Operators and Matrices
Volume 15, Number 2 (2021), 615–626
doi:10.7153/oam-2021-15-41

ANALYTIC EXTENSION OF n–NORMAL OPERATORS

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(*Communicated by G. Misra*)

Abstract. Normal operators and n-normal operators played a pivotal role in the development of operator theory. In order to generalize these classes of operators, we introduce new classes of operators which we call analytic extension of n- normal operator and F-quasi-n-normal operator. We show that every analytic extension of n-normal operator and F-quasi-n-normal operator have scalar extensions. We also show that an analytic extension of n-normal operator has a nontrivial invariant subspace. Some spectral properties are also presented.

1. Introduction

Let $B(\mathscr{H})$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space $\mathscr{H}$. Throughout this paper $R(T)$, $N(T)$, $\sigma(T)$ denotes range, null space and spectrum of $T \in B(\mathscr{H})$ respectively. An operator $T \in B(\mathscr{H})$ is said to be analytic if there exists a nonconstant analytic function F on a neighborhood of $\sigma(T)$ such that $F(T) = 0$. An operator $T \in B(\mathscr{H})$ is said to be algebraic if there is a nonconstant polynomial p such that $p(T) = 0$. Recall that an operator $T \in B(\mathscr{H})$ is said to be *normal* if $T^*T = TT^*$. In [1] , S. A. Alzuraiqi and A. B. Patel introduced n-normal operators.

DEFINITION 1.1. An operator $T \in B(H)$ is said to be n-normal if

$$T^*T^n = T^nT^* \tag{1.1}$$

for some $n \in \mathbb{N}$.

This definition seems natural. S. A. Alzraiqi and A. B. Patel proved characterizations of 2-normal, 3-normal and n-normal operators on $\mathbb{C}^2$. Also, they made several examples of n-normal operators and proved that T is n-normal if and only if T^n is normal. Also, they proved that if T is 2-normal with the following condition

$$\sigma(T) \cap (-\sigma(T)) = \emptyset; \tag{1.2}$$

Mathematics subject classification (2020): Primary 47A11; Secondary 47A15, 47B20.
Keywords and phrases: n-power normal operator, subscalar operator, invariant subspace, hyperinvariant subspace.

Paper OaM-15-41

then T is subscalar. Recently, the authors in [5] have studied spectral properties of an n-normal operator T satisfying the following condition (1.2).

$$\sigma(T) \cap (-\sigma(T)) \subset \{0\}. \tag{1.3}$$

It is a little weaker assumption than this condition (1.2). Recently the authors in [6], studied several properties of n-normal. In particular, they proved that if T is 2-normal with (1.3), then T is polaroid. They also studied subscalarity of n-normal operators under certain conditions.

In order to generalize the classes of quasi-n-normal and k-quasi-n-normal operators, we introduce the class of F-quasi-n-normal operators as follows:

DEFINITION 1.2. An operator $T \in B(\mathcal{H})$ is said to be F-quasi-n-normal if $F(T)^*(T^nT^* - T^*T^n)F(T) = 0$ for some nonconstant analytic function F on some neighborhood of $\sigma(T)$, and p-quasi-n-normal if there exists a nonconstant polynomial p such that $p(T)^*(T^nT^* - T^*T^n)p(T) = 0$. In particular, if $p(z) = z^k$ for some positive integer k or $p(z) = z$, then T is said to be k-quasi-n-normal operator or quasi-n-normal operator, respectively.

If $T \in B(\mathcal{H})$ is analytic, then $F(T) = 0$ for some nonconstant analytic function F on a bounded neighborhood U of spectrum of T. Since F cannot have infinitely many zeros in U, we write $F(z) = G(z)p(z)$, where the function G is analytic and does not vanish on U and p is a nonconstant polynomial with zeros in U. By Riesz-Dunford functional calculus, $G(T)$ is invertible and the invertibility of $G(T)$ induces that $p(T) = 0$, which means that T is algebraic (See [4]). We say that T is analytic with order n when p has degree n.

In order to generalize the class of n-normal operators, we introduce analytic extensions of n-normal operators as follows:

DEFINITION 1.3. An operator $T \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is said to be an analytic extension of n-normal operator if $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where T_1 is an n-normal operator and T_3 is analytic of order n, where n is a positive integer. This means that $T \in B(\mathcal{H})$ is said to be an analytic extension of an n-normal operator if there exists an invariant subspace $\mathcal{M}$ such that $T|_{\mathcal{M}}$ is n-normal and $T^*|_{\mathcal{M}^\perp}$ is algebraic.

Let $0 \leqslant m \leqslant \infty$. Recall that an operator $T \in B(\mathcal{H})$ is said to be a scalar operator of order m if there exists a continuous unital moromorphism of topological algebra $\Phi : C_0^m(\mathbb{C}) \to B(\mathcal{H})$ such that $\Phi(z) = T$, where z stands for the identity function on C_0^m, the space of all compactly supported functions continuously differentiable of order m. An operator T is said to be *subscalar* of order m if T is similar to the restriction of a scalar operator of order m to an invariant subspace ([14]). M. Putinar [17] proved subscalarity for hyponormal operators.

Let $H^\infty(U)$ denote the space of all bounded analytic functions on a bounded open set U in $\mathbb{C}$. A subset σ of $\mathbb{C}$ is dominating in U if $\|f\| = \sup_{x \in \sigma \cap U} |f(x)|$ holds for each function $f \in H^\infty(U)$. Recall [3], a subset σ is thick if there is a bounded open set

U in $\mathbb{C}$ such that σ is dominating in U. In [3], S. Brown [3] proved if T is hyponormal operator with thick spectra then T has non trivial invariant subspace. Eschmeier [7] showed that a Banach space operator T has a nontrivial invariant subspace if T has the property (β) with thick spectra.

In this paper we prove that analytic extension of n-normal operators are subscalar without any additional condition and we present several properties of these classes of operators. We also show that an analytic extension of n-normal operator has a nontrivial invariant subspace. Some spectral properties of such operators are also presented.

2. Preliminaries

Let $\mathbb{C}$ denote the set of complex numbers and let D be a bounded open disk in $\mathbb{C}$. We denote by $L^2(D,\mathscr{H})$ the Hilbert space of measurable functions $f : D \to \mathscr{H}$ such that

$$||f||_{2,D} = \left(\int_D ||f(z)||^2 d\mu(z) \right)^{\frac{1}{2}} < \infty,$$

where $d\mu(z)$ be the planar Lebesgue measure.

The Bergman space for D, denoted by $A^2(D,\mathscr{H})$, is a subspace of $L^2(D,\mathscr{H})$ in which each function is analytic in D (ie., $\frac{\partial f}{\partial \bar{z}} = 0$). Let $\mathscr{O}(D,\mathscr{H})$ be the Fréchet space of $\mathscr{H}$-valued analytic functions on D with respect to uniform topology. Note that

$$A^2(D,H) = L^2(D,\mathscr{H}) \cap \mathscr{O}(D,\mathscr{H})$$

is a Hilbert space. The following function space $W^m(D,\mathscr{H})$ is a Sobolev type space with respect to $\bar{\partial}$ and of order m

$$W^m(D,\mathscr{H}) = \{f \in L^2(D,\mathscr{H}) : \bar{\partial}^i f \in L^2(D,\mathscr{H}), \text{for } i = 1,2,\ldots,m\}.$$

Note that $W^2(D,\mathscr{H})$ is a Hilbert space with respect to the norm

$$||f||^2_{W^m} = \sum_{i=0}^{m} ||\bar{\partial}^i f||^2_{2,D},$$

$W^m(D,\mathscr{H})$ becomes a Hilbert space contained continuously in $L^2(D,\mathscr{H})$. A bounded linear operator S on $\mathscr{H}$ is called scalar of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbb{C}) \to B(\mathscr{H})$$

such that $\Phi(z) = S$, where z is the identity function on $\mathbb{C}$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace. Let U be a (connected) bounded open subset of $\mathbb{C}$, and let m be a nonnegative integer. The linear operator M_f of multiplication by f on $W^m(U,\mathscr{H})$ is continuous, has a spectral distribution of order m, and is defined by the functional calculus

$$\Phi_M : C_0^m(\mathbb{C}) \to B(W^m(U,\mathscr{H})), \Phi_M(f) = M_f.$$

Therefore, M_f is a scalar operator of order m. Let

$$V : W^m(U,\mathscr{H}) \to \oplus_0^\infty L^2(U,\mathscr{H})$$

be the operator defined by $V(f) = (f,\overline{\partial}f,\ldots,\overline{\partial}^m f), f \in W^m(U,\mathscr{H})$. Then V is an isometry such that $VM_z = (\oplus_0^m M_z)V$. Therefore, M_z is a subnormal operator.

An operator $T \in B(\mathscr{H})$ is said to have the *single-valued extension property (SVEP)* if for every open subset $\mathscr{U}$ of $\mathbb{C}$ and any analytic function $f : \mathscr{U} \to \mathscr{H}$ such that $(T-z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on $\mathscr{U}$. A Hilbert space operator $T \in B(\mathscr{H})$ is said to satisfy *Bishop's property* (β) if, for every open subset $\mathscr{U}$ of $\mathbb{C}$ and every sequence $f_n : \mathscr{U} \longrightarrow \mathscr{H}$ of analytic functions with $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of $\mathscr{U}$, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of $\mathscr{U}$.

For $T \in B(\mathscr{H})$ and $x \in \mathscr{H}$, the *local resolvent set* of T at x, $\rho_T(x)$, is the set of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(\lambda)$ defined in a neighborhood of z_0, with values in $\mathscr{H}$, which verifies $(T-\lambda)f(\lambda) \equiv x$. The set $\sigma_T(x)$, the compliment of $\rho_T(x)$ is called the *local spectrum* of T at x. The local *spectral subspace* of T denoted by $\mathscr{H}_T(G)$ is the set

$$\mathscr{H}_T(G) = \{x \in \mathscr{H} : \sigma_T(x) \subset G\}$$

for each subset G of $\mathbb{C}$.

If $T \in B(H)$ satisfies Bishop's property (β), then T has the (SVEP). For more details see [13, 15, 16].

3. Subscalarity

In this section we prove that an analytic extension of n-normal operator is subscalar of order $2n+2$. We begin with the following useful lemma due to Putinar [17].

LEMMA 3.1. (See [17, Proposition 2.1]) *For a bounded open disk D in the complex plane $\mathbb{C}$, there is a constant C_D such that for an arbitrary operator $T \in B(\mathscr{H})$ and $f \in W^2(D,\mathscr{H})$ we have*

$$||(I-P)f||_{2,D} \leqslant C_D(||(T-z)\overline{\partial}f||_{2,D} + ||(T-z)\overline{\partial}^2 f||_{2,D}),$$

where P denote the orthogonal projection of $L^2(D,\mathscr{H})$ on to the Bergman space $A^2(D,\mathscr{H})$

Now we prove the following lemma which will be used for the sequel.

LEMMA 3.2. *Let $T \in B(H)$ be n-normal operator, then T has the Bishop's property (β).*

Proof. Let $T \in B(H)$ be n-normal. It is easy to see that $T^n(T^*)^n = (T^*)^nT^n$ for some $n \in \mathbb{N}$. Hence T^n is normal. Now, since T^n is normal, by applying [13] T has the Bishop's property (β). $\square$

LEMMA 3.3. *Let $T \in B(\mathscr{H})$ be an n-normal operator and let $\{f_j\}$ be a sequence in $W^m(D,\mathscr{H})$ $(m \geqslant 2)$ such that*

$$\lim_{j\to\infty} ||(T-z)\bar{\partial}^i f_j||_{2,D} = 0$$

for $i = 1,2,\ldots,m$, where D is a bounded disc in $\mathbb{C}$. Then,

$$\lim_{j\to\infty} ||\bar{\partial}^i f_j||_{2,D_0} = 0$$

for $i = 1,2,\ldots,m-2$, where $D_0 \subsetneq D$.

Proof. Let $T \in B(\mathscr{H})$ be an n-normal operator. From Lemma 3.1, there exists a constant C_D such that

$$||(I-P)f||_{2,D} \leqslant C_D(||(T-z)\bar{\partial} f||_{2,D} + ||(T-z)\bar{\partial}^2 f||_{2,D}) \tag{3.1}$$

for $i = 1,2,\ldots .m$. From (3.1), we have

$$\lim_{j\to\infty} ||(I-P)\bar{\partial}^i f_j||_{2,D} = 0 \tag{3.2}$$

for $i = 1,2,\ldots,m-2$. Thus we have

$$\lim_{j\to\infty} ||(T-z)P\bar{\partial}^i f_j||_{2,D} = 0 \tag{3.3}$$

holds for $i = 1,2,\ldots,m-2$. From Lemma 3.2, n-normal operator satisfies Bishop's property(β) and hence by (3.3), we have

$$\lim_{j\to\infty} ||P\bar{\partial}^i f_j||_{2,D_0} = 0 \tag{3.4}$$

for $i = 1,2,\ldots,m-2$, where $D_0 \subsetneq D$.

From (3.2) and (3.4) , we get

$$\lim_{j\to\infty} ||\bar{\partial}^i f_j||_{2,D_0} = 0$$

for $i = 1,2,\ldots,m-2$. $\square$

LEMMA 3.4. *Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(\mathscr{H}_1 \oplus \mathscr{H}_2)$, where T_1 is an n-normal operator and T_3 is analytic with order n. For any bounded disc D which contains $\sigma(T)$, define the map $A : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \frac{W^{2n+2}(D,\mathscr{H}_1)\oplus W^{2n+2}(D,\mathscr{H}_2)}{(T-z)W^{2(n+1)}(D,\mathscr{H}_1)\oplus W^{2n+2}(D,\mathscr{H}_2)}$ by*

$$Ax = \widetilde{1\otimes x}(\equiv 1\otimes x + \overline{(T-z)W^{2n+2}(D,\mathscr{H}_1)\oplus W^{2n+2}(D,\mathscr{H}_2)}),$$

where $1\otimes x$ denotes the constant function sending any $z \in D$ to $x \in \mathscr{H}_1 \oplus \mathscr{H}_2$. Then, A is injective with closed range.

Proof. Let $f_j = f_{j,1} \oplus f_{j,2} \in W^{2n+2}(D,\mathscr{H}_1) \oplus W^{2n+2}(D,\mathscr{H}_2)$ and let $x_j = x_{j,1} \oplus x_{j,2} \in \mathscr{H}_1 \oplus \mathscr{H}_2$ be sequences such that

$$\lim_{j\to\infty} ||(T-z)f_j + 1\otimes x_j||_{W^{2n+2}(D,\mathscr{H}_1)\oplus W^{2n+2}(D,\mathscr{H}_2))} = 0. \tag{3.5}$$

From (3.5), we write

$$\lim_{j\to\infty} ||(T_1-z)f_{j,1} + T_2 f_{j,2} + 1\otimes x_{j,1}||_{W^{2n+2}} = 0, \tag{3.6}$$

$$\lim_{j\to\infty} ||(T_3-z)f_{j,2} + 1\otimes x_{j,2}||_{W^{2n+2}} = 0. \tag{3.7}$$

Then from the definition of the norm of Sobolev space, (3.6) and (3.7) yields,

$$\lim_{j\to\infty} ||(T_1-z)\bar{\partial}^i f_{j,1} + T_2\bar{\partial}^i f_{j,2}||_{2,D} = 0 \tag{3.8}$$

and

$$\lim_{j\to\infty} ||(T_3-z)\bar{\partial}^i f_{j,2}||_{2,D} = 0 \tag{3.9}$$

for $i = 1,2,\ldots,2(n+1)$.

Write $F(z) = G(z)p(z)$, where G is non vanishing analytic function on a neighborhood of $\sigma(T)$ and nonconstant polynomial p. Let $z_1,z_2,z_3,\ldots,z_n$ be zeros of $p(z)$. Set $q_s = (z-z_{(s+1)}\ldots(z-z_n)$, $s = 0,1,2,3\ldots,n-1$.

Now we need to prove that for all $s = 0,1,2,\ldots,n-1$ the following equation hold

$$\lim_{j\to\infty} ||q_s T_3^{n-s}\bar{\partial}^i f_{j,2}||_{2,D_s} = 0 \tag{3.10}$$

for $i = 1,2,\ldots,2n+2-2s$, where $\sigma(T) \subsetneqq D_n \subsetneqq D_{n-1} \subsetneqq \ldots\ldots \subsetneqq D_1 \subset D$. We use induction on s for the proof (3.10). Since T_3 is analytic of order n, (3.10) is true for $s = 0$. Suppose that

$$\lim_{j\to\infty} ||q_s(T_3^{n-s})\bar{\partial}^i f_{j,2}||_{2,D_s} = 0$$

holds for $0 < s < n$ and $i = 1,2,\ldots,2n+2-2s$.

From (3.9) and (3.10), we obtain that

$$0 = \lim_{j\to\infty} ||q_{s+1}(T_3^{n-s-1}-z)\bar{\partial}^i f_{j,2}||_{2,D_s} = \lim_{j\to\infty} ||(z_{s+1}-z)q_s T_3^{n-s-1}\bar{\partial}^i f_{j,2}||_{2,D_s} \tag{3.11}$$

holds for $i = 1,2,\ldots,2n-2s+2$.

From [12, lemma 3.2], it follows that

$$\lim_{j\to\infty} ||T_3^{n-s-1}\bar{\partial}^i f_{j,2}||_{2,D_{s+1}} = 0 \tag{3.12}$$

holds for $i = 1,2$, where $\sigma(T) \subsetneqq D_{s+1} \subsetneqq D_s$. Which completes the proof of (3.10). Now consider $s = n$ in (3.10), so we have

$$\lim_{j\to\infty} ||\bar{\partial}^i f_{j,2}||_{2,D_n} = 0 \tag{3.13}$$

for $i = 1,2$. So from (3.8) and (3.9), we have

$$\lim_{j\to\infty} ||(T_1 - z)\bar{\partial}^i f_{j,1}||_{2,D_n} = 0$$

for $i = 1,2$. It follows from Lemma 3.1 that

$$\lim_{j\to\infty} ||(I - P_{H_1})f_{j,1}||_{2,D_t} = 0, \tag{3.14}$$

where $\sigma(T) \subsetneqq D_t \subsetneqq D_n$ and $P_{\mathscr{H}_1}$ denotes the orthogonal projection of $L^2(D_t, \mathscr{H}_1)$ onto $A^2(D_t, \mathscr{H}_1)$.

From (3.13) and Lemma 3.1 with zero operator, it follows that

$$\lim_{j\to\infty} ||(I - P_{H_2})f_{j,2}||_{2,D_t} = 0, \tag{3.15}$$

where $P_{\mathscr{H}_2}$ denotes the orthogonal projection of $L^2(D_t, \mathscr{H}_2)$ onto $A^2(D_t, \mathscr{H}_2)$. Set $Pf_j := P_{\mathscr{H}_1} f_{j,1} \oplus P_{\mathscr{H}_2} f_{j,2}$. Then from (3.5), (3.14) and (3.15), we have

$$\lim_{j\to\infty} ||(T - z)Pf_j + 1 \otimes x_j||_{2,D_t} = 0.$$

Let γ be a closed curve in D_k surrounding $\sigma(T)$. Then, $\lim_{j\to\infty} ||Pf_j + (T - z)^{-1}(1 \otimes x_j)(z)|| = 0$ uniformly for all $z \in \gamma$. Then by Riesz-Dunford functional calculus, we get $\lim_{j\to\infty} ||\frac{1}{2\pi i}\int_\gamma Pf_j(z)dz + x_j|| = 0$. But Cauchy's theorem yields that $\frac{1}{2\pi i}\int_\gamma Pf_j(z)dz = 0$. Thus we have

$$\lim_{j\to\infty} ||x_j|| = 0.$$

This completes the proof. □

Now we are ready to prove that every analytic extension of an n-normal operator has a scalar extension.

THEOREM 3.5. *If T is an analytic extension of n-normal operator, then T is subscalar of order $2n+2$, where n is a positive integer.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(\mathscr{H}_1 \oplus \mathscr{H}_2)$, where T_1 is an n-normal operator and T_3 is analytic with order n. For any bounded disc D which contains $\sigma(T)$, the map

$$A : \mathscr{H} \oplus \mathscr{K} \to \frac{W^{2n+2}(D, \mathscr{H}_1) \oplus W^{2n+2}(D, \mathscr{H}_2)}{\overline{(T - z)W^{2n+2}(D, \mathscr{H}_1) \oplus W^{2n+2}(D, \mathscr{H}_2)}}$$

by

$$Ax = \widetilde{1 \otimes x}(\equiv 1 \otimes x + \overline{(T-z)W^{2n+2)}(D,\mathscr{H}_1) \oplus W^{2n+2}(D,\mathscr{H}_2)},$$

where $1 \otimes x$ denotes the constant function sending any $z \in D$ to $x \in \mathscr{H}_1 \oplus \mathscr{H}_2$ is injective with closed range by Lemma 3.4. Consider M, which is the operator of multiplication by z on $W^{2n+2}(D,\mathscr{H}_1) \oplus W^{2n+2}(D,\mathscr{H}_2)$. Then M is scalar operator of order $2n+2$ and has spectral distribution

$$\Phi : C_0^{2n+2}(\mathbb{C}) \to W^{2n+2}(D,\mathscr{H}_1) \oplus W^{2n+2}(D,\mathscr{H}_2)$$

defined by $\Phi(\nu)x = \nu x$ for $\nu \in C_0^{2n+2}(\mathbb{C})$ and $x \in W^{2n+2}(D,\mathscr{H}_1) \oplus W^{2n+2}(D,\mathscr{H}_2)$. Since $\overline{(T-z)W^{2n+2}(D,\mathscr{H}_1) \oplus W^{2n+2}(D,\mathscr{H}_2)}$ is invariant under M, $\tilde{M}$ is scalar operator of order $2n+2$ with $\tilde{\Phi}$ as a spectral distribution. From the definition of map A, we have $AT = \tilde{M}A$. In particular $R(A)$ is an invariant subspace for $\tilde{M}$. Since T is similar to restriction $\tilde{M}|_{R(A)}$, it follows that T is subscalar of order $2n+2$. $\square$

The following corollaries are immediate.

COROLLARY 3.6. *Let T be an analytic extension of an n-normal operator. Then T satisfies the Bishop's property (β).*

COROLLARY 3.7. *Let T be an analytic extension of an n-normal operator. Then T satisfies the single valued extension property* (SVEP).

Recall that an operator $T \in B(\mathscr{H})$ is called isoloid if every isolated point of spectrum of T is an eigenvalue. An operator $T \in B(\mathscr{H})$ is said to be polaroid if every $\lambda \in iso\sigma(T)$ is a pole of the resolvent of T.

COROLLARY 3.8. *Let T be a analytic extension of n-normal operator. Then T is polaroid.*

Proof. Assume that T is an analytic extension of n-normal operator. Then T is subscalar by Theorem 3.5. Hence, it follows from [16, Corollary 2.2] that T is polaroid. $\square$

LEMMA 3.9. *Let $T \in B(\mathscr{H}_1 \oplus \mathscr{H}_2)$ be an analytic extension of n-normal operator, i.e.,*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

is an operator matrix on $\mathscr{H}_1 \oplus \mathscr{H}_2$, where T_1 is n-normal and $F(T_3) = 0$ for a nonconstant analytic function F on a neighborhood D of $\sigma(T_3)$. Then $\sigma(T) = \sigma(T_1) \cup \sigma(T_3)$ and $\sigma(T_3)$ is a subset of $\{z \in \mathbb{C} : p(z) = 0\}$ where $F(z) = G(z)p(z)$, G is analytic and does not vanish on D, and p is a polynomial.

Proof. Since $p(T_3)=0$, choose a minimal polynomial q such that $q(T_3)=0$ and $q(z)$ is a factor of $p(z)$. Then $\{z\in\mathbb{C}: q(z)=0\}$ is nonempty and is contained in $\{z\in\mathbb{C}: p(z)=0\}$. First we will show that $\sigma(T_3)=\sigma_p(T_3)=\{z\in\mathbb{C}: q(z)=0\}$. Since $q(T_3)=0$, we have $q(\sigma(T_3))=\sigma(q(T_3))=\{0\}$ by the spectral mapping theorem. This means that $\sigma(T_3)=\{z\in\mathbb{C}: q(z)=0\}$. Moreover if we assume that $z_1\cdots,z_k$ are all the roots of $q(z)=0$, not necessarily distinct, then $(T_3-z_1)(T_3-z_2)\cdots(T_3-z_k)x=0$ for all $x\in H_2$. By the minimality of the degree of q, we can select a vector $x_0\in H_2$ such that $(T_3-z_2)\cdots(T_3-z_k)x_0\neq 0$, and so $z_1\in\sigma(p(T_3)$. Similarly, $z_i\in\sigma_p(T_3)$ for all $i=1;2;\cdots;k$. Hence $\sigma(T_3)=\sigma_p(T_3)=\{z\in\mathbb{C}: q(z)=0\}$. Since $\{z\in\mathbb{C}: q(z)=0\}$ is a finite set, $\sigma(T_1)\cap\sigma(T_3)$ is also finite, which implies that $\sigma(T_1)\cap\sigma(T_3)$ has no interior point. By using [9], we get $\sigma(T)=\sigma(T_1)\cup\sigma(T_3)$. □

THEOREM 3.10. *Let $T\in B(\mathscr{H}_1\oplus\mathscr{H}_2)$ be an analytic extension of n-normal operator, i.e.,*

$$T=\begin{pmatrix} T_1 & T_2 \\ 0 & T_3\end{pmatrix},$$

is an operator matrix on $\mathscr{H}_1\oplus\mathscr{H}_2$, where T_1 is n-normal and $F(T_3)=0$ for a non-constant analytic function F on a neighborhood D of $\sigma(T_3)$. Then the following statements hold

(i) $H_T(E)\subseteq H_{T_1}(E)\oplus\{0\}$ for every subset E of $\mathbb{C}$.

(ii) $\sigma_{T_3}(x_2)\subset\sigma_T(x_1\oplus x_2)$ and $\sigma_{T_1}(x_1)=\sigma_T(x_1\oplus 0)$, where $x_1\oplus x_2\in\mathscr{H}_1\oplus\mathscr{H}_2$.

(iii) $R_{T_1}(F)\oplus 0\subset H_T(F)$, where $R_{T_1}(F):=\{y\in H_1 : \sigma_{T_1}(y)\subset F\}$ for any subset $F\subset\mathbb{C}$.

Proof. (i) Let E be any subset of $\mathbb{C}$ and let $x_1\in H_{T_1}(E)$. Since T has SVEP by Lemma 3.2, there exists an H-valued analytic function f_1 on $\mathbb{C}\setminus E$ such that $(T_1-z)f_1(z)\equiv x_1$ on $\mathbb{C}\setminus E$. Hence $(T-z)(f_1(z)\oplus 0)\equiv x_1\oplus 0$ on $\mathbb{C}\setminus E$, and so $x_1\oplus 0\in H_T(E)$.

(ii) Let $x_1\oplus x_2\in\mathscr{H}_1\oplus\mathscr{H}_2$. If $z_0\in\rho_T(x_1\oplus x_2)$, then there exists an $\mathscr{H}$-valued analytic function defined on a neighborhood U of z_0 such that $(T-\lambda)f(\lambda)=x_1\oplus x_2$ for all $\lambda\in U$. We can write $f=f_1\oplus f_2$ where $f_1\in O(U;H_1)$ and $f_2\in O(U;H_2)$. Then we get

$$(T-\lambda)f(\lambda)=\begin{pmatrix} T_1-\lambda & T_2 \\ 0 & T_3-\lambda\end{pmatrix}\begin{pmatrix} f_1(\lambda) \\ f_2(\lambda)\end{pmatrix}=\begin{pmatrix} x_1 \\ x_2\end{pmatrix},$$

Thus $(T_3-\lambda)f_2(\lambda)=x_2$. Hence $z_0\in\rho_{T_3}(x_2)$. On the other hand, if $z_0\in\rho_T(x_1\oplus 0)$, then there exists an $\mathscr{H}$- valued analytic function defined on a neighborhood U of z_0 such that $(T-\lambda)g(\lambda)=x_1\oplus 0$ for all $\lambda\in U$. We can write $g=g_1\oplus g_2$ where $g_1\in\mathscr{O}(U,\mathscr{H}_1)$ and $g_2\in\mathscr{O}(U,\mathscr{H}_2)$. Then we get

$$(T-\lambda)(g(\lambda))=\begin{pmatrix} T_1-\lambda & T_2 \\ 0 & T_3-\lambda\end{pmatrix}\begin{pmatrix} g_1(\lambda) \\ g_2(\lambda)\end{pmatrix}=\begin{pmatrix} x_1 \\ 0\end{pmatrix}.$$

Thus $(T_1-\lambda)g_1(\lambda)+T_2g_2(\lambda)\equiv x_1$ and $(T_3-\lambda)g_2(\lambda)\equiv 0$. Since T_3 is algebraic of order n, it has SVEP, which implies that $g_2(\lambda)\equiv 0$. Thus $(T_1-\lambda)g_1(\lambda)\equiv x_1$, and so

$z_0 \in \rho_{T_1}(x_1)$. Conversely, let $z_0 \in \rho_{T_1}(x_1)$. Then there exists a function $g_1 \in O(U,\mathscr{H}_1)$ for some neighborhood U of z_0 such that $(T_1-\lambda)g_1(\lambda) \equiv x_1$. Then $(T-\lambda)(g_1(\lambda)\oplus 0) \equiv x_1 \oplus 0$. Hence $z_0 \in \rho_T(x_1\oplus 0)$.

(iii) If $x_1 \in R_{T_1}(F)$, then $\sigma_{T_1}(x_1) \subset F$. Since $\sigma_{T_1}(x_1) = \sigma_{T_1}(x_1\oplus 0)$ by (ii), $\sigma_{T_1}(x_1\oplus 0) \subset F$. Thus $x_1\oplus 0 \in H_T(F)$, and hence $R_{T_1}(F)\oplus 0 \subset H_T(F)$. $\square$

LEMMA 3.11. *Let $T \in B(H)$ be F-quasi-n-normal and let $\mathscr{M}$ be a reducing subspace for T. Then the restriction $T\mid_{\mathscr{M}}$ is a p-quasi-n-normal operator.*

Proof. Since T is an F-quasi-n-normal operator for some function F analytic and nonconstant on a neighborhood of $\sigma(T)$. Let $F(z) = G(z)p(z)$ where G is a nonvanishing analytic function on a neighborhood of $\sigma(T)$ and p is a nonconstant polynomial. Since $\mathscr{M}$ is a T-reducing subspace, we can write

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_3 \end{pmatrix},$$

on $\mathscr{M}\oplus\mathscr{M}^{\perp}$; where $T_1 = T|_M$ and $T_3 = (I-P)T(I-P)|_{\mathscr{M}^{\perp}}$, and P denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. Since T is F-quasi-n-normal, $F(T)^*(T^nT^*)F(T) = F(T)^*T^*T^nF(T)$. Therefore

$$0 = G(T)^*\begin{pmatrix} p(T_1)^*(T_1^nT_1^*-T_1^*T_1^n)p(T_1) & A \\ B & C \end{pmatrix} G(T)$$

for some operators $A;B$ and C by Riesz-Dunfords functional calculus. Since $G(T)$ is invertible, $p(T_1)^*(T_1^nT_1^*-T_1^*T_1^n)p(T_1) = 0$. This implies that $T_1 = T\mid_{\mathscr{M}}$ is p-quasi-n-normal. $\square$

THEOREM 3.12. *If T is an F -quasi-n-normal operator, then T is subscalar. In particular, every k-quasi-n-normal operator is subscalar of order $2k+2$.*

Proof. Suppose that $T \in B(\mathscr{H})$ be F-quasi-n-normal for some analytic function F on a neighborhood of $\sigma(T)$. If the range of $F(T)$ is norm dense in $\mathscr{H}$, then T is n-normal. Hence T is subscalar of order 2. Hence it suffices to assume that the range of $F(T)$ is not norm dense in $\mathscr{H}$. Since $F(T)$ commutes with T, $\overline{R(F(T))}$ is a T-invariant subspace. Thus T can be expressed as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix},$$

on $\overline{R(F(T))}\oplus N(F(T)^*)$; where $T_1 = T|_{\overline{R(F(T))}}$ and $T_3 = (I-P)T(I-P)|_{N(F(T)^*)}$, and P denotes the projection of H onto $\overline{R(F(T))}$. Note that $F(z) = G(z)p(z)$ where G is a nonvanishing analytic function on a neighborhood of $\sigma(T)$ and p is a nonconstant polynomial. Then $G(T)$ is invertible and thus we obtain that $N(F(T)) = N(p(T))$. Since $p(T_3) = (I-P)p(T)(I-P)|_{N(F(T)^*)}$, it follows for any $x \in N(F(T)^*)$,

$$\langle p(T_3)x;y\rangle = \langle p(T)x;y\rangle = \langle x;p(T)^*y\rangle = 0$$

for all $y \in N(F(T)^*)$. Hence $p(T_3) = 0$. Thus T_3 is analytic. Since $P(T_1^n T_1^* - T_1^* T_1^n)P = 0$, $PT_1^n T_1^* P - PT_1^* T_1^n P = 0$. Hence $T_1^n T_1^* - T_1^* T_1^n = 0$. This shows that T_1 is n-normal. If T_3 is analytic of order k, then T is subscalar of order $2k+2$ by Theorem 3.5. $\square$

COROLLARY 3.13. *Every F-quasi-n-normal operator has the Bishop's property (β).*

COROLLARY 3.14. *Every k-quasi-n-normal operator $T \in B(\mathscr{H})$ is subscalar of order $2k+2$. In particular, T has the Bishop's property (β).*

It is known that a normal operator has a nontrivial invariant subspace. In the following theorem, we will show that an analytic extension of n-normal operator also has a nontrivial invariant subspace.

THEOREM 3.15. *Let T be a n-normal operator. Then T has a nontrivial invariant subspace.*

Proof. Let T be a n-normal operator. Then T^n is normal. Hence, T^n has no hypercyclic vector by [11, Corollary 4.5]. Hence, T has no hypercyclic vector by [2]. Therefore, T has a nontrivial closed invariant subspace by [10]. $\square$

THEOREM 3.16. *Let T be an analytic extension of n-normal operator. Then T has a nontrivial invariant subspace.*

Proof. Let T be an analytic extension of n-normal operator. Then there is a closed subspace $\mathscr{M}$ invariant under T such that $T_1 = T|_{\mathscr{M}^\perp}$ is n-normal. If $\mathscr{M}^\perp = \{0\}$, then T is an n-normal. So, T has a nontrivial invariant subspace by Theorem 3.16. Now, if $\mathscr{M}^\perp \neq \{0\}$, then $\mathscr{M}$ is a non trivial proper invariant subspace for T. $\square$

Acknowledgements. The author would like to thank the referee for his valuable comments and suggestions, which considerably helped to improve the paper.

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(Received February 11, 2020)

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