$$B = \begin{bmatrix} J - I_{\frac{n}{2}\times\frac{n}{2}} & J - I_{\frac{n}{2}\times\frac{n}{2}} \\ D & D \end{bmatrix}$$

where J is the $\frac{n}{2}\times\frac{n}{2}$ matrix of all ones and D is the $\frac{n}{2}\times\frac{n}{2}$ matrix such that

$$D = \begin{bmatrix} 0 & 1 & 1\,1\,1 \dots & 1 & -(\frac{n-4}{2}) \\ 1 & 0 & 1\,1\,1 \dots & 1 & -(\frac{n-4}{2}) \\ 1 & 1 & 0\,1\,1 \dots & 1 & -(\frac{n-4}{2}) \\ \vdots & \ddots & \vdots & & \vdots \\ 1 & 1 & 1\,1\,1 \dots & 0 & -(\frac{n-4}{2}) \\ 1 & 1 & 1\,1\,1 \dots & -(\frac{n-4}{2}) & 0 \end{bmatrix}.$$

Next, we define $A = BB^T$. It can be checked that M is a matrix representation of $S(G)$. Since $A = BB^T$, we have

$$A = BB^T = \begin{bmatrix} J-I & J-I \\ D & D \end{bmatrix} \begin{bmatrix} J-I & D^T \\ J-I & D^T \end{bmatrix} = \begin{bmatrix} 2(J-I)^2 & 2(J-I)D^T \\ 2D(J-I) & 2DD^T \end{bmatrix}$$

where $2(J-I)^2$ has no zero entry, $(J-I)D^T$ has zero entries on the diagonal, $D(J-I)$ has zero entries on the diagonal, DD^T has no zero entries on the diagonal and the entry a_{ij} of A is zero if $|i-j| = \frac{n}{2}$. Since I is positive definite and $A - BI^{-1}B^T = A - BB^T = A - A = 0$, by Schur complement for positive semidefiniteness we have that M is psd and

$$\text{rank}(M) = \text{rank}(I_{n\times n}) + \text{rank}(A - BI_{n\times n}^{-1}B^T) = n + \text{rank}(0) = n.$$

Thus, $\text{mr}_+^{\mathbb{R}}(S(G)) \leqslant n$. By Result 5 we have $n \leqslant \alpha(S(G)) \leqslant \text{mr}_+^{\mathbb{R}}(S(G))$. Thus, $n \leqslant \text{mr}_+^{\mathbb{R}}(S(G)) \leqslant n$. Therefore, $\text{mr}_+^{\mathbb{R}}(S(G)) = n = |G|$. □

In the next two theorems we find the minimum semidefinite rank of the complement of the shadow graph $S(G)$.

THEOREM 4. *Suppose G is a simple connected graph such that $\overline{G}$ is connected. Then, either*

$$\text{mr}_+^{\mathbb{R}}(\overline{S(G)}) = \text{mr}_+^{\mathbb{R}}(\overline{G}) \;\; or \;\; \text{mr}_+^{\mathbb{R}}(\overline{S(G)}) = \text{mr}_+^{\mathbb{R}}(\overline{G}) + 1.$$

Proof. Let $V(\overline{S(G)}) = \{u_1,\dots,u_n,v_1,\dots,v_n\}$ where u_i $(1 \leqslant i \leqslant n)$ are the vertices of G that are labeled first followed by the corresponding shadow vertices v_i $(1 \leqslant i \leqslant n)$. Since $\overline{G}$ is an induced subgraph of $\overline{S(G)}$, we have $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)})$. Next, it suffices to show that $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G})+1$. Let $A=[a_{ij}]$ be an $n\times n$ real symmetric positive semidefinite matrix corresponding to $\overline{G}$ with $\mathrm{rank}(A) = \mathrm{mr}_+^{\mathbb{R}}(\overline{G})$. Let x be a real number such that $x > \max\{|a_{ij}| : 1 \leqslant i,j \leqslant n\}$ and J be the $n\times n$ matrix of all ones. Then we define a 2×2 block matrix M as

$$M = \begin{bmatrix} A & A \\ A & A+xJ \end{bmatrix}.$$

Next, we claim that M is a matrix corresponding to $\overline{S(G)}$. Recall that all the diagonal entries of A are positive because $\overline{G}$ is connected by assumption. The block $M_{1,1}=A$ corresponds to $\overline{G}$. In the block $M_{1,2}=A=[a_{ij}]$ for $1\leqslant i,j\leqslant n$, a_{ij} is nonzero if and only if u_i is adjacent to v_j in $\overline{S(G)}$. Since u_i is adjacent to v_i in $\overline{S(G)}$, the diagonal entries of A are nonzero. Moreover, u_i is adjacent to v_j in $\overline{S(G)}$ for $i\neq j$ if and only if u_i is adjacent to u_j in $\overline{G}$ for $i\neq j$. In the block $M_{2,2}$ each entry corresponds to the adjacency between v_i and v_j. Since $\{v_1,\dots,v_n\}$ form an independent set in $S(G)$, they induce a complete subgraph in $\overline{S(G)}$. Therefore, each off-diagonal entry in $A+xJ$ must be nonzero. By the choice of x, every entry in $A+xJ$ is nonzero. Next, we show that M is psd. For $\overrightarrow{v} = \begin{bmatrix}\overrightarrow{p}\\ \overrightarrow{q}\end{bmatrix}$ in $\mathbb{R}^{2n}$ where $\overrightarrow{p},\overrightarrow{q}\in\mathbb{R}^n$, we have

$$\begin{aligned}
\overrightarrow{v}^T M \overrightarrow{v} &= \begin{bmatrix}\overrightarrow{p}^T & \overrightarrow{q}^T\end{bmatrix} \begin{bmatrix} A & A \\ A & A+xJ \end{bmatrix} \begin{bmatrix}\overrightarrow{p}\\ \overrightarrow{q}\end{bmatrix} \\
&= \overrightarrow{p}^T A\overrightarrow{p} + \overrightarrow{q}^T A\overrightarrow{p} + \overrightarrow{p}^T A\overrightarrow{q} + \overrightarrow{q}^T Aq + \overrightarrow{q}^T(xJ)\overrightarrow{q}. \\
&= (\overrightarrow{p}^T + \overrightarrow{q}^T)A(\overrightarrow{p}+\overrightarrow{q}) + \overrightarrow{q}^T(xJ)\overrightarrow{q}.
\end{aligned}$$

Since A and xJ are psd matrices and $\overrightarrow{v}$ is any vector in $\mathbb{R}^{2n}$, we conclude that $\overrightarrow{v}^T M\overrightarrow{v} \geqslant 0$ and hence M is a psd matrix. Moreover,

$$\mathrm{rank}(M) = \mathrm{rank}\left(\begin{bmatrix} A & A \\ A & A\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & xJ\end{bmatrix}\right) \leqslant \mathrm{rank}(A) + \mathrm{rank}(xJ) = \mathrm{mr}_+^{\mathbb{R}}(\overline{G})+1.$$

Therefore, $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{rank}(M) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G})+1$. □

We now extend the proof of Theorem 4 to the case where $\overline{G}$ is disconnected or $\overline{G}$ contains isolated vertices.

THEOREM 5. *Let G be a simple connected graph such that $\overline{G}$ is disconnected. If $G_1, G_2, \dots, G_k$ are the connected components of $\overline{G}$ with each component having two or more vertices and if there are r isolated vertices in $\overline{G}$, then*

$$\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \left(\sum_{i=1}^{k} \mathrm{mr}_+^{\mathbb{R}}(G_i)\right) + r + 1.$$

Proof. For $1 \leqslant i \leqslant k$, let A_i be a real symmetric psd matrix corresponding to G_i with $\operatorname{rank}(A_i) = \operatorname{mr}_+^{\mathbb{R}}(G_i)$. We define $A = [a_{ij}] = \left(\bigoplus_{i=1}^k A_i\right) \bigoplus I_r$ where I_r is the $r \times r$ identity matrix. Let x be a real number such that $x > \max\{|a_{ij}| : 1 \leqslant i, j \leqslant n\}$. Then we define a 2×2 block matrix M as

$$M = \begin{bmatrix} A & A \\ A & A + xJ \end{bmatrix}$$

where J is the matrix of all ones. It can be verified that M is a matrix corresponding to $\overline{S(G)}$. Since the direct sum of psd matrices is psd, A is psd. From the previous proof we know that M is psd. Moreover,

$$\begin{aligned} \operatorname{rank}(M) &= \operatorname{rank}\left(\begin{bmatrix} A & A \\ A & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & xJ \end{bmatrix}\right) \\ &\leqslant \operatorname{rank}(A) + \operatorname{rank}(xJ) \\ &= \operatorname{rank}\left[\left(\bigoplus_{i=1}^{k} A_i\right) \bigoplus I_r\right] + 1 \\ &= \left(\sum_{i=1}^{k} \operatorname{mr}_+^{\mathbb{R}}(G_i)\right) + r + 1. \end{aligned}$$

Therefore, $\operatorname{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \left(\sum_{i=1}^k \operatorname{mr}_+^{\mathbb{R}}(G_i)\right) + r + 1.$ $\square$

REMARK 2. Since $\overline{G}$ is an induced subgraph of $\overline{S(G)}$, we get $\sum_{i=1}^{k} \operatorname{mr}_+^{\mathbb{R}}(G_i) \leqslant \operatorname{mr}_+^{\mathbb{R}}(\overline{S(G)})$ in Theorem 5. Moreover, if there are no isolated vertices in $\overline{G}$, then the conclusion of Theorem 5 is same as that of Theorem 4.

Next, we give an example of a graph G for which the upper bound in Theorem 5 is achieved.

EXAMPLE 4. Let $G = P_4 \vee K_3$. Then $\overline{G} = P_4 \cup 3K_1$. Let

$$V(\overline{S(G)}) = \{u_1, \ldots, u_7, v_1, \ldots, v_7\}$$

where for $1 \leqslant i \leqslant 7$, u_i are the vertices in $\overline{G}$ and v_i are the shadow vertices of u_i. The set of vertices $\{u_1, u_2, u_3, u_4\}$ forms an induced path P_4 in $\overline{G}$ and u_5, u_6, u_7 are isolated vertices in $\overline{G}$. In $\overline{S(G)}$, the set of vertices $\{u_1, u_2, v_1, v_2\}$, $\{u_2, u_3, v_2, v_3\}$, $\{u_3, u_4, v_3, v_4\}$ and $\{v_1, \ldots, v_7\}$ form complete subgraphs and u_5, u_6, u_7 are pendant vertices. It can be verified that the clique cover number $\operatorname{cc}(\overline{S(G)}) = 7$. Since $\overline{S(G)}$ is a chordal graph, using Result 4, we have $\operatorname{mr}_+^{\mathbb{R}}(\overline{S(G)}) = 7 = \operatorname{mr}_+^{\mathbb{R}}(P_4) + 3 + 1$.

DEFINITION 6. A graph G is said to be *unicyclic* if it has exactly one induced subgraph that is a cycle.

The following two propositions show that the upper bound in Theorem 4 is attained when $\overline{G}$ is either a tree or a unicyclic graph.

PROPOSITION 2. *Suppose G is a simple graph with $|G| \geqslant 3$ such that $\overline{G}$ is a tree. Then* $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) = \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$.

Proof. Let u, v be two of the pendant vertices in $\overline{G}$ with shadow vertices u' and v', respectively. Let K be the graph induced in $\overline{S(G)}$ by $V(\overline{G}) \cup \{u', v'\}$.

Case 1. Let us assume that the pendant vertices u and v satisfy $N(u) = N(v) = \{w\}$ in $\overline{G}$ and $P_{u,v}$ is a path from u to v in $\overline{G}$. Let $P' = V(\overline{G}) \setminus V(P_{u,v})$. Since the graph induced by P' in K is a forest, by sequentially removing the pendant vertices of P' orthogonally in K we obtain the subgraph J of K induced by $V(P_{u,v}) \cup \{u', v'\}$. The subgraph J is isomorphic to the graph in Figure 4. Since J is a chordal graph from Result 4, $\mathrm{mr}_+^{\mathbb{R}}(J) = 3$ and hence $\mathrm{mr}_+^{\mathbb{R}}(K) = |\overline{G}| - 3 + \mathrm{mr}_+^{\mathbb{R}}(J) = |\overline{G}| = \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$. Since K is an induced subgraph of $\overline{S(G)}$ we have $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1 = \mathrm{mr}_+^{\mathbb{R}}(K) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$ where the last inequality is from Theorem 4.

Case 2. Suppose $N(u) \neq N(v)$. Then, as in case 1, if we orthogonally remove the vertices of the forest induced by P' in K we get a graph induced by $V(P_{u,v}) \cup \{u', v'\}$. By orthogonally removing the degree 2 vertices in $V(P_{u,v})$ we obtain the subgraph H in the Figure 5. Using orthogonal removal of u and v in H we get $\mathrm{mr}_+^{\mathbb{R}}(H) = 4$. Hence $\mathrm{mr}_+^{\mathbb{R}}(K) = |\overline{G}| - 4 + \mathrm{mr}_+^{\mathbb{R}}(H) = |\overline{G}| = \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$. As before we get $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) = \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$.

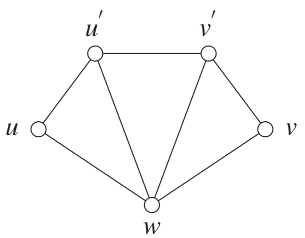


Figure 4:

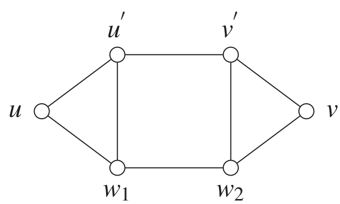


Figure 5:

□

PROPOSITION 3. *Suppose G is a simple graph with $|G| \geqslant 3$ such that $\overline{G}$ is a unicyclic graph. Then* $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) = \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$.

Proof. Let $u_1 u_2 \ldots u_{n-1} u_n u_1$ be the cycle C induced in $\overline{G}$, which is an induced subgraph of $\overline{S(G)}$.

Case 1. Suppose u_1 is a vertex of degree 2 in $\overline{G}$. Then $V(\overline{G}) \setminus \{u_1\}$ is a tree. From Proposition 2 we get an induced subgraph L of $\overline{S(G)}$ such that $\mathrm{mr}_+^{\mathbb{R}}(L) = |\overline{G}| - 1$. Since

$\overline{G}$ is unicyclic, by [15] we have $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) = |\overline{G}| - 2$. Hence $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1 = |\overline{G}| - 1 = \mathrm{mr}_+^{\mathbb{R}}(L) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$ using Theorem 4.

Case 2. Suppose there are no vertices on C of degree 2 in $\overline{G}$. Then there is a tree joined to every vertex of C in $\overline{G}$. Suppose T_1 and T_2 are trees joined to u_1 and u_2 of C, respectively.

Case 2.1 When T_1 and T_2 are not single vertices. Let v and w be pendant vertices in T_1 and T_2, respectively and v' and w' be the corresponding shadow vertices. Let K be the graph induced in $\overline{S(G)}$ by $V(\overline{G}) \cup \{v', w'\}$. Let $P_{v,w}$ be the path in $\overline{G}$ containing the edge u_1u_2 that is a part of the cycle C. Then $V(P_{v,w}) \cup \{v', w'\}$ induces a cycle along with two triangles obtained by the edges joining the shadow vertices v' and w' to the unique neighbors of the pendant vertices v and w, respectively. By orthogonally removing the pendant vertices of the forest in $V(K) \setminus \{V(C) \cup V(P_{v,w}) \cup \{v', w'\}\}$ and then orthogonally removing the degree two vertices $\{u_3, \ldots, u_{n-1}\}$ of C we obtain a graph H that is isomorphic to the graph in Figure 6. By orthogonally removing the degree two vertices v, w and u_n and deleting the resulting multiple edges on the cycle in Figure 7 we get three paths. Thus, $\mathrm{mr}_+^{\mathbb{R}}(H) = |P_{v,w}|$. Recall that the number of vertices deleted orthogonally from the forest is $|\overline{G}| - |C| - |P_{v,w}| + 2$ where u_1, u_2 are counted in both C and $P_{v,w}$. Therefore, $\mathrm{mr}_+^{\mathbb{R}}(K) = (|\overline{G}| - |C| - |P_{v,w}| + 2) + (|C| - 3) + \mathrm{mr}_+^{\mathbb{R}}(H) = (|\overline{G}| - |C| - |P_{v,w}| + 2) + (|C| - 3) + |P_{v,w}| = |\overline{G}| - 1$. Since $\overline{G}$ is unicyclic, $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) = |\overline{G}| - 2$. Hence $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1 = |\overline{G}| - 1 = \mathrm{mr}_+^{\mathbb{R}}(K) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$ using Theorem 4.

Case 2.2 When T_1 and T_2 are single vertices. Let v and w be vertices in T_1 and T_2, respectively and v' and w' be the corresponding shadow vertices. Proceeding as above, we have $V(P_{v,w}) \cup \{v', w'\}$ induces a cycle along with two triangles obtained by the edges joining the shadow vertices v' and w' to the unique neighbors of the vertices v and w, respectively. By orthogonally removing the pendant vertices of the forest in $V(K) \setminus \{V(C) \cup V(P_{v,w}) \cup \{v', w'\}\}$ and then orthogonally removing the degree two vertices $\{u_3, \ldots, u_{n-1}\}$ of C we obtain a graph H that is isomorphic to the graph in Figure 8. Recall that $|P_{v,w}| = 4$. By orthogonally removing the degree 2 vertices v, w and u_n and deleting the resulting multiple edges on the cycle in Figure 8 we get a path P_2 and two isolated vertices. Thus, $\mathrm{mr}_+^{\mathbb{R}}(H) = \mathrm{mr}_+^{\mathbb{R}}(P_2) + 3 = 4 = |P_{v,w}|$. Recall that the number of vertices deleted orthogonally from the forest is $|\overline{G}| - |C| - |P_{v,w}| + 2$ where u_1, u_2 are counted in both C and $P_{v,w}$. Therefore, $\mathrm{mr}_+^{\mathbb{R}}(K) = (|\overline{G}| - |C| - |P_{v,w}| + 2) + (|C| - 3) + \mathrm{mr}_+^{\mathbb{R}}(H) = (|\overline{G}| - |C| - |P_{v,w}| + 2) + (|C| - 3) + |P_{v,w}| = |\overline{G}| - 1$. Since $\overline{G}$ is unicyclic, $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) = |\overline{G}| - 2$. Hence $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1 = |\overline{G}| - 1 = \mathrm{mr}_+^{\mathbb{R}}(K) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1$ using Theorem 4.

Case 2.3 When T_1 and T_2 are trees such that T_2 is a single vertex. Let v and w be vertices in T_1 and T_2, respectively and v' and w' be the corresponding shadow vertices. By orthogonally removing the pendant vertices of the forest in $V(K) \setminus \{V(C) \cup V(P_{v,w}) \cup \{v', w'\}\}$ and then orthogonally removing the degree two vertices $\{u_3, \ldots, u_{n-1}\}$ of C we obtain a graph H that is isomorphic to the graph in Figure 9. By orthogonally removing the degree 2 vertices v, w and u_n in H and deleting the resulting multiple edges on the cycle in H we get 2 paths and one isolated vertex. The number of vertices on those two paths are 2 and $|P_{v,w}| - 3$. Thus, $\mathrm{mr}_+^{\mathbb{R}}(H) = 3 + \mathrm{mr}_+^{\mathbb{R}}(P_2) +$

$(|P_{v,w}|-4) = |P_{v,w}|$. Recall that the number of vertices deleted orthogonally from the forest is $|\overline{G}|-|C|-|P_{v,w}|+2$ where u_1, u_2 are counted in both C and $P_{v,w}$. Therefore, $\text{mr}_+^{\mathbb{R}}(K) = (|\overline{G}|-|C|-|P_{v,w}|+2)+(|C|-3)+\text{mr}_+^{\mathbb{R}}(H) = (|\overline{G}|-|C|-|P_{v,w}|+2)+(|C|-3)+|P_{v,w}| = |\overline{G}|-1$. Since $\overline{G}$ is unicyclic, $\text{mr}_+^{\mathbb{R}}(\overline{G}) = |\overline{G}|-2$. Hence $\text{mr}_+^{\mathbb{R}}(\overline{G})+1 = |\overline{G}|-1 = \text{mr}_+^{\mathbb{R}}(K) \leqslant \text{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \text{mr}_+^{\mathbb{R}}(\overline{G})+1$ using Theorem 4. □

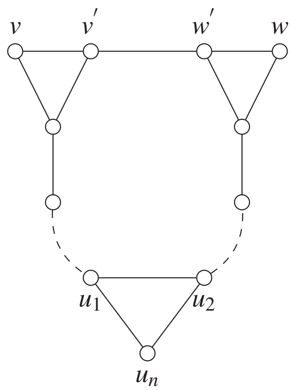


Figure 6:

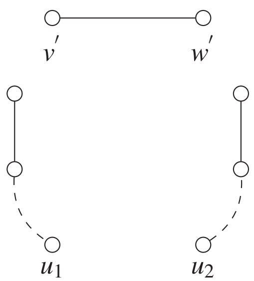


Figure 7:

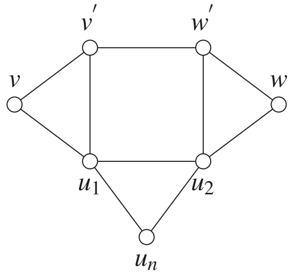


Figure 8:

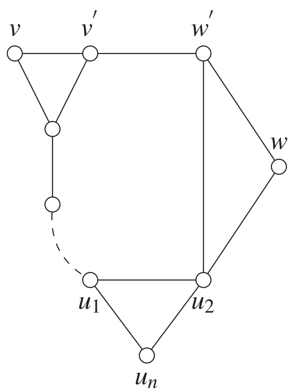


Figure 9:

4. Shadow graph $S(G)$ and GCC_+

In this section we show that $S(G)$ satisfies GCC_+ when G is a tree or a unicyclic graph or a complete graph. Whenever G is a k-tree or a chordal graph whose complement has no isolated vertices, we show that $S(G)$ satisfies GCC_+. Also, we show that when G is a partial k-tree $(k \geqslant 2)$ where G has a subgraph K_{k+1} and $\overline{G}$ has no isolated vertices, then $S(G)$ satisfies GCC_+.

THEOREM 6. *The shadow graph $S(T)$ of a tree T satisfies GCC_+.*

Proof. Let T be a tree. If $|T| = 2$ then $S(T) = P_4$ and $\overline{S(T)} = P_4$. Since P_4 is a tree, we have $\operatorname{mr}_+^{\mathbb{R}}(P_4) = 3$ and

$$\operatorname{mr}_+^{\mathbb{R}}(S(T)) + \operatorname{mr}_+^{\mathbb{R}}(\overline{S(T)}) = 6 = |S(T)| + 2.$$

If $|T| = 3$ then $T = P_3$. The graphs of $S(P_3)$ and $\overline{S(P_3)}$ are shown in Figures 10 and 11, respectively.

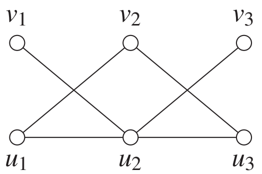


Figure 10: $S(P_3)$

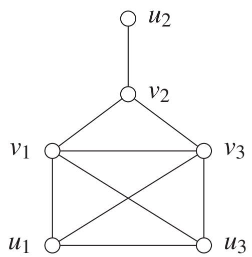


Figure 11: $\overline{S(P_3)}$

In $S(P_3)$, since v_1 and v_3 are pendant vertices, using Result 1, we have $\operatorname{mr}_+^{\mathbb{R}}(S(P_3)) = 4$. Since $\overline{S(P_3)}$ is chordal, using Result 4, we have $\operatorname{mr}_+^{\mathbb{R}}(\overline{S(P_3)}) = \operatorname{cc}(\overline{S(P_3)}) = 3$. Thus,

$$\operatorname{mr}_+^{\mathbb{R}}(S(P_3)) + \operatorname{mr}_+^{\mathbb{R}}(\overline{S(P_3)}) = 7 = |S(P_3)| + 1.$$

Suppose $|T| \geqslant 4$. First, we consider the case when T is a star with $|T| = n$. Let $V(S(T)) = \{u_1, \dots, u_{n-1}, x, v_1, \dots, v_{n-1}, x'\}$ where $u_1, \dots, u_{n-1}, x$ are vertices of T and x is the center vertex of T. For $1 \leqslant i \leqslant n-1$, v_i is the shadow vertex of u_i and x' is the shadow vertex of x. Since for $1 \leqslant i \leqslant n-1$, v_i is a vertex of degree one in $S(T)$, after deleting the vertices v_i of degree one in $S(T)$ the resulting graph is a complete bipartite graph $K_{2,n-1}$. By Result 1 and ([5], Theorem 2.1) we have

$$\operatorname{mr}_+^{\mathbb{R}}(S(T)) = \operatorname{mr}_+^{\mathbb{R}}(K_{2,n-1}) + n - 1 = (n-1) + (n-1) = 2n - 2. \tag{1}$$

In $\overline{S(T)}$, x is a pendant vertex joined to its shadow vertex x'. The subgraph induced by the set of vertices $\{u_1, \dots, u_{n-1}, v_1, \dots, v_{n-1}\}$ is $K_{n-1} \vee K_{n-1} = K_{2(n-1)}$ and the subgraph induced by the set of vertices $\{v_1, v_2, \dots, v_{n-1}, x'\}$ is K_n. Since $\overline{S(T)}$ is chordal, using Result 4 we have $\operatorname{mr}_+^{\mathbb{R}}(\overline{S(T)}) = \operatorname{cc}(\overline{S(T)}) = 3$. Therefore, from (1) we have

$$\operatorname{mr}_+^{\mathbb{R}}(S(T)) + \operatorname{mr}_+^{\mathbb{R}}(\overline{S(T)}) = (2n-2) + 3 = 2n + 1 = |S(T)| + 1.$$

Next, assume T is not a star and $|T| \geqslant 4$. By Theorem 3.,

$$\mathrm{mr}_+^{\mathbb{R}}(S(T)) \leqslant |S(T)| - 2. \tag{2}$$

By ([13], Theorem 3.16) we have $\mathrm{mr}_+^{\mathbb{R}}(\overline{T}) \leqslant 3$. Since $\overline{T}$ is connected, using Theorem 4 we get

$$\mathrm{mr}_+^{\mathbb{R}}(\overline{S(T)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{T}) + 1 \leqslant 4. \tag{3}$$

By Equations (2) and (3) we have

$$\mathrm{mr}_+^{\mathbb{R}}(S(T)) + \mathrm{mr}_+^{\mathbb{R}}(\overline{S(T)}) \leqslant (|S(T)| - 2) + 4 = |S(T)| + 2. \quad \square$$

In Theorem 8 we show that the shadow graph $S(G)$ of a unicyclic graph satisfies GCC_+. We first show that K_4 is a minor of $S(G)$ when G is unicyclic. We then use the minor monotone property of the Colin de Verdière type parameter ν of a graph G to get bounds on $\mathrm{mr}_+^{\mathbb{R}}(S(G))$ (Refer to Section 2.2).

OBSERVATION 2. For every graph G, $\nu(G) \leqslant \mathrm{M}_+^{\mathbb{R}}(G)$.

LEMMA 1. ([11], Theorem 3) *If H is a minor of G, then $\nu(H) \leqslant \nu(G)$.*

We now recall the following well known result.

LEMMA 2. *Let K_s be a complete graph on s vertices. If $s \geqslant 2$ then $\nu(K_s) = s-1$ and $\nu(K_1) = 1$.*

Proof. For $s \geqslant 2$, consider the $s \times s$ matrix J of all ones. Since J is symmetric and the eigenvalues are s (with multiplicity one) and zero (with multiplicity $s-1$), the matrix J is a psd matrix with nullity $s-1$. To satisfy the Hadamard product of J with any symmetric matrix X is the zero matrix, X is necessarily the zero matrix. Thus, J satisfies the Strong Arnold Property. So, $\nu(K_s) \geqslant s-1$. Since $s \geqslant 2$ and $\mathrm{mr}_+^{\mathbb{R}}(K_s) = 1$, we have $\nu(K_s) = s-1$. It is easy to show $\nu(K_1) = 1$. $\square$

THEOREM 7. *The shadow graph $S(G)$ of a complete graph G where $|G| \geqslant 2$, satisfies GCC_+.*

Proof. Let $V(S(G)) = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ where u_i is a vertex in G for $1 \leqslant i \leqslant n$ and v_i is the shadow vertex of u_i. Since $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ form cliques of size n in $S(G)$ and $\overline{S(G)}$ respectively, $S(G)$ and $\overline{S(G)}$ contain a complete graph K_n as an induced subgraph. By Observation 2, Lemma 1 and Lemma 2 we have $n-1 = \nu(K_n) \leqslant \nu(S(G)) \leqslant \mathrm{M}_+^{\mathbb{R}}(S(G))$. Using $\mathrm{mr}_+^{\mathbb{R}}(S(G)) = |S(G)| - \mathrm{M}_+^{\mathbb{R}}(S(G))$ we get $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant n+1$. Similarly, $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant n+1$. Therefore, $\mathrm{mr}_+^{\mathbb{R}}(S(G)) + \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant (n+1) + (n+1) = |S(G)| + 2$. $\square$

THEOREM 8. *The shadow graph $S(G)$ of a unicyclic graph G satisfies GCC_+.*

Proof. We consider four cases as following:

Case 1. Suppose G is a cycle C_3. The proof is provided in Theorem 7 as the case of the shadow graph of a complete graph.

Case 2. Suppose G is the vertex sum of C_3 and a star graph S_{n-2} where $|G| = n \geqslant 4$. Let $V(C_3) = \{u_1, u_2, u_3\}$ where u_1 is the shared vertex of the vertex sum and u_1 is the center of the star. Also, let v_1 be the corresponding shadow vertex of u_1. Assume w is a vertex of S_{n-2} where w is adjacent to u_1. Consider a partition $(V_0, V_1, V_2, V_3, V_4)$ of $V(S(G))$ where

$$V_0 = V(S(G)) \setminus \{V(C_3) \cup \{v_1, w\}\}, V_1 = \{u_1\}, V_2 = \{u_2\}, V_3 = \{u_3\}, V_4 = \{v_1, w\}.$$

Let H be the minor obtained from $S(G)$ by deleting V_0 and contracting the induced subgraph $G[V_4]$ to a single vertex. The graph H is isomorphic to K_4. Thus, $S(G)$ contains K_4 as a minor. By Observation 2, Lemma 1 and Lemma 2 we have $3 = \nu(K_4) \leqslant \nu(S(G)) \leqslant \mathrm{M}_+^{\mathbb{R}}(S(G))$. Using $|S(G)| - \mathrm{M}_+^{\mathbb{R}}(S(G)) = \mathrm{mr}_+^{\mathbb{R}}(S(G))$, we get $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - 3$. In $\overline{G}$, u_1 is an isolated vertex and the subgraph induced by the set of vertices $V(\overline{G}) \setminus \{u_1\}$ is the graph $2K_1 \vee K_{n-3}$. By ([14], Proposition 2.6) we have $\mathrm{mr}_+^{\mathbb{R}}(V(\overline{G}) \setminus \{u_1\}) = \mathrm{mr}_+^{\mathbb{R}}(2K_1 \vee K_{n-3}) = 2$. Since $\overline{G}$ contains an isolated vertex, using Theorem 5 we have $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(2K_1 \vee K_{n-3}) + 1 + 1 = 4$. Hence $\mathrm{mr}_+^{\mathbb{R}}(S(G)) + \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant |S(G)| + 1$.

Case 3. Suppose G contains a cycle C_3 and G is not the vertex sum of C_3 and S_{n-2} with $|G| = n \geqslant 5$. Let C_3 be an induced cycle of G with $V(C_3) = \{u_1, u_2, u_3\}$. Since $|G| \geqslant 5$, there exists a vertex w in G such that w is adjacent to only one vertex in C_3, say u_1, since G is unicyclic. Let v_1 be the shadow vertex of u_1. Consider the partition of $V(S(G))$ as the same partition as case 2. and then use the edge contraction v_1w. Thus, $S(G)$ will contain K_4 as a minor. As above we get $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - 3$. By ([15], Corollary 3.4) $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant 4$. Next, we claim that $\overline{G}$ does not contain any isolated vertices. Suppose x is an isolated vertex in $\overline{G}$. If $x \notin V(C_3)$, then there are at least two cycles in G formed by the set of vertices $\{u_1, u_2, u_3\}$ and $\{x, u_1, u_2\}$ which contradicts to G is a unicyclic graph. If $x \in V(C_3)$, since G is a unicyclic graph, it implies that G must be the vertex sum of C_3 and S_{n-2} which contradicts to the assumption of this case. Thus, $\overline{G}$ does not contain any isolated vertices. Since $\overline{G}$ does not contain any isolated vertices, by Remark 2 we have $\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1 \leqslant 5$. Thus, $\mathrm{mr}_+^{\mathbb{R}}(S(G)) + \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant |S(G)| + 2$.

Case 4. Suppose G contains an induced subgraph C_n where $n \geqslant 4$. Let $V(C_n) = \{u_1, \ldots, u_n\}$ and v_1, v_n be the shadow vertices of u_1, u_n, respectively. Consider a partition $(V_0, V_1, V_2, V_3, V_4)$ of $V(S(G))$ such that $V_0 = V(S(G)) \setminus [V(C_n) \cup \{v_1, v_n\}], V_1 = \{u_1, v_n\}, V_2 = \{v_1, u_n\}, V_3 = \{u_2, \ldots, u_{n-2}\}, V_4 = \{u_{n-1}\}$. Let H be the minor obtained from $S(G)$ by deleting V_0, contracting each induced subgraph $G[V_1], G[V_2]$ to a single vertex. In $G[V_3]$ for $3 \leqslant i \leqslant n-2$, use edge contractions $u_{i-1}u_i$ inductively and for each edge contraction we identify u_{i-1} and u_i and label the new vertex as u_i. The graph H is isomorphic to a complete graph K_4. Thus, $S(G)$ has K_4 as a minor. By Observation 2, Lemma 1 and Lemma 2 we have $3 = \nu(K_4) \leqslant \nu(S(G)) \leqslant \mathrm{M}_+^{\mathbb{R}}(S(G))$.

Using $|S(G)| - \mathrm{M}_{+}^{\mathbb{R}}(S(G)) = \mathrm{mr}_{+}^{\mathbb{R}}(S(G))$, we obtain

$$\mathrm{mr}_{+}^{\mathbb{R}}(S(G)) \leqslant |S(G)| - 3.$$

By ([15], Corollary 3.4) we have $\mathrm{mr}_{+}^{\mathbb{R}}(\overline{G}) \leqslant 4$. Using Remark 2 we get

$$\mathrm{mr}_{+}^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_{+}^{\mathbb{R}}(\overline{G}) + 1 \leqslant 5.$$

Therefore,

$$\mathrm{mr}_{+}^{\mathbb{R}}(S(G)) + \mathrm{mr}_{+}^{\mathbb{R}}(\overline{S(G)}) \leqslant (|S(G)| - 3) + 5 = |S(G)| + 2. \quad \square$$

LEMMA 3. *Let G be a connected graph with $|G| \geqslant 3$. Suppose G is not a complete graph and G contains a maximum clique of size m. Then, the shadow graph $S(G)$ contains a complete graph K_{m+1} as a minor.*

Proof. Let Q be a maximum clique in G with $V(Q) = \{u_1, \ldots, u_m\}$. Since G is a connected graph, there exists $w \in V(G)$ such that w is adjacent to at least one of the vertices in $V(Q)$, namely u_1. Let v be the shadow vertex of u_1 in $S(G)$. Consider a partition $(V_0, V_1, \ldots, V_{m+1})$ of $V(S(G))$ where $V_0 = V(S(G)) \setminus \{V(Q) \cup \{w, v\}\}$, $V_i = \{u_i\}$ for $1 \leqslant i \leqslant m$ and $V_{m+1} = \{w, v\}$. Let H be the minor obtained from $S(G)$ by deleting V_0 and contracting the edge in $G[V_{m+1}]$. The graph H is a complete graph K_{m+1} on $\{u_1, \ldots, u_m, w\}$ with possible multiple edges. From the definition of a minor we can replace any multiple edges by single edges. Thus, $S(G)$ contains a complete graph K_{m+1} as a minor. $\square$

DEFINITION 7. ([7], p. 167) We give a recursive description of a k-tree.

i) A clique with k vertices is a k-tree.

ii) If $T = (V, E)$ is a k-tree and Q is a clique of T with k vertices and $x \notin V$, then $T' = (V \cup \{x\}, E \cup \{cx : c \in Q\})$ is a k-tree.

Recall that the size of a maximum clique in a graph G is called the *clique number* of G, denoted by $\omega(G)$.

OBSERVATION 3. For a k-tree T, $\omega(T) = k$ if T is a complete graph and $\omega(T) = k+1$ otherwise.

THEOREM 9. *Suppose G is a k-tree such that $\overline{G}$ does not contain any isolated vertices. Then the shadow graph $S(G)$ satisfies GCC_{+}.*

Proof. By Theorem 6 the shadow graph of a 1-tree satisfies GCC_{+}. Suppose G is a k-tree with $k \geqslant 2$. By Observation 3, every maximum clique in G has size $\omega(G) = k+1$. By Lemma 3, the shadow graph $S(G)$ contains a $K_{\omega(G)+1} = K_{k+2}$ as a minor. From Observation 2, Lemma 1 and Lemma 2 we have

$$k + 1 = \nu(K_{k+2}) \leqslant \nu(S(G)) \leqslant \mathrm{M}_{+}^{\mathbb{R}}(S(G)).$$

Using $|S(G)| - \mathrm{M}_+^{\mathbb{R}}(S(G)) = \mathrm{mr}_+^{\mathbb{R}}(S(G))$, we get

$$\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - k - 1.$$

By ([21], Corollary 3) we have $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant k+2$. As there are no isolated vertices in $\overline{G}$ by assumption, using Remark 2 we get

$$\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1 \leqslant k+3.$$

Therefore,

$$\mathrm{mr}_+^{\mathbb{R}}(S(G)) + \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant (|S(G)| - k - 1) + (k+3) = |S(G)| + 2. \quad \square$$

DEFINITION 8. ([12], p. 103) A graph is a *partial k-tree* if it is a subgraph of a k-tree.

THEOREM 10. *Let G be a partial k-tree with $k \geqslant 2$. If G has a complete subgraph K_{k+1} and $\overline{G}$ does not contain any isolated vertices, then the shadow graph $S(G)$ satisfies GCC_+.*

Proof. By Lemma 3 the shadow graph $S(G)$ contains a complete graph K_{k+2} as a minor. Thus, $k+1 = \nu(K_{k+2}) \leqslant \nu(S(G)) \leqslant \mathrm{M}_+^{\mathbb{R}}(S(G))$. Using $|S(G)| - \mathrm{M}_+^{\mathbb{R}}(S(G)) = \mathrm{mr}_+^{\mathbb{R}}(S(G))$, we have

$$\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - k - 1.$$

By ([21], Theorem 5) we have $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant k+2$. As there are no isolated vertices in $\overline{G}$ by assumption, using Remark 2 we get

$$\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) + 1 \leqslant k+3.$$

Therefore,

$$\mathrm{mr}_+^{\mathbb{R}}(S(G)) + \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant (|S(G)| - k - 1) + (k+3) = |S(G)| + 2. \quad \square$$

THEOREM 11. *Suppose G is a chordal graph such that $\overline{G}$ does not contain any isolated vertices. Then the shadow graph $S(G)$ satisfies GCC_+.*

Proof. Since G is not a complete graph, by Lemma 3 we have $S(G)$ contains a complete graph $K_{\omega(G)+1}$ as a minor. Thus, we have

$$\omega(G) = \nu(K_{\omega(G)+1}) \leqslant \nu(S(G)) \leqslant \mathrm{M}_+^{\mathbb{R}}(S(G)).$$

Using $|S(G)| - \mathrm{M}_+^{\mathbb{R}}(S(G)) = \mathrm{mr}_+^{\mathbb{R}}(S(G))$, we obtain

$$\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - \omega(G).$$

By ([19], Proposition 6) we have $\nu(\overline{G}) \geqslant |G| - \omega(G) - 1$. Therefore, we have $|G| - \omega(G) - 1 \leqslant \nu(\overline{G}) \leqslant \mathrm{M}_+^{\mathbb{R}}(\overline{G})$. Using $|\overline{G}| - \mathrm{M}_+^{\mathbb{R}}(\overline{G}) = \mathrm{mr}_+^{\mathbb{R}}(\overline{G})$, we get $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant$

$\omega(G)+1$. As there are no isolated vertices in $\overline{G}$ by assumption, using Remark 2 we obtain

$$\mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G})+1 \leqslant (\omega(G)+1)+1 = \omega(G)+2.$$

Therefore,

$$\mathrm{mr}_+^{\mathbb{R}}(S(G)) + \mathrm{mr}_+^{\mathbb{R}}(\overline{S(G)}) \leqslant (|S(G)|-\omega(G)) + (\omega(G)+2) = |S(G)|+2. \quad \square$$

5. Shadow graph $\mathrm{Shad}(G)$ and GCC_+

A different definition of a shadow graph, denoted $\mathrm{Shad}(G)$, appears in Chartrand, Lesniak, and Zhang's book [10]. We show that if G satisfies GCC_+ and $\overline{G}$ does not contain any isolated vertices, then $\mathrm{Shad}(G)$ satisfies GCC_+.

DEFINITION 9. ([10], p. 412) Let G be a graph with $V(G) = \{u_1, u_2, \dots, u_n\}$. The *shadow graph* denoted $\mathrm{Shad}(G)$ is that graph with vertex set $V(G) \cup \{v_1, v_2, \dots, v_n\}$, where v_i is called the *shadow vertex* of u_i and where v_i is adjacent to both v_j and u_j if u_i is adjacent to u_j for $1 \leqslant i, j \leqslant n$.

EXAMPLE 5. The following are $\mathrm{Shad}(G)$ where G is the path P_5 and the cycle C_4. The shadow vertices are represented as black vertices.

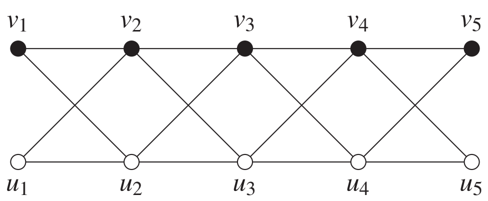


Figure 12: $\mathrm{Shad}(P_5)$

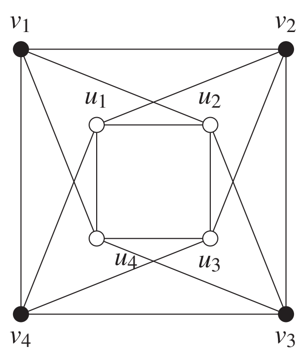


Figure 13: $\mathrm{Shad}(C_4)$

REMARK 3. By the definition of $\mathrm{Shad}(G)$, it can be obtained by taking two copies of G, say G_1 and G_2 and joining each vertex u_i in G_1 to the vertex v_j in G_2 if and only if the corresponding vertex v_i in G_2 is adjacent to v_j.

PROPOSITION 4. *Let G be a connected graph. Then* $\mathrm{mr}_+^{\mathbb{R}}(\mathrm{Shad}(G)) \leqslant \mathrm{mr}_+^{\mathbb{R}}(G) + |G|$.

Proof. Let $V(\mathrm{Shad}(G)) = \{u_1,\dots,u_n,v_1,\dots,v_n\}$ where v_i is the shadow vertex of u_i for $1 \leqslant i \leqslant n$. Note that $\{u_1,\dots,u_n\}$ and $\{v_1,\dots,v_n\}$ are sets of vertices of two copies of G. Let $A \in \mathcal{S}_+(G,\mathbb{R})$ with $\mathrm{rank}(A) = \mathrm{mr}_+^{\mathbb{R}}(G)$. Denote $A = [a_{ij}]$ and $D = \mathrm{diag}(a_{ii})$ for $1 \leqslant i \leqslant n$. Define a 2×2 block matrix

$$M = \begin{bmatrix} A+D^2 & A-D \\ A-D & A+I \end{bmatrix}$$

where M_{11} and M_{22} correspond to the adjacency of the vertices in G. First, we claim that M is a matrix for $\mathrm{Shad}(G)$. Since A is a matrix for G with positive diagonal entries, adding a diagonal matrix with positive diagonal entries will not affect the adjacency between u_i and u_j and we have that the diagonal entries of the resulting matrix are still positive. The entries in M_{12} and M_{21} correspond to the adjacency between u_i and v_j. Note that u_i is adjacent to v_j if and only if u_i is adjacent to u_j for $i \neq j$. Since v_i is not adjacent to u_i, the diagonal entries of M_{12} and M_{21} must be zero. We have that the diagonal entries of $A-D = M_{12} = M_{21}$ are zero. Since $M_{11}^T = (A+D^2)^T = A+D^2 = M_{11}$ and $M_{22}^T = (A+I)^T = A+I = M_{22}$ and $M_{12}^T = M_{21} = A-D$, we have M is symmetric. Since A is psd, $A = B^TB$ for some matrix B. Therefore,

$$\begin{aligned}\begin{bmatrix} B^T & -D \\ B^T & I \end{bmatrix}\begin{bmatrix} B & B \\ -D & I \end{bmatrix} &= \begin{bmatrix} B^TB+D^2 & B^TB-D \\ B^TB-D & B^TB+I \end{bmatrix} \\ &= \begin{bmatrix} A+D^2 & A-D \\ A-D & A+I \end{bmatrix} \\ &= M.\end{aligned}$$

Thus, M is psd. Moreover, we have

$$\begin{aligned}\mathrm{rank}(M) = \mathrm{rank}\begin{bmatrix} A+D^2 & A-D \\ A-D & A+I \end{bmatrix} &\leqslant \mathrm{rank}\begin{bmatrix} A & A \\ A & A \end{bmatrix} + \mathrm{rank}\begin{bmatrix} D^2 & -D \\ -D & I \end{bmatrix} \\ &= \mathrm{mr}_+^{\mathbb{R}}(G) + |G|. \quad \square\end{aligned}$$

PROPOSITION 5. *Let G be a simple connected graph such that $\overline{G}$ is disconnected. If $G_1,G_2,\dots,G_k$ are the connected components of $\overline{G}$ with each component having two or more vertices and if there are r isolated vertices in $\overline{G}$, then*

$$\mathrm{mr}_+^{\mathbb{R}}(\overline{\mathrm{Shad}(G)}) \leqslant \left(\sum_{i=1}^{k} \mathrm{mr}_+^{\mathbb{R}}(G_i)\right) + r.$$

Proof. Denote $V(\overline{\mathrm{Shad}(G)}) = \{u_1,\dots,u_n,v_1,\dots,v_n\}$ where v_i is the shadow vertex of u_i for $1 \leqslant i \leqslant n$. Since $\overline{G}$ is an induced subgraph of $\overline{\mathrm{Shad}(G)}$, we have $\mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{\mathrm{Shad}(G)})$. Next, we claim that $\mathrm{mr}_+^{\mathbb{R}}(\overline{\mathrm{Shad}(G)}) \leqslant \left(\sum_{i=1}^{k} \mathrm{mr}_+^{\mathbb{R}}(G_i)\right) + r$. Let $A_i \in \mathcal{S}_+(G_i,\mathbb{R})$ with $\mathrm{rank}(A_i) = \mathrm{mr}_+^{\mathbb{R}}(G_i)$. We define $A = [a_{ij}] = \left(\bigoplus_{i=1}^{k} A_i\right) \bigoplus I_r$

where I_r is the $r \times r$ identity matrix. Then we define a 2×2 block matrix

$$M = \begin{bmatrix} A & A \\ A & A \end{bmatrix}$$

where $M_{1,1}$ and $M_{2,2}$ correspond to the adjacency of the vertices $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$, respectively and $M_{1,2}$ corresponds to the adjacency between u_i and v_j. Note that A is symmetric and all the diagonal entries of A are positive. It can be verified that M is a matrix representation of $\overline{\mathrm{Shad}(G)}$. Since A is symmetric, we have M is also symmetric. Moreover, we have M is psd since A is psd. Now $\mathrm{rank}(M) = \mathrm{rank}(A) = \left(\sum_{i=1}^{k} \mathrm{mr}_+^{\mathbb{R}}(G_i)\right) + r$. Therefore, $\mathrm{mr}_+^{\mathbb{R}}(\overline{\mathrm{Shad}(G)}) \leqslant \left(\sum_{i=1}^{k} \mathrm{mr}_+^{\mathbb{R}}(G_i)\right) + r$. □

REMARK 4. If $\overline{G}$ does not contain any isolated vertices, by Proposition 5 we have

$$\mathrm{mr}_+^{\mathbb{R}}(\overline{\mathrm{Shad}(G)}) \leqslant \sum_{i=1}^{k} \mathrm{mr}_+^{\mathbb{R}}(G_i) = \mathrm{mr}_+^{\mathbb{R}}(\overline{G}).$$

THEOREM 12. *Let G be a simple connected graph and $\overline{G}$ does not contain any isolated vertices. If G satisfies GCC_+, then* $\mathrm{Shad}(G)$ *satisfies GCC_+.*

Proof. By Proposition 4 and Remark 4 we have $\mathrm{mr}_+^{\mathbb{R}}(\mathrm{Shad}(G)) \leqslant \mathrm{mr}_+^{\mathbb{R}}(G) + |G|$ and $\mathrm{mr}_+^{\mathbb{R}}(\overline{\mathrm{Shad}(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(\overline{G})$. Therefore, $\mathrm{mr}_+^{\mathbb{R}}(\mathrm{Shad}(G)) + \mathrm{mr}_+^{\mathbb{R}}(\overline{\mathrm{Shad}(G)}) \leqslant \mathrm{mr}_+^{\mathbb{R}}(G) + |G| + \mathrm{mr}_+^{\mathbb{R}}(\overline{G})$. When G satisfies GCC_+ we get $\mathrm{mr}_+^{\mathbb{R}}(G) + \mathrm{mr}_+^{\mathbb{R}}(\overline{G}) \leqslant |G| + 2$. Hence

$$\mathrm{mr}_+^{\mathbb{R}}(\mathrm{Shad}(G)) + \mathrm{mr}_+^{\mathbb{R}}(\overline{\mathrm{Shad}(G)}) \leqslant 2|G| + 2 = |\mathrm{Shad}(G)| + 2. \quad \square$$

It has been shown in ([13], [15], [19], [2], [21]) respectively that a tree, a unicyclic graph, a chordal graph, a graph G with $\delta(G) \geqslant |G| - 3$, a partial 3-tree, and a k-connected partial k-tree satisfy GCC_+.

COROLLARY 2. *Suppose G is a tree, a unicyclic graph, a chordal graph, a graph G with $\delta(G) \geqslant |G| - 3$, a partial 3-tree, and a k-connected partial k-tree such that $\overline{G}$ does not contain any isolated vertices. Then* $\mathrm{Shad}(G)$ *satisfies GCC_+.*

6. Shadow graph $S(G)$ and the delta conjecture

In this section we will prove that the shadow graphs $S(G)$ when G are trees, unicyclic graphs, k-trees, partial k-trees and chordal graphs satisfy the delta conjecture.

CONJECTURE 1. [8] *For a connected graph G, $\mathrm{mr}_+^{\mathbb{R}}(G) \leqslant |G| - \delta(G)$, where $\delta(G)$ is the minimum degree of the vertices in G.*

LEMMA 4. *Let G be a connected graph and $S(G)$ be the shadow graph of G. Then $\delta(S(G)) = \delta(G)$.*

Proof. Let $V(S(G)) = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ where $|G| = n$ and v_i is the shadow vertex of u_i for $1 \leqslant i \leqslant n$. Note that for each j, $N_{S(G)}(v_j) = N_G(u_j)$ so that $d_{S(G)}(v_j) = d_G(u_j)$. Moreover, $d_G(u_i) \leqslant d_{S(G)}(u_i)$ for $1 \leqslant i \leqslant n$. Therefore,

$$\begin{aligned}\delta(S(G)) &= \min\{d_{S(G)}(u_i), d_{S(G)}(v_j), 1 \leqslant i, j \leqslant n\} \\ &= \min\{d_G(u_j), 1 \leqslant j \leqslant n\} \\ &= \delta(G). \quad \square\end{aligned}$$

Clearly, if G is a connected graph with $\delta(G) = 1$ or $\delta(G) = 2$, then $S(G)$ satisfies the delta conjecture by Theorem 3. That is, the shadow graph $S(G)$ of a tree and the shadow graph $S(G)$ of a unicyclic graph satisfy the delta conjecture.

PROPOSITION 6. *Let G be a complete graph where $|G| \geqslant 2$. The shadow graph $S(G)$ satisfies the delta conjecture.*

Proof. In the proof of Theorem 7 we have $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |G| + 1$. Thus, $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |G| + 1 = 2|G| - (|G| - 1) = 2|G| - \delta(G) = |S(G)| - \delta(S(G))$. $\square$

THEOREM 13. *Let G be a k-tree where $k \geqslant 2$. Then the shadow graph $S(G)$ satisfies the delta conjecture.*

Proof. If G is a complete graph, then by Proposition 6 the shadow graph $S(G)$ satisfies the delta conjecture. Assume G is not a complete graph. Since G is a k-tree, $\delta(G) = k$. In the proof of Theorem 9 we have $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - k - 1$. Therefore,

$$\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - k - 1 < |S(G)| - k = |S(G)| - \delta(G) = |S(G)| - \delta(S(G)). \quad \square$$

THEOREM 14. *Let G be a partial k-tree where $k \geqslant 2$. If, in addition, G has a complete subgraph K_{k+1}, then the shadow graph $S(G)$ satisfies the delta conjecture.*

Proof. Since a partial k-tree is a subgraph of a k-tree, we have $\delta(G) \leqslant k$. In the proof of Theorem 10. we have $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - k - 1$. Therefore,

$$\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - k - 1 < |S(G)| - k \leqslant |S(G)| - \delta(G) = |S(G)| - \delta(S(G)). \quad \square$$

THEOREM 15. *Let G be a chordal graph. Then the shadow graph $S(G)$ satisfies the delta conjecture.*

Proof. If G is a complete graph, then by Proposition 6 the shadow graph $S(G)$ satisfies the delta conjecture. Assume G is not a complete graph. Let $\omega(G)$ be the size of a largest clique in G. Since G is chordal, G has a simplicial vertex ([23], p. 290), say v. Since the closed neighborhood $N[v]$ forms a clique in G, we have $|N[v]| \leqslant \omega(G)$. Thus, $\delta(G) \leqslant d_G(v) \leqslant \omega(G) - 1$. In the proof of Theorem 11. we have $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - \omega(G)$. Therefore, we get $\mathrm{mr}_+^{\mathbb{R}}(S(G)) \leqslant |S(G)| - \omega(G) \leqslant |S(G)| - \delta(G) - 1 < |S(G)| - \delta(G) = |S(G)| - \delta(S(G))$. $\square$

Acknowledgement. The first author thanks Central Michigan University for its support.

REFERENCES

[1] FRANCESCO BARIOLI, WAYNE BARRETT, SHAUN M. FALLAT, H. TRACY HALL, LESLIE HOGBEN, BRYAN SHADER, P. VAN DEN DRIESSCHE AND HEIN VAN DER HOLST, *Zero forcing parameters and minimum rank problems*, Linear Algebra Appl., 433 (2): 401–411, 2010.

[2] FRANCESCO BARIOLI, WAYNE BARRETT, SHAUN M. FALLAT, H. TRACY HALL, LESLIE HOGBEN AND HEIN VAN DER HOLST, *On the graph complement conjecture for minimum rank*, Linear Algebra Appl., 436 (12): 4373–4391, 2012.

[3] FRANCESCO BARIOLI, SHAUN M. FALLAT, LON H. MITCHELL AND SIVARAM K. NARAYAN, *Minimum semidefinite rank of outerplanar graphs and the tree cover number*, Electron. J. Linear Algebra, 22: 10–21, 2011.

[4] J. A. BONDY AND U. S. R. MURTY, *Graph theory*, vol. 244 of Graduate Texts in Mathematics, Springer, New York, 2008.

[5] MATTHEW BOOTH, PHILIP HACKNEY, BENJAMIN HARRIS, CHARLES R. JOHNSON, MARGARET LAY, TERRY D. LENKER, LON H. MITCHELL, SIVARAM K. NARAYAN, AMANDA PASCOE AND BRIAN D. SUTTON, *On the minimum semidefinite rank of a simple graph*, Linear Multilinear Algebra, 59 (5): 483–506, 2011.

[6] MATTHEW BOOTH, PHILIP HACKNEY, BENJAMIN HARRIS, CHARLES R. JOHNSON, MARGARET LAY, LON H. MITCHELL, SIVARAM K. NARAYAN, AMANDA PASCOE, KELLY STEINMETZ, BRIAN D. SUTTON AND WENDY WANG, *On the minimum rank among positive semidefinite matrices with a given graph*, SIAM J. Matrix Anal. Appl., 30 (2): 731–740, 2008.

[7] ANDREAS BRANDSTÄDT, VAN BANG LE AND JEREMY P. SPINRAD, *Graph classes: a survey*, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[8] RICHARD BRUALDI, LESLIE HOGBEN AND BRYAN SHADER, *AIM Workshop Spectra of Families of Matrices described by Graphs, Digraphs, and Sign Patterns Final Report: Mathematical Results* (revised), `https://aimath.org/pastworkshops/matrixspectrum.html.07 2018`

[9] G. CHARTRAND AND P. ZHANG, *A First Course in Graph Theory*, Dover books on Mathematics, Dover Publications, 2012.

[10] GARY CHARTRAND, LINDA LESNIAK AND PING ZHANG, *Graphs & Digraphs*, Chapman & Hall/CRC, 6th edition, 2015.

[11] YVES COLIN DE VERDIÈRE, *Multiplicities of eigenvalues and tree-width of graphs*, J. Combin. Theory Ser. B, 74 (2): 121–146, 1998.

[12] JONATHAN L. GROSS, JAY YELLEN AND PING ZHANG, *Handbook of graph theory*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, second edition, 2014.

[13] *AIM Minimum Rank-Special Graphs Work Group*, *Zero forcing sets and the minimum rank of graphs*, Linear Algebra Appl., 428 (7): 1628–1648, 2008.

[14] PHILIP HACKNEY, BENJAMIN HARRIS, MARGARET LAY, LON H. MITCHELL, SIVARAM K. NARAYAN AND AMANDA PASCOE, *Linearly independent vertices and minimum semidefinite rank*, Linear Algebra Appl., 431 (8): 1105–1115, 2009.

[15] LESLIE HOGBEN, *Orthogonal representations, minimum rank, and graph complements*, Linear Algebra Appl., 428 (11–12): 2560–2568, 2008.

[16] LESLIE HOGBEN, *Handbook of linear algebra*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, second edition, 2014.

[17] ROGER A. HORN AND CHARLES R. JOHNSON, *Matrix analysis, 2nd edition*, Cambridge University Press, Cambridge, 2013.

[18] RUSSELL MERRIS, *A survey of graph Laplacians*, Linear and Multilinear Algebra, 39 (1-2): 19–31, 1995.

[19] LON H. MITCHELL, *On the graph complement conjecture for minimum semidefinite rank*, Linear Algebra Appl., 435 (6): 1311–1314, 2011.

[20] LON H. MITCHELL, SIVARAM K. NARAYAN AND ANDREW M. ZIMMER, *Lower bounds for minimum semidefinite rank from orthogonal removal and chordal supergraphs*, Linear Algebra Appl., 436 (3): 525–536, 2012.

[21] JOHN SINKOVIC AND HEIN VAN DER HOLST, *The minimum semidefinite rank of the complement of partial k-trees*, Linear Algebra Appl., 434 (6): 1468–1474, 2011.

[22] HEIN VAN DER HOLST, *Graphs whose positive semi-definite matrices have nullity at most two*, Linear Algebra Appl., 375: 1–11, 2003.

[23] DOUGLAS B. WEST, *Introduction to graph theory, 2nd edition*, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.

(Received February 8, 2020)

Monsikarn Jansrang
Mahidol Wittayanusorn School
Salaya Nakhon Pathom, Thailand, 73170
e-mail: monsikarn.jan@mwit.ac.th

Sivaram K. Narayan
Central Michigan University
Mount Pleasant, Michigan USA, 48859
e-mail: naray1sk@cmich.edu