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PROPERTIES OF J–SELF–ADJOINT OPERATORS

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Abstract. In this paper, we consider operators $T \in \mathscr{L}(\mathscr{H})$ such that $(JT)^* = JT$ for some anti-unitary J with $J^2 = -I$; in this case, we say that T is J-self-adjoint. We show that the Aluthge transform of a J-self-adjoint operator is skew-complex symmetric. As an application, we prove that w-hyponormal operators which are J-self-adjoint must be normal. Moreover, we obtain that if $T \in \mathscr{L}(\mathscr{H})$ is a J-self-adjoint operator with property (β), then $T + A$ is decomposable where $A \in \mathscr{L}(\mathscr{H})$ is an algebraic operator commuting with T. We also give examples of J-self-adjoint operators.

1. Introduction

Let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathscr{H}$. If $T \in \mathscr{L}(\mathscr{H})$, we write $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $\sigma_{comp}(T)$, $\sigma_{su}(T)$, $\sigma_{le}(T)$, $\sigma_{re}(T)$, and $\sigma_e(T)$ for the resolvent set, spectrum, point spectrum, approximate point spectrum, compression spectrum, surjective spectrum, left essential spectrum, right essential spectrum, and essential spectrum of T, respectively.

An operator $J : \mathscr{H} \to \mathscr{H}$ is said to be *anti-unitary* if J is anti-linear and $J^*J = JJ^* = I$, where J^* stands for the adjoint of J, which is uniquely determined by the relation $\langle J^*x, y\rangle = \overline{\langle x, Jy\rangle}$ for $x, y \in \mathscr{H}$. We say that an operator $T \in \mathscr{L}(\mathscr{H})$ is *J-self-adjoint* if there exists an anti-unitary operator $J : \mathscr{H} \to \mathscr{H}$ satisfying $J^2 = -I$ and $(JT)^* = JT$.

An anti-linear operator $C : \mathscr{H} \to \mathscr{H}$ is said to be a *conjugation* if $C^2 = I$ and C is isometric, i.e., $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathscr{H}$. If $C : \mathscr{H} \to \mathscr{H}$ is a conjugation, then the operator matrix $\mathscr{J}$ on $\mathscr{H} \oplus \mathscr{H}$ given by

$$\mathscr{J} = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}$$

is anti-unitary and $\mathscr{J}^2 = -I$.

We say that $T \in \mathscr{L}(\mathscr{H})$ is *complex symmetric with conjugation* C if $T^* = CTC$ for some conjugation C. The class of complex symmetric operators contains all normal

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operators, Hankel operators, compressed Toeplitz operators, algebraic operators of order 2, and some Volterra integration operator, and there are a lot of consequences and applications about complex symmetric operators (see [14], [15], [16], [19], [20], [21], [22], [29], etc.). If T is complex symmetric with conjugation C, then C is anti-unitary with $C^* = C$ and $(CT)^* = CT$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *skew-complex symmetric* if $T^* = -CTC$ for some conjugation C.

If $T = U|T|$ denotes the polar decomposition of an operator $T \in \mathcal{L}(\mathcal{H})$, the *Aluthge transform* of T is defined as $\widetilde{T} := |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. This transform has several properties which are transmitted to the original operators. For example, by [23, Corollary 1.16], if $\widetilde{T}$ has a nontrivial invariant subspace, then so does T. Thus, many authors have been interested in this operator transform and its applications (see [3], [4], [6], [7], [17], [18], [23], [24], etc.).

For $0 < p < \infty$, we say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *p-hyponormal* if $(T^*T)^p \geqslant (TT^*)^p$. In particular, 1-hyponormal operators and $\frac{1}{2}$-hyponormal operators are called *hyponormal* and *semi-hyponormal*, respectively. We call $T \in \mathcal{L}(\mathcal{H})$ *w-hyponormal* if $|\widetilde{T}| \geqslant |T| \geqslant |(\widetilde{T})^*|$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *paranormal* if $\|T^2x\| \geqslant \|Tx\|^2$ for all unit vectors $x \in \mathcal{H}$. p-Hyponormal operators are w-hyponormal and w-hyponormal operators are paranormal (see [12]). In addition, if $T \in \mathcal{L}(\mathcal{H})$ is p-hyponormal, then $\widetilde{T}$ is $(p+\frac{1}{2})$-hyponormal (see [3]). Thus, if $T \in \mathcal{L}(\mathcal{H})$ is w-hyponormal, then $\widetilde{T}$ is semi-hyponormal and $\widetilde{\widetilde{T}}$ is hyponormal.

In this paper, we show that the Aluthge transform of a J-self-adjoint operator is skew-complex symmetric. As an application, we prove that w-hyponormal operators which are J-self-adjoint must be normal. Moreover, we obtain that if $T \in \mathcal{L}(\mathcal{H})$ is a J-self-adjoint operator with property (β), then $T + A$ is decomposable where $A \in \mathcal{L}(\mathcal{H})$ is an algebraic operator commuting with T. We also give examples of J-self-adjoint operators.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open subset G of $\mathbb{C}$, the only analytic solution $f : G \to \mathcal{H}$ of the equation $(T-z)f(z) \equiv 0$ on G is the zero function on G. For $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, *the local resolvent set* $\rho_T(x)$ of T at x is defined to be the union of every open set G in $\mathbb{C}$ for which there exists an analytic function $f : G \to \mathcal{H}$ such that $(T-z)f(z) \equiv x$ on G. Since the analytic function $g(z) := (T-z)^{-1}x$ on $\rho(T)$ satisfies that $(T-z)g(z) \equiv x$ on G for every open set G in $\mathbb{C}$ containing $\rho(T)$, it holds that $\rho(T) \subset \rho_T(x)$ and any analytic function f appearing in the definition of $\rho_T(x)$ can be regarded as an extension of g. It is well known that if T has the single-valued extension property, then the function g is uniquely extended to $\rho_T(x)$. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called *the local spectrum* of T at x, and define *the local spectral subspace* of T by $\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$ for each subset F of $\mathbb{C}$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. We say that $T \in \mathscr{L}(\mathscr{H})$ has *Dunford's property* (C) if $\mathscr{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. From [8] or [27], we know that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}$$

and each of the converse implications fails to hold, in general.

We say that an operator $T \in \mathscr{L}(\mathscr{H})$ is *decomposable* provided that for every open cover $\{G_1, G_2\}$ of $\mathbb{C}$, there are T-invariant subspaces $\mathscr{M}_1$ and $\mathscr{N}$ such that $\mathscr{H} = \mathscr{M}_1 + \mathscr{M}_2$, $\sigma(T|_{\mathscr{M}_1}) \subset G_1$, and $\sigma(T|_{\mathscr{M}_2}) \subset G_2$. An operator T is said to have *the decomposition property* (δ) if for any open cover $\{G_1, G_2\}$ of $\mathbb{C}$, each vector $x \in \mathscr{H}$ is written as $x = x_1 + x_2$ where $(T-z)f_1(z) \equiv x_j$ on $\mathbb{C} \setminus \overline{G_j}$, with $\mathscr{H}$-valued analytic function f_j on $\mathbb{C} \setminus \overline{G_j}$, for $j = 1, 2$. We remark that $T \in \mathscr{L}(\mathscr{H})$ is decomposable precisely when T has properties (β) and (δ), i.e., both T and T^* have Bishop's property (β) (see [1], [8], or [27]).

3. Main results

In this section, we prove that every J-self-adjoint operator has skew-complex symmetric Aluthge transform and give several applications of this result. We begin with the following lemma.

LEMMA 3.1. Let $T \in \mathscr{L}(\mathscr{H})$ be J-self-adjoint. Then the following statements hold:

(i) T^* is J^*-self-adjoint;
(ii) $TJ^* = JT^*$ and $J^*T = T^*J$;
(iii) If $T = U|T|$ is the polar decomposition, then $\ker(T) = \ker(U^*J^*) = \ker(U^*J)$.

Proof. (i) Since T is J-self-adjoint, we have

$$TJ = J^*(JT)J = J^*(JT)^*J = J^*(T^*J^*)J = J^*T^*,$$

i.e., $(J^*T^*)^* = J^*T^*$. Since J^* is anti-unitary with $J^{*2} = -I$, the adjoint T^* is J^*-self-adjoint.

(ii) It follows from (i) that

$$TJ^* = -J(JT)J^* = -JT^*J^{*2} = JT^*$$

and

$$J^*T = -J^*(TJ)J = -J^{*2}T^*J = T^*J.$$

(iii) If $U^*J^*x = 0$, then (i) implies that

$$Tx = (TJ)J^*x = J^*T^*J^*x = J^*|T|U^*J^*x = 0.$$

Hence, we get that $\ker(T) \supset \ker(U^*J^*)$.

Conversely, if $Tx = 0$, then $0 = JTx = T^*J^*x$ by (i). Since $\ker(T^*) = \ker(U^*)$, we obtain that $U^*J^*x = 0$, and so $\ker(T) \subset \ker(U^*J^*)$. Thus $\ker(T) = \ker(U^*J^*)$.

If $U^*Jx = 0$, then $Jx \in \ker(U^*) = \ker(T^*)$, i.e., $T^*Jx = 0$. Since $T^* = JTJ$ and $J^2 = -I$, it follows that $0 = T^*Jx = JTJ^2x = -JTx$, which ensures that $Tx = 0$. This means that $\ker(T) \supset \ker(U^*J)$. By applying this procedure reversely, we can show that $\ker(T) \subset \ker(U^*J)$. $\square$

We say that an anti-linear operator $W : \mathscr{H} \to \mathscr{H}$ is a *partial conjugation* if it is a conjugation on $\ker(W)^\perp$. In the following theorem, we provide a representation for the polar decomposition of J-self-adjoint operators.

THEOREM 3.2. Let $T \in \mathscr{L}(\mathscr{H})$ be J-self-adjoint. If $T = U|T|$ is the polar decomposition, then $|T| = J|T^*|J^*$ and U is a J^*-self-adjoint operator factorized as $U = JW$ where $W := J^*U = U^*J$ is a partial conjugation supported by $\overline{\operatorname{ran}(|T|)}$ such that $|T|W = W|T|$.

Proof. Observe that

$$T = J^*T^*J^* = J^*|T|U^*J^*.$$

Since U^*U is the orthogonal projection of $\mathscr{H}$ onto $\overline{\operatorname{ran}(|T|)}$, we get that

$$T = J^*(U^*U)|T|U^*J^* = (J^2J^*U^*J)(JU|T|U^*J^*) = (JU^*J)(J|T^*|J^*).$$

Set $V := JU^*J$ and $P := J|T^*|J^*$. Since $P \geqslant 0$ and

$$P^2 = J|T^*|^2J^* = (JT)(T^*J^*) = T^*J^*JT = |T|^2,$$

we have $|T| = P = J|T^*|J^*$. In addition, since $V^* = J^*UJ^*$ and $U^*UU^* = U^*$, we see that

$$VV^*V = (JU^*J)(J^*UJ^*)(JU^*J) = J(U^*UU^*)J = JU^*J = V,$$

which implies that V is a partial isometry. According to Lemma 3.1, we know that $\ker(V) = \ker(U^*J) = \ker(T)$, and thus $U = V = JU^*J$. In other words, U is J^*-self-adjoint. If $W := J^*U = U^*J$, then $U = JW$ and it follows from Lemma 3.1 that

$$|T|W = |T|U^*J = T^*J = J^*T = J^*U|T| = W|T|.$$

Moreover, $W^* = W$ and $W^2 = U^*JJ^*U = U^*U$ is the orthogonal projection of $\mathscr{H}$ onto $\overline{\operatorname{ran}(|T|)}$, and so W is isometric on $\overline{\operatorname{ran}(|T|)}$. Since

$$\ker(W)^\perp = \ker(J^*U)^\perp = \ker(U)^\perp = \ker(|T|)^\perp = \overline{\operatorname{ran}(|T|)},$$

we conclude that W is a partial conjugation supported by $\overline{\operatorname{ran}(|T|)}$. $\square$

COROLLARY 3.3. Let $T \in \mathscr{L}(\mathscr{H})$ be J-self-adjoint. Then T is normal if and only if $|T|J = J|T|$.

Proof. Let $T = U|T|$ be the polar decomposition. By Theorem 3.2, it holds that $|T| = J|T^*|J^*$ and $U = JW$ where $W := J^*U = U^*J$ is a partial conjugation supported by $\overline{\operatorname{ran}(|T|)}$ such that $|T|W = W|T|$. Hence, if T is normal, then $|T| = |T^*| = J^*|T|J$, or equivalently, $|T|J = J|T|$.

Conversely, if $|T|J = J|T|$, then

$$\begin{aligned}|T^*|^2 &= U|T|^2U^* = J(W|T|^2W)J^* = J(W^2|T|^2)J^* \\ &= J|T|^2J^* = |T|^2JJ^* = |T|^2,\end{aligned}$$

and thus T is normal. □

In [15, page 3916], S. Garcia and M. Putinar pointed out that each partial conjugation can be extended to a conjugation; in detail, if W is a partial conjugation on $\mathscr{H}$, then $C := W \oplus W'$ acting on $\mathscr{H} = \ker(W)^\perp \oplus \ker(W)$ is a conjugation on the entire space $\mathscr{H}$, where W' is any partial conjugation supported by $\ker(W)$. This fact leads to the following decomposition of J-self-adjoint operators.

COROLLARY 3.4. If $T \in \mathscr{L}(\mathscr{H})$ is a J-self-adjoint operator, then it is decomposed as $T = V|T|$ where V is a unitary operator that is J^*-self-adjoint; furthermore, the map $C := J^*V = V^*J$ is a conjugation such that $|T|C = C|T|$.

Proof. From Theorem 3.2, write $T = U|T|$ where $U = JW$ and W is a partial conjugation, supported by $\overline{\operatorname{ran}(|T|)}$, commuting with $|T|$. Take a partial conjugation W' with support $\ker(W)$ so that $C = W \oplus W'$ is a conjugation on $\mathscr{H} = \ker(W)^\perp \oplus \ker(W) = \overline{\operatorname{ran}(|T|)} \oplus \ker(|T|)$. Set $V := JC$. Then $V^*V = CJ^*JC = I$ and $VV^* = JCCJ^* = I$, and thus V is unitary. Since $C^* = C$, we have $C = J^*V = V^*J$, i.e., V is J^*-self-adjoint. Writing $|T| = |T| \oplus 0$ on $\mathscr{H} = \overline{\operatorname{ran}(|T|)} \oplus \ker(|T|)$, we obtain that

$$T = U|T| = JW|T| = JC|T| = V|T|.$$

Moreover, since $|T|W = W|T|$, the conjugation C commutes with $|T|$. □

Let $T \in \mathscr{L}(\mathscr{H})$ be a J-self-adjoint operator having polar decomposition $T = U|T|$. Under the same notations as in Theorem 3.2 and Corollary 3.4, note that

$$\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}(JW)|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}(JC)|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}. \tag{1}$$

In the following theorem, we prove that the Aluthge transform of a J-self-adjoint operator is skew-complex symmetric.

THEOREM 3.5. If $T \in \mathscr{L}(\mathscr{H})$ is J-self-adjoint, then its Aluthge transform $\widetilde{T}$ is skew-complex symmetric.

Proof. Suppose that T is J-self-adjoint. Corollary 3.4 permits us to factorize T as $T = V|T|$ where V is a unitary operator which is J^*-self-adjoint and $C = J^*V$ is a conjugation commuting with $|T|$. Since $C|T| = |T|C$ and $C^2 = I$, it follows by (1) that

$$\begin{aligned}C\widetilde{T}C &= |T|^{\frac{1}{2}}CVC|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}CJ|T|^{\frac{1}{2}} = |T|^{\frac{1}{2}}CJ(-J^{*2})|T|^{\frac{1}{2}} \\ &= -|T|^{\frac{1}{2}}CJ^*|T|^{\frac{1}{2}} = -|T|^{\frac{1}{2}}V^*|T|^{\frac{1}{2}} = -(\widetilde{T})^*,\end{aligned}$$

which completes the proof. □

From Theorem 3.5, we assert that every w-hyponormal operator that is J-self-adjoint must be normal.

COROLLARY 3.6. Let $T \in \mathscr{L}(\mathscr{H})$ be J-self-adjoint. Then T is w-hyponormal if and only if it is normal.

Proof. If T is normal, then it is clearly w-hyponormal. Conversely, assume that T is w-hyponormal. Since $\widetilde{T}$ is semi-hyponormal, the square $(\widetilde{T})^2$ is w-hyponormal by [6]. Since T is J-self-adjoint, it follows from Theorem 3.5 that $\widetilde{T}$ is skew-complex symmetric and so its square $(\widetilde{T})^2$ is complex symmetric. According to [29, Theorem 3.2], the only complex symmetric w-hyponormal operators are normal operators. Hence, $(\widetilde{T})^2$ must be normal. From [5], the Aluthge transform $\widetilde{T}$ is normal, and so is T by [7]. □

We now apply Theorem 3.5 to derive local spectral properties of J-self-adjoint operators.

LEMMA 3.7. Let $T \in \mathscr{L}(\mathscr{H})$. If T has property (β) (resp. property (δ)) if and only if $\widetilde{T}$ has property (β) (resp. property (δ)).

Proof. It is not difficult to show that if $A, B \in \mathscr{L}(\mathscr{H})$, then AB has property (β) if and only if BA does. Hence, taking $A = U|T|^{\frac{1}{2}}$ and $B = |T|^{\frac{1}{2}}$, we see that T has property (β) if and only if $\widetilde{T}$ does. Moreover, since $T^* = |T|^{\frac{1}{2}}(|T|^{\frac{1}{2}}U^*)$ has property (β) exactly when $(\widetilde{T})^* = (|T|^{\frac{1}{2}}U^*)|T|^{\frac{1}{2}}$ has property (β), the duality of properties (β) and (δ) completes the proof. □

Recall that $A \in \mathscr{L}(\mathscr{H})$ is said to bed *algebraic* if $p(A) = 0$ for some nonconstant polynomial p.

THEOREM 3.8. Let $T \in \mathscr{L}(\mathscr{H})$ be a J-self-adjoint operator. If T has property (β), then $T + A$ is decomposable where A is an algebraic operator in $\mathscr{L}(\mathscr{H})$ commuting with T.

Proof. Note that $(\widetilde{T})^2$ is complex symmetric by Theorem 3.5. According to Lemma 3.7, the Aluthge transform $\widetilde{T}$ has property (β). Since $(\widetilde{T})^2$ has property (β) from [27, Theorem 3.3.9], it follows that $(\widetilde{T})^2$ is decomposable by [20]. Since $(\widetilde{T})^2$ and $(\widetilde{T})^{2*}$ have property (β), we get that $\widetilde{T}$ and $\widetilde{T}^*$ satisfy the same property using [27, Theorem 3.3.9] again. Therefore, Lemma 3.7 implies that T and T^* have property (β).

Next, take any algebraic operator $A \in \mathscr{L}(\mathscr{H})$ such that $AT = TA$, and let $p(z) = (z-\gamma_1)(z-\gamma_2)\cdots(z-\gamma_k)$ be a nonconstant polynomial such that $p(A) = 0$. Suppose that $\{f_n\}$ is any sequence of analytic functions on an open set G such that

$$\lim_{n\to\infty} \|(T + A - z)f_n(z)\| = 0$$

uniformly on compact sets in G. Setting

$$p_0(z) = 1 \text{ and } p_j(z) = (z-\gamma_1)(z-\gamma_2)\cdots(z-\gamma_j) \text{ for } j = 1,2,\cdots,k,$$

we will verify that

$$\lim_{n\to\infty} \|p_j(A)f_n(z)\| = 0 \text{ uniformly on compact sets in } G \tag{2}$$

for all $j = 0,1,2,\cdots,k$. Equation (2) holds obviously for $j = k$. If (2) is true for some integer j with $1 \leqslant j \leqslant k$, then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|p_{j-1}(A)(T+A-\gamma_j+\gamma_j-z)f_n(z)\| \\ &= \lim_{n\to\infty} \|(T+\gamma_j-z)p_{j-1}(A)f_n(z)\| \end{aligned}$$

uniformly on compact sets in G. Since T has property (β), so does $T+\gamma_k$, and thus $\lim_{n\to\infty}\|p_{j-1}(A)f_n(z)\| = 0$ uniformly on compact sets in G. Thus, by induction, we conclusion that (2) holds for all $j = 0,1,2,\cdots,k$. In particular, $\lim_{n\to\infty}\|f_n(z)\| = 0$ uniformly on compact sets in G. Accordingly, $T+A$ has property (β). Since T^* has property (β) and A^* is an algebraic operator commuting with T^*, T^*+A^* has property (β). Hence, $T+A$ is decomposable. □

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a vector $x \in \mathcal{H}$, the *local spectral radius* of T at x is defined as

$$r_T(x) := \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}.$$

It is known that $r(T) = \max\{r_T(x) : x \in \mathcal{H}\}$ for any $T \in \mathcal{L}(\mathcal{H})$, where $r(T)$ denotes the spectral radius of T (see [27, Proposition 3.3.14]). An operator $T \in \mathcal{L}(\mathcal{H})$ is called *power regular* if $\lim_{n\to\infty}\|T^n x\|^{\frac{1}{n}}$ exists for every $x \in \mathcal{H}$. We say that an element $x \in \mathcal{H}$ is a *cyclic* vector for an operator $T \in \mathcal{L}(\mathcal{H})$ if the linear span of the orbit $\{T^n x : n = 0,1,2,\cdots\}$ is dense in $\mathcal{H}$.

COROLLARY 3.9. Let $T \in \mathcal{L}(\mathcal{H})$ be J-self-adjoint. If T has property (β), then the following assertions hold:

(i) Both T and T^* are power regular. Moreover, $r_T(x) = \lim_{n\to\infty}\|T^n x\|^{\frac{1}{n}}$ and $r_{T^*}(x) = \lim_{n\to\infty}\|T^{*n}x\|^{\frac{1}{n}}$ for all $x \in \mathcal{H}$.

(ii) If $x \in \mathcal{H}$ is a cyclic vector for T^*, then $\sigma_{T^*}(x) = \sigma(T^*)$ and $r_{T^*}(x) = r(T^*)$.

Proof. Since both T and T^* have property (β) from Theorem 3.8, the result (i) follows by [27, Proposition 3.3.17]. Moreover, since T^* has Dunford's property (C), we obtain (ii) using [27, page 238]. □

The *mean transform* of an operator $T \in \mathcal{L}(\mathcal{H})$, firstly introduced in [26], is defined as $\widehat{T} := \frac{1}{2}(U|T| + |T|U)$ where $T = U|T|$ is the polar decomposition. There are several connections between T and $\widehat{T}$ (see [24] for more details). In the following proposition, we give some local spectral relation between J-self-adjoint operators and their mean transforms.

PROPOSITION 3.10. Let $T \in \mathcal{L}(\mathcal{H})$ be J-self-adjoint with $|T|J|T| = |T|^2J$. If T has property (β), then both $\widehat{T}$ and $\widehat{(T^*)}$ have property (β).

Proof. According to Theorem 3.2, the polar decomposition of T is given by $T = U|T|$ where $|T| = J|T^*|J^*$ and $U = JW$ for some partial conjugation W commuting with T. Since $|T|J|T| = |T|^2J$, it holds that

$$|T|U|T| = |T|JW|T| = |T|J|T|W = |T|^2JW = |T|^2U.$$

Due to [24], it follows that $\widehat{T}$ has property (β).

Now, let $\{f_n\}$ be a sequence of $\mathcal{H}$-valued functions analytic on an open set G such that $\lim_{n\to\infty} \|(\widehat{(T^*)} - z)f_n(z)\| = 0$ uniformly on compact sets in G. Since $W = J^*U = U^*J$, $|T^*| = J^*|T|J$, and $W|T| = |T|W$, we obtain that

$$\begin{aligned} J^*\widehat{T}J^* &= \frac{1}{2}(W|T|J^* + J^*|T|JWJ^*) = \frac{1}{2}(|T|WJ^* + |T^*|WJ^*) \\ &= \frac{1}{2}(|T|U^* + |T^*|U^*) = \widehat{(T^*)}. \end{aligned}$$

Hence

$$0 = \lim_{n\to\infty} \|J\big(\widehat{(T^*)} - z\big)J(J^* f_n(z))\| = \lim_{n\to\infty} \|(\widehat{T} + \overline{z})(J^* f_n(z))\|$$

uniformly on compact sets in G. For each n, define the function $g_n(\zeta) = J^* f_n(-\overline{\zeta})$ for $\zeta \in -G^* := \{-\overline{z} : z \in G\}$. Then $\lim_{n\to\infty} \|(\widehat{T} - \zeta)g_n(\zeta)\| = 0$ uniformly on compact sets in $-G^*$. Note that each g_n is analytic on the open set $-G^*$; indeed, if $\zeta_0 \in -G^*$, then $-\overline{\zeta_0} \in G$. Writing $f_n(z) = \sum_{n=0}^{\infty}(z + \overline{\zeta_0})^n a_n$ on a neighborhood of $-\overline{\zeta_0}$ contained in G, where $\{a_n\} \subset \mathcal{H}$, we see that for $\zeta \in -G^*$,

$$g_n(\zeta) = J^* f_n(-\overline{\zeta}) = J^*\Big(\sum_{n=0}^{\infty}(-\overline{\zeta} + \overline{\zeta_0})^n a_n\Big) = \sum_{n=0}^{\infty}(-1)^n(\zeta - \zeta_0)^n J^* a_n.$$

This means that g_n is analytic at every point ζ_0 in $-G^*$. Since $\widehat{T}$ has property (β), we get that $\lim_{n\to\infty} \|g_n\| = 0$ uniformly on compact sets in G, which ensures that $\{f_n\}$ converges in norm to 0 uniformly on compact sets in G. Thus, $\widehat{(T^*)}$ has property (β). $\square$

We next examine Dunford's property (C) of J-self-adjoint operators.

PROPOSITION 3.11. If $T \in \mathcal{L}(\mathcal{H})$ is J-self-adjoint, then the following properties hold:

(i) $\sigma_T(x) = -\big(\sigma_{T^*}(Jx)\big)^*$ for all $x \in \mathcal{H}$.

(ii) $J\mathcal{H}_T(F) = \mathcal{H}_{T^*}(-F^*)$ for any subset F of $\mathbb{C}$.

Proof. (i) Let $x \in \mathcal{H}$ be given and let G be any open set in $\mathbb{C}$. If $f : G \to \mathcal{H}$ is an analytic function such that $(T - z)f(z) = x$ for all $z \in G$, then

$$Jx = J(T - zJJ^*)f(z) = (T^* + \overline{z})J^* f(z)$$

for $z \in G$, i.e.,

$$(T^* - \zeta)J^* f(-\overline{\zeta}) = Jx \tag{3}$$

for $\zeta \in -G^*$. Since $J^* f(-\overline{\zeta})$ is analytic for $\zeta \in -G^*$ (see the proof of Theorem 3.10), we have $-\big(\rho_T(x)\big)^* \subset \rho_{T^*}(Jx)$ for all $x \in \mathscr{H}$. Hence

$$\big(\sigma_{T^*}(Jx)\big)^* \subset \mathbb{C} \setminus \big(-\rho_T(x)\big) = -\big(\mathbb{C} \setminus \rho_T(x)\big) = -\sigma_T(x) \tag{4}$$

for all $x \in \mathscr{H}$. Since T^* is J^*-self-adjoint by Lemma 3.1, we obtain from (4) that $\big(\sigma_T(J^*x)\big)^* \subset -\sigma_{T^*}(x)$ for all $x \in \mathscr{H}$. Replacing x with Jx and taking complex conjugate, we get that

$$\sigma_T(x) \subset -\big(\sigma_{T^*}(Jx)\big)^* \tag{5}$$

for all $x \in \mathscr{H}$. Thus, we complete the proof from (4) and (5).

(ii) Suppose that F is a subset of $\mathscr{H}$. If $x \in \mathscr{H}_T(F)$, then

$$-\big(\sigma_{T^*}(Jx)\big)^* = \sigma_T(x) \subset F$$

by (i). Since $\sigma_{T^*}(Jx) \subset -F^*$, it holds that $Jx \in \mathscr{H}_{T^*}(-F^*)$, and so

$$J\mathscr{H}_T(F) \subset \mathscr{H}_{T^*}(-F^*).$$

Applying the above argument to the adjoint T^*, we deduce the inclusion

$$J^*\mathscr{H}_{T^*}(-F^*) \subset \mathscr{H}_T(F).$$

Therefore, $J\mathscr{H}_T(F) = \mathscr{H}_{T^*}(-F^*)$. □

COROLLARY 3.12. Let $T \in \mathscr{L}(\mathscr{H})$ be J-self-adjoint. Then T has Dunford's property (C) if and only if its adjoint T^* does.

Proof. Assume that $T \in \mathscr{L}(\mathscr{H})$ is a J-self-adjoint operator satisfying Dunford's property (C). Let F be any closed subset of $\mathbb{C}$. Then $\mathscr{H}_T(-F^*)$ is closed. Since $\mathscr{H}_{T^*}(F) = J\mathscr{H}_T(-F^*)$ from Proposition 3.11 and J is anti-unitary, the subspace $\mathscr{H}_{T^*}(F)$ is closed. Hence, we conclude that T^* has Dunford's property (C). The converse also holds by Lemma 3.1. □

We say that an operator $T \in \mathscr{L}(\mathscr{H})$ has *Dunford's boundedness condition* (B) if it has the single-valued extension property and there exists a constant $K > 0$ such that $\|x_1\| \leqslant K\|x_1 + x_2\|$ for any $x_1, x_2 \in \mathscr{K}$ with $\sigma_T(x_1) \cap \sigma_T(x_2) = \emptyset$, where K is independent of x_1 and x_2.

COROLLARY 3.13. Let $T \in \mathscr{L}(\mathscr{H})$ be J-self-adjoint. Then the following assertions hold:

(i) T has Dunford's boundedness condition (B) if and only if T^* does.

(ii) If T has the single-valued extension property and possesses the property that $\sigma_T(P_F x) \subset \sigma_T(x)$ for all $x \in \mathscr{H}$ and each closed set F in $\mathbb{C}$, where P_F denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_T(F)$, then both T and T^* have Dunford's boundedness condition (B).

Proof. (i) It suffices to prove one implication. If T has Dunford's boundedness condition (B), choose a constant $K > 0$ such that $\|x_1\| \leqslant K\|x_1 + x_2\|$ for any $x_1, x_2 \in \mathscr{H}$ with $\sigma_T(x_1) \cap \sigma_T(x_2) = \emptyset$. Let y_1 and y_2 be arbitrary vectors in $\mathscr{H}$ with $\sigma_{T^*}(y_1) \cap \sigma_{T^*}(y_2) = \emptyset$. It follows from Proposition 3.11 that $\sigma_T(J^*y_1) \cap \sigma_T(J^*y_2) = \emptyset$, and thus $\|J^*y_1\| \leqslant K\|J^*y_1 + J^*y_2\|$. This implies that

$$\|y_1\| = \|J^*y_1\| \leqslant K\|J^*(y_1 + y_2)\| = K\|y_1 + y_2\|.$$

In addition, we can obtain that T^* has the single-valued extension property. Thus, T^* satisfies Dunford's boundedness condition (B).

(ii) Let $x_1, x_2 \in \mathscr{H}$ be such that $\sigma_T(x_1) \cap \sigma_T(x_2) = \emptyset$. Set $F_j = \sigma_T(x_j)$ for $j = 1,2$. By the hypothesis, we have $\sigma_T(P_{F_2}x_1) \subset \sigma_T(x_1) = F_1$. Moreover, it is obvious that $\sigma_T(P_{F_2}x_1) \subset F_2$ by the definition of P_{F_2}. Hence

$$\sigma_T(P_{F_2}x_1) \subset F_1 \cap F_2 = \sigma_T(x_1) \cap \sigma_T(x_2) = \emptyset.$$

Since T has the single-valued extension property, we get that $P_{F_2}x_1 = 0$ by [27, Proposition 1.2.16], that is, $x_1 \perp \mathscr{H}_T(F_2)$. But $\sigma_T(x_2) = F_2$, and so x_2 clearly belongs to $\mathscr{H}_T(F_2)$. Then $\langle x_1, x_2 \rangle = 0$, which implies that $\|x_1 + x_2\| \geqslant \|x_1\|$. Thus, T has Dunford's boundedness condition (B), and so does T^* from (i). $\square$

For an operator $T \in \mathscr{L}(\mathscr{H})$, the *quasinilpotent part* of T is defined by

$$H_0(T) := \{x \in \mathscr{H} : \lim_{n\to\infty} ||T^n x||^{\frac{1}{n}} = 0\}.$$

COROLLARY 3.14. Let $T \in \mathscr{L}(\mathscr{H})$ be J-self-adjoint. If $H_0(T - \lambda)$ is closed for all $\lambda \in \mathbb{C}$, then T^* has the single-valued extension property and $\mathscr{H}_{T^*}(\{\lambda\})$ is closed for each $\lambda \in \mathbb{C}$.

Proof. Suppose that T is J-self-adjoint and $H_0(T - \lambda)$ is closed for each $\lambda \in \mathbb{C}$. Since T has the single-valued extension property by [1, Theorem 2.31], so does $\widetilde{T}$ by some application of the proof of Lemma 3.7. As in the proof of Theorem 3.8, we see that T^* has the single-valued extension property. Fix any $\lambda \in \mathbb{C}$. From [2, Theorem 1.5], we get that $\mathscr{H}_T(\{\lambda\}) = H_0(T - \lambda)$. Proposition 3.11 implies that

$$\mathscr{H}_{T^*}(\{\lambda\}) = J\mathscr{H}_T(\{-\overline{\lambda}\}) = JH_0(T + \overline{\lambda}).$$

Since $H_0(T + \overline{\lambda})$ is closed and J maps a closed subspace onto a closed one, we conclude that the local spectral subspace $\mathscr{H}_{T^*}(\{\lambda\})$ is closed. $\square$

Similarly to complex symmetric operators, there exist connections between the spectra of a J-self-adjoint operator and its adjoint. Given any set E in $\mathbb{C}$, write $E^* := \{\overline{z} : z \in E\}$ and $-E := \{-z : z \in E\}$.

PROPOSITION 3.15. Let $T \in \mathscr{L}(\mathscr{H})$ be J-self-adjoint. Then

$$\sigma_\Delta(T^*) = -\sigma_\Delta(T)^* \tag{6}$$

where $\sigma_\Delta \in \{\sigma_p, \sigma_a, \sigma_{comp}, \sigma_{su}, \sigma_{le}, \sigma_{re}, \sigma_e, \sigma\}$.

Proof. We first deal with the left essential spectrum. If $\alpha \in \sigma_{le}(T)$, then there is a sequence $\{x_n\}$ of unit vectors in $\mathscr{H}$ such that $x_n \to 0$ weakly and $\lim_{n\to\infty} \|(T-\alpha)x_n\| = 0$. Observe that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \|J(T-\alpha)x_n\| = \lim_{n\to\infty} \|J(T-\alpha JJ^*)x_n\| \\ &= \lim_{n\to\infty} \|(T^*J^* - \overline{\alpha}J^2J^*)x_n\| = \lim_{n\to\infty} \|(T^*+\overline{\alpha})J^*x_n\|. \end{aligned}$$

It is evident that $\|J^*x_n\| = \|x_n\| = 1$ for all n and $J^*x_n \to 0$ weakly, and so $-\overline{\alpha} \in \sigma_{le}(T^*)$, meaning that

$$-\sigma_{le}(T)^* \subset \sigma_{le}(T^*). \tag{7}$$

Since T^* is J^*-self-adjoint by Lemma 3.1, equation (7) holds when we replace T with T^*, which yields that

$$\sigma_{le}(T^*) \subset -\sigma_{le}(T)^*. \tag{8}$$

From (7) and (8), it follows that

$$\sigma_{le}(T^*) = -\sigma_{le}(T)^*.$$

By a similar method, one can see that (6) is also true for the cases $\sigma_\Delta = \sigma_p, \sigma_{ap}$. Since $\sigma_{comp}(A^*) = \sigma_p(A)^*$, $\sigma_{su}(A^*) = \sigma_a(A)^*$, and $\sigma_{re}(A^*) = \sigma_{le}(A)^*$ where A is any operator in $\mathscr{L}(\mathscr{H})$, we obtain (6) for $\sigma_\Delta = \sigma_\Delta = \sigma_{comp}, \sigma_{su}, \sigma_{re}$. Moreover, since $\sigma_e(A) = \sigma_{le}(A) \cup \sigma_{re}(A)$ and $\sigma(A) = \sigma_a(A) \cup \sigma_{comp}(A)$ for any operator $A \in \mathscr{L}(\mathscr{H})$, equation (6) holds for $\sigma_\Delta = \sigma_e, \sigma$. So, we complete the proof. $\square$

COROLLARY 3.16. If $T \in \mathscr{L}(\mathscr{H})$ is J-self-adjoint, then the following properties hold:

(i) $\sigma_{comp}(T) = -\sigma_p(T)$, $\sigma_{su}(T) = -\sigma_a(T)$, and $\sigma_{re}(T) = -\sigma_{le}(T)$.
(ii) $\sigma(T) = -\sigma(T)$ and $\sigma_e(T) = -\sigma_e(T)$.
(iii) $\sigma(T) = \sigma_a(T) \cup \big(-\sigma_p(T)\big) = \sigma_p(T) \cup \big(-\sigma_a(T)\big) = \sigma_p(T) \cup \sigma_{su}(T)$.
(iv) $\sigma_e(T) = \sigma_{le}(T) \cup \big(-\sigma_{le}(T)\big) = \sigma_{re}(T) \cup \big(-\sigma_{re}(T)\big)$.
(v) $\ker(T-\alpha) = J\ker(T^* + \overline{\alpha})$ for each $\alpha \in \mathbb{C}$.
(vi) $\ker(T^2-\alpha) = J^*\ker(T^{*2} - \overline{\alpha})$ for each $\alpha \in \mathbb{C}$.

Proof. (i) Proposition 3.15 implies that

$$\sigma_{comp}(T) = -\sigma_{comp}(T^*)^* = -\sigma_p(T).$$

Similarly, we get the remaining identities in (i).

(ii) We obtain from Proposition 3.15 that

$$\sigma_e(T) = -\sigma_e(T^*)^* = -\sigma_e(T) \text{ and } \sigma(T) = -\sigma(T^*)^* = -\sigma(T).$$

(iii) By (i), it follows that

$$\sigma(T) = \sigma_a(T) \cup \sigma_{comp}(T) = \sigma_a(T) \cup \big(-\sigma_p(T)\big).$$

Hence, the proof is complete due to (ii).

(iv) Since $\sigma_e(T) = \sigma_{le}(T) \cup \sigma_{re}(T)$ and $\sigma_{re}(T) = -\sigma_{le}(T)$ by (i), we deduce the result.

(v) As an application of the proof of Proposition 3.15, we see that

$$J^* \ker(T-\alpha) \subset \ker(T^* + \overline{\alpha}), \text{ i.e., } \ker(T-\alpha) \subset J \ker(T^* + \overline{\alpha})$$

for $\alpha \in \mathbb{C}$. Since T^* is J^*-self-adjoint by Lemma 3.1, it also holds that

$$J \ker(T^* + \overline{\alpha}) \subset \ker(T-\alpha)$$

for $\alpha \in \mathbb{C}$, which verifies (v).

(vi) Let $\alpha \in \mathbb{C}$ be arbitrary. If $x \in \ker(T^2 - \alpha)$, then

$$\overline{\alpha} Jx = J(\alpha x) = JT^2 x = (JT)Tx = T^*(J^*T)x = T^{*2} Jx$$

by Lemma 3.1, and so $Jx \in \ker(T^{*2} - \overline{\alpha})$. Hence $J \ker(T^2 - \alpha) \subset \ker(T^{*2} - \overline{\alpha})$. Similarly, we get that $J^* \ker(T^{*2} - \overline{\alpha}) \subset \ker(T^2 - \alpha)$. Therefore it holds that $\ker(T^2 - \alpha) = J^* \ker(T^{*2} - \overline{\alpha})$. $\square$

4. Examples

In this section, we give several examples and study their spectral properties of J-self-adjoint operators. In particular, we find J-self-adjoint operators that are not complex symmetric (see Proposition 4.5 and Example 4.6). We first consider 2×2 operator matrices which are $\mathcal{J}$-self-adjoint where

$$\mathcal{J} = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}$$

for some conjugation $C : \mathcal{H} \to \mathcal{H}$.

PROPOSITION 4.1. Let $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ be a 2×2 operator matrix in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, and let $\mathcal{J} = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}$ where C is any conjugation on $\mathcal{H}$. Then T is $\mathcal{J}$-self-adjoint if and only if both T_2 and T_3 are complex symmetric with the conjugation C and $T_4 = -CT_1^*C$. In particular, if all of T_1, T_2, and T_3 are complex symmetric with the same conjugation C, then $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & -T_1 \end{pmatrix}$ is $\mathcal{J}$-self-adjoint.

Proof. It is easy to see that T is $\mathcal{J}$-self-adjoint if and only if $T^* = \mathcal{J} T \mathcal{J}$, namely

$$\begin{pmatrix} T_1^* & T_3^* \\ T_2^* & T_4^* \end{pmatrix} = \begin{pmatrix} -CT_4C & CT_3C \\ CT_2C & -CT_1C \end{pmatrix}. \tag{9}$$

Since $T_4^* = -CT_1C$ is equivalent to $T_1^* = -CT_4C$, equation (9) holds exactly when both T_2 and T_3 are complex symmetric with conjugation C and $T_4 = -CT_1^*C$. $\square$

COROLLARY 4.2. Let $T_1 \in \mathscr{L}(\mathscr{H})$ be a normal operator, and let $A \in \mathscr{L}(\mathscr{H})$ be a nonzero operator such that $AT_1 = T_1A = 0$. Then the operator matrix

$$\begin{pmatrix} T_1 & A \\ T_1 & -T_1 \end{pmatrix}$$

is decomposable.

Proof. Since every normal operator is complex symmetric by [14], choose a conjugation C on $\mathscr{H}$ satisfying $CT_1C = T_1^*$. Then

$$T = \begin{pmatrix} T_1 & 0 \\ T_1 & -T_1 \end{pmatrix}$$

is $\mathscr{J}$-self-adjoint with $\mathscr{J} = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}$. In addition, it is easy to see that T has property (β). Since $N := \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$ is nilpotent of order 2 and $NT = TN$, we complete the proof from Theorem 3.8. □

According to Proposition 4.1, one can construct J-self-adjoint operators using complex symmetric operators. In order to give concrete examples, consider weighted composition operators on the Hilbert-Hardy space H^2 of the open unit disk $\mathbb{D}$. The Hardy space H^2 is regarded as a closed subspace of $L^2 = L^2(\partial\mathbb{D}, m)$ where m denotes the (normalized) Lebesgue measure on the unit circle $\partial\mathbb{D}$. For an analytic function f on $\mathbb{D}$ and an analytic self-map φ of $\mathbb{D}$, the operator $W_{f,\varphi} : H^2 \to H^2$ given by $W_{f,\varphi}h = f \cdot (h \circ \varphi)$ is called a *weighted composition operator*. In particular, $C_\varphi := W_{1,\varphi}$ is said to be a *composition operator*. If φ is any analytic self-map of $\mathbb{D}$ and $f \in H^2$ for which $W_{f,\varphi}$ is bounded on H^2, then $W_{f,\varphi}^* K_\beta = \overline{f(\beta)} K_{\varphi(\beta)}$ for $\beta \in \mathbb{D}$, where $K_\beta := \frac{1}{1-\overline{\beta}z}$ so-called the *reproducing kernel* of H^2 at a point β in $\mathbb{D}$. We refer the readers to [9], [10], [11], [19], and [28] for more details on weighted composition operators on H^2. In [19], the authors characterized complex symmetric weighted composition operators on H^2 with a specific conjugation. Using this characterization, we give the following example.

EXAMPLE 4.3. Let $\mathscr{C} : H^2 \to H^2$ be the conjugation given by $\mathscr{C}h = \widehat{h}$ where $\widehat{h}(z) := \overline{h(\overline{z})}$ for $z \in \mathbb{D}$. Suppose that $\psi_j(z) = a_j + \frac{b_j z}{1-a_j z}$ and $g_j(z) = \frac{c_j}{1-a_j z}$ with constants $a_j \in \mathbb{D}$ and $b_j, c_j \in \mathbb{C}$ for $j = 1, 2$. Then each W_{g_j,ψ_j} is complex symmetric with conjugation $\mathscr{C}$ by [19, Theorem 3.3]. Hence, given analytic self-map φ of $\mathbb{D}$ and $f \in H^2$ for which $W_{f,\varphi}$ is bounded on H^2, Proposition 4.1 implies that $\begin{pmatrix} W_{f,\varphi} & W_{g_1,\psi_1} \\ W_{g_2,\psi_2} & -\mathscr{C}W_{f,\varphi}^*\mathscr{C} \end{pmatrix}$ is $\mathscr{J}$-self-adjoint with respect to $\mathscr{J} = \begin{pmatrix} 0 & -\mathscr{C} \\ \mathscr{C} & 0 \end{pmatrix}$. Since $\mathscr{C}K_\beta = K_{\overline{\beta}}$ for each point β in $\mathbb{D}$, we compute that

$$\mathscr{C}W_{f,\varphi}^*\mathscr{C}K_\beta = \mathscr{C}W_{f,\varphi}^*K_{\overline{\beta}} = \mathscr{C}\left(\overline{f(\overline{\beta})}K_{\varphi(\overline{\beta})}\right) = \widehat{f}(\beta)K_{\widehat{\varphi}(\beta)} = W_{\widehat{f},\widehat{\varphi}}^* K_\beta$$

for $\beta \in \mathbb{D}$. Since the linear span of reproducing kernels is dense in H^2, we have $\mathscr{C}W_{f,\varphi}^*\mathscr{C} = W_{\widehat{f},\widehat{\varphi}}^*$. Thus

$$\begin{pmatrix} W_{f,\varphi} & W_{g_1,\psi_1} \\ W_{g_2,\psi_2} & -W_{\widehat{f},\widehat{\varphi}}^* \end{pmatrix}$$

is $\mathscr{J}$-self-adjoint.

If $\varphi(z) = \frac{az+b}{cz+d}$ is a linear fractional self-map of $\mathbb{D}$ where a,b,c,d are complex numbers with $ad-bc \neq 0$, then Cowen's adjoint formula states that $C_\varphi^* = T_g C_\sigma T_h^*$ where $g(z) = \frac{1}{-\overline{b}z+\overline{d}}$, $\sigma(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}$, and $h(z) = cz+d$ (see [9]). Taking $f \equiv 1$ in Example 4.3, we obtain the following $\mathscr{J}$-self-adjoint block matrix of operators:

$$\begin{pmatrix} C_\varphi & W_{g_1,\psi_1} \\ W_{g_2,\psi_2} & -C_{\widehat{\varphi}}^* \end{pmatrix} \tag{10}$$

where φ is any analytic self-map of $\mathbb{D}$. If φ is a linear self-map of $\mathbb{D}$, then Cowen's adjoint formula allows us to replace $C_{\widehat{\varphi}}^*$ in (10) with some weighted composition operator.

EXAMPLE 4.4. Assume that $\varphi(z) = az+b$ where $|a|+|b| \leqslant 1$. Then φ is an analytic self-map of $\mathbb{D}$. Since $\widehat{\varphi}(z) := \overline{\varphi(\overline{z})} = \overline{a}z+\overline{b}$, apply Cowen's adjoint formula to $C_{\widehat{\varphi}}^*$, as follows:

$$C_{\widehat{\varphi}}^* = T_g C_\sigma = W_{g,\sigma}$$

with $g(z) = \frac{1}{1-bz}$ and $\sigma(z) = \frac{az}{1-bz}$. Therefore, the block matrix of weighted composition operators $\begin{pmatrix} C_\varphi & W_{g_1,\psi_1} \\ W_{g_2,\psi_2} & -W_{g,\sigma} \end{pmatrix}$ is $\mathscr{J}$-self-adjoint, where the maps ψ_j and g_j as well as the anti-unitary $\mathscr{J}$ are defined as in Example 4.3. In particular, substituting $W_{g_j,\psi_j} = I$ for $j = 1,2$ (i.e., $a_j = 0$ and $b_j = c_j = 1$), we get that

$$\begin{pmatrix} C_\varphi & I \\ I & -W_{g,\sigma} \end{pmatrix} = \begin{pmatrix} C_{az+b} & I \\ I & W_{\frac{-1}{1-bz},\frac{az}{1-bz}} \end{pmatrix}$$

is $\mathscr{J}$-self-adjoint.

We next find J-self-adjoint operators that are not complex symmetric.

COROLLARY 4.5. Suppose that C is a conjugation on $\mathscr{H}$ and A is any operator in $\mathscr{L}(\mathscr{H})$ such that $\mathscr{E}_p(A) \neq -\mathscr{E}_p(A)$ where $\mathscr{E}_p(A) := \sigma_p(A)^* \cup \big(-\sigma_p(A^*)\big)$. Then the operator matrix

$$T = \begin{pmatrix} A & 0 \\ 0 & -CA^*C \end{pmatrix}$$

is J-self-adjoint but not complex symmetric.

Proof. We obtain from Proposition 4.1 that $T = \begin{pmatrix} A & 0 \\ 0 & -CA^*C \end{pmatrix}$ is $\mathscr{J}$-self-adjoint where $\mathscr{J} = \begin{pmatrix} 0 & -C \\ C & 0 \end{pmatrix}$. Since $(CA^*C)^* = CAC$, one can see that

$$\begin{cases} \sigma_p(T) = \sigma_p(A) \cup \big(-\sigma_p(CA^*C) \big) \\ \sigma_p(T^*) = \sigma_p(A^*) \cup \big(-\sigma_p(CAC) \big). \end{cases}$$

We will use

$$\sigma_p(CBC) = \sigma_p(B)^* \tag{11}$$

where B is any operator in $\mathscr{L}(\mathscr{H})$. Indeed, if $\alpha \in \sigma_p(CBC)$, then $(CBC - \alpha)x = 0$ for some nonzero vector $x \in \mathscr{H}$, and so $0 = C(CBC - \alpha)x = (B - \overline{\alpha})Cx$. Since C is a conjugation, Cx must be a nonzero vector in $\mathscr{H}$, so that $\overline{\alpha} \in \sigma_p(B)$. Hence $\sigma_p(CBC) \subset \sigma_p(B)^*$. Replacing B with CBC, we get that $\sigma_p(B) \subset \sigma_p(CBC)^*$. Thus $\sigma_p(CBC) = \sigma_p(B)^*$. According to (11), we obtain that

$$\begin{cases} \sigma_p(T) = \sigma_p(A) \cup \big(-\sigma_p(A^*)^* \big) \\ \sigma_p(T^*) = \sigma_p(A^*) \cup \big(-\sigma_p(A)^* \big), \end{cases}$$

which implies that $\sigma_p(T)^* \neq \sigma_p(T^*)$ by the given hypothesis. By [20, Lemma 4.1], we can draw the conclusion that T is not complex symmetric. □

The following example illuminates Corollary 4.5.

EXAMPLE 4.6. Let $A := S + \alpha$ for some nonzero $\alpha \in \mathbb{C}$ where S is a unilateral shift on $\mathscr{H}$. Since $\sigma_p(A) = \emptyset$ and $\sigma_p(A^*) = \sigma_p(S^* + \overline{\alpha})$ is the open disk of radius 1 centered at $\overline{\alpha}$, we have $\mathscr{E}_p(A) \neq -\mathscr{E}_p(A)$ where $\mathscr{E}_p(A)$ is given as in Corollary 4.5. Hence, it follows from Corollary 4.5 that the operator matrix $T = \begin{pmatrix} A & 0 \\ 0 & -CA^*C \end{pmatrix}$ is J-self-adjoint but not complex symmetric, where C is any conjugation on $\mathscr{H}$.

For $u \in L^\infty = L^\infty(\partial\mathbb{D}, m)$, the *Toeplitz operator* T_u is defined by

$$T_u h = P_+(uh) \text{ for } h \in H^2$$

where P_+ stands for the orthogonal projection of L^2 onto the Hardy space H^2. In the following theorem, we show that every J-self-adjoint Toeplitz operator has no eigenvalues.

THEOREM 4.7. Let $u \in L^\infty$ be nonconstant. If T_u is J-self-adjoint, then the following assertions hold:

(i) $\sigma_p(T_u) = \emptyset$; hence, both T_u and T_u^* have the single-valued extension property.
(ii) $\sigma(T_u) = \sigma_e(T_u)$.

Proof. (i) Since T_u^* is J^*-self-adjoint by Lemma 3.1 and the single-valued extension property holds for each operator in $\mathcal{L}(\mathcal{H})$ whose point spectrum has empty interior (see [27, page 15]), it is enough to prove that $\sigma_p(T_u) = \emptyset$. We want to show that $\sigma_p(T_u^2) = \sigma_p(T_{u^2}) = \emptyset$, which yields that $\sigma_p(T_u) = \emptyset$ by the spectral mapping theorem. If $\ker(T_u^2 - \alpha) \neq \{0\}$ for some $\alpha \in \mathbb{C}$, then $\ker(T_u^{*2} - \overline{\alpha}) \neq \{0\}$ by Corollary 3.16, which contradicts to the Coburn alternative theorem. Hence, we have that $\ker(T_u^2 - \alpha) = \{0\}$ for all $\alpha \in \mathbb{C}$, meaning that $\sigma_p(T_u^2) = \emptyset$. Since $\sigma_p(T_u) = \emptyset$ by the spectral mapping theorem, the Toeplitz operator T_u has the single-valued extension property. Since $\sigma_p(T^*) = -\sigma_p(T)^* = \emptyset$, the adjoint T_u^* has the single-valued extension property, too.

(ii) Since T_u is J-self-adjoint and T_u^* is J^*-self-adjoint, it follows from (i) that $\sigma_p(T_u) = \sigma_p(T_u^*) = \emptyset$. This yields that

$$\sigma(T_u) = \sigma_e(T_u) \cup \sigma_p(T_u) \cup \sigma_p(T_u^*) = \sigma_e(T_u),$$

as we desired. □

From Theorem 4.7, we find skew-diagonal block Toeplitz operators with the single-valued extension property.

COROLLARY 4.8. Let u and v be nonconstant functions in L^∞. If T_u and T_v are commuting Toeplitz operators which are complex symmetric with the same conjugation, then $T = \begin{pmatrix} 0 & T_u \\ T_v & 0 \end{pmatrix}$ is a J-self-adjoint operator with the single-valued extension property and

$$\sigma(T) = \sigma_a(T) = -\sigma_a(T) = \bigcup\{\sigma_T(x) : x \in \mathcal{H}\} = \bigcup\{-\sigma_T(x) : x \in \mathcal{H}\}.$$

Proof. From Proposition 4.1, the block Toeplitz operator $T = \begin{pmatrix} 0 & T_u \\ T_v & 0 \end{pmatrix}$ is J-self-adjoint. We know from [27, Theorem 3.3.9] that if T^2 has the single-valued extension property, then so does T. Thus, we consider the square

$$T^2 = \begin{pmatrix} T_uT_v & 0 \\ 0 & T_vT_u \end{pmatrix}.$$

Since T_u and T_v commute, one of the following statements holds:

(i) both T_u and T_v are analytic;
(ii) both T_u and T_v are co-analytic;
(iii) there are $\alpha, \beta \in \mathbb{C}$, not both zero, such that $\alpha u + \beta v$ is constant on $\partial\mathbb{D}$.

If (i) holds, then $T_uT_v = T_{uv}$ is subnormal, which ensures from [25] that T^2 has the single-valued extension property. If (ii) happens, then T^{2*} has property (β) by [25], and so is T^2 due to Theorem 3.8. Since property (β) guarantees the single-valued extension property, the square T^2 has the single-valued extension property. Suppose that (iii) holds, and set $\alpha u + \beta v \equiv \gamma$ on $\partial\mathbb{D}$. Here, we may assume that $\beta \neq 0$. Then

$$\sigma_p(T_uT_v) = \sigma_p(T_{uv}) = \sigma_p(T_{\frac{1}{\beta}u(\gamma - \alpha u)}) = q(\sigma_p(T_u))$$

where $q(\lambda) = \frac{1}{\beta}\lambda(\gamma - \alpha\lambda)$. Since u is a nonconstant function in L^∞ such that T_u is complex symmetric, it follows from Theorem 4.7 that $\sigma_p(T_u) = \emptyset$, and so we have $\sigma_p(T_u T_v) = q(\sigma_p(T_u)) = \emptyset$. Hence, $T_u T_v$ has the single-valued extension property, implying that T^2 has the single-valued extension property, and so does T as remarked above.

Since T and T^* have the single-valued extension property, [27, Proposition 1.3.2] yields that

$$\sigma(T) = \sigma_a(T) = \sigma_{su}(T) = \bigcup\{\sigma_T(x) : x \in \mathscr{H}\}.$$

In addition, we obtain from Corollary 3.16 that $\sigma_a(T) = -\sigma_{su}(T)$, which completes the proof. □

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