Operators
and
Matrices
Volume 15, Number 2 (2021), 659–675
doi:10.7153/oam-2021-15-44

RATIONAL ELIMINATION ALGORITHM AND APPLICATIONS

BÉCHIR DALI AND MOHAMED ELGHAOUI

(*Communicated by R. Vandebril*)

Abstract. Given a matrix $A \in \mathbb{R}^{m\times n}$, we develop an algorithm called the rational elimination algorithm which ressembles the algorithm elimination except that the pivot (the leading coefficient) is a sequence of real independent numbers over $\mathbb{Q}$. This algorithm is used to calculate the rank of A over $\mathbb{Q}$ and to seek rational solutions to any linear system $Ax = b$ with $b \in \mathbb{R}^m$. We also present a criterion for testing the density of additive subgroups of $\mathbb{R}^n$ which needs the rank of a certain matrix over $\mathbb{Q}$. Finally, we apply such algorithm for testing the regularity of an orbit under the linear continuous action of some subgroup of $GL(V)$ where V is finite dimensional real vector space.

1. Introduction

When solving a linear system $Ax = b$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, the existence of solution depends only on the ranks of the matrices A and $(A|b)$. These ranks can be easily obtained by using Gauss elimination method. In this method the matrix A is reduced to echelon row form. When both A and b have their entries in $\mathbb{Z}$, and integral solutions are sought, then the Hermite Normal Form (HNF) can be used ([8]). In theoretical computer science, the generalized multi-knapsack feasibility problem arises when we seek a binary solution to real coefficients linear system [6]. In this paper we are interested in solving real linear system in $\mathbb{Q}^n$, to this end an algorithm analogue to the usual Gauss elimination is presented. The main feature of this algorithm is that the leading coefficient (pivot) in the usual Gauss process is now a tuple of rationally independent entries.

We now briefly describe the contents of the paper. In Section 2 we present our algorithm for a matrix $A \in \mathbb{R}^{m\times n}$, and we generalize it in Section 3. In Section 4 we give applications of the algorithm of column reduction, more precisely we define the $\mathrm{rank}_{\mathbb{Q}}(A)$ as the dimension of the $\mathbb{Q}$-vector space spanned by the columns of A. Next we apply the algorithm of reduction to find rational solutions of a real linear system. In this Section we also apply the obtained results concerning the $\mathbb{Q}$-rank of a real matrix to test directly the density of additive subgroups of $\mathbb{R}^n$ (and $\mathbb{C}^n$) and then we give an explicit description of the hypercyclicity test of finitely generated abelian subgroups of $GL(n,\mathbb{C})$. Finally we apply our algorithm to the test of regularity of an orbit under

Mathematics subject classification (2020): 15A06, 15A24, 15B36, 11J72.
Keywords and phrases: Irrationality, linear dependence over a field, rational numbers, rational elimination algorithm.
This work is supported by the research unit: *Dynamical Systems and their Applications, UR17ES21*.

the action of a group $G = \exp\left(\sum_{j=1}^{k} \mathbb{R}M_j\right)$, where exp denotes the matrix exponential mapping and $M_1, \ldots, M_k$ are linearly independent pairwise commuting square real matrices acting on a finite dimensional real vector space V.

2. The rational elimination algorithm

Our goal in this section is to introduce an algorithm by which we obtain a column echelon rational form for any real entries matrix.

DEFINITION 2.1. A matrix $M \in \mathbb{R}^{m\times n}$ is in column echelon rational form if it satisfies the following conditions:

- All zero columns, if any, are at the rightmost of the matrix.
- There exist s positive integers $r_1, r_2, \ldots, r_s$ such that the first non zero entry in each of the columns $1 + \sum_{k=1}^{i-1} r_k, \ldots, \sum_{k=1}^{i} r_k$, for $2 \leqslant i \leqslant s$, are on the same row and form a sequence of real independent numbers over $\mathbb{Q}$ called the leading sequence and r_i its length.
- Each leading sequence is in a row strictly below the previous leading sequence above it.

A similar definition holds for a matrix in row echelon rational form.

As an example, the following matrix

$$\begin{pmatrix} \sqrt{2} & \pi & 1+\sqrt{2} & 0 & 0 & 0 & 0 \\ 1 & 2 & -\pi & 0 & 0 & 0 & 0 \\ 4 & \sqrt{3} & -17 & 2 & \sqrt{3} & 0 & 0 \\ \pi & -5 & 6 & -11 & e^{\pi} & 0 & 0 \\ -1 & 3 & \sqrt{7} & 11 & 13 & 0 & 0 \end{pmatrix}$$

is in column echelon rational form where:

$$r_1 = 3, \quad r_2 = 2.$$

So let $A = (a_{i,j})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$ be a non-zero matrix (with real entries). Without loss of generality we may assume that the first row of A is non zero and choose $a_{1,j_1}, \ldots, a_{1,j_{r_1}}$ to be a longest sequence extracted from the list $\{a_{1,1}, \ldots, a_{1,n}\}$ such that its elements are independent over $\mathbb{Q}$. Set $I_1 := \{j_1, \ldots, j_{r_1}\}$.

Then we move the columns $(C_j)_{j\in I_1}$ to the leftmost of the matrix. In matrix form, $A^{(0)} := A$ is transformed to:

$$A^{(0)} \to A^{(0)}P^{(1)} := \widetilde{A}^{(1)} = (\tilde{a}_{i,j}),$$

where $P^{(1)}$ is a square permutation matrix.

Now, we write the scalars $\tilde{a}_{1,j}$ for every $j = 1+r_1,\ldots,n$ as a rational linear combination of the real numbers $\{\tilde{a}_{1,k},\ k = 1,\ldots,r_1\}$, *i.e*;

$$\tilde{a}_{1,j} = \sum_{k=1}^{r_1} \gamma_{j,k}^{(1)}\, \tilde{a}_{1,k};$$

where $\gamma_{j,1}^{(1)},\ldots,\gamma_{j,r_1}^{(1)} \in \mathbb{Q}$.

The idea is now to reduce to zero the entries $\tilde{a}_{1,j},\ j = 1+r_1,\ldots,n$ using the columns elementary operations:

$$C_j \leftarrow C_j - \sum_{k=1}^{r_1} \gamma_{j,k}^{(1)}\, C_k, \qquad j = 1+r_1,\ldots,n.$$

We can translate matricially the whole step as:

$$A^{(0)} \to A^{(0)}P^{(1)} \to A^{(0)}P^{(1)}Q^{(1)} := A^{(1)} = (a_{i,j}^{(1)}),$$

where $Q^{(1)}$ is the square upper-triangular matrix:

$$Q^{(1)} = \begin{pmatrix} I_{r_1} & \begin{pmatrix} -\gamma_{1+r_1,1}^{(1)} & \cdots & -\gamma_{n,1}^{(1)} \\ \vdots & \ddots & \vdots \\ -\gamma_{1+r_1,r_1}^{(1)} & \cdots & -\gamma_{n,r_1}^{(1)} \end{pmatrix} \\ 0 & I_{n-r_1} \end{pmatrix}.$$

The next step is to consider the sub-matrix $(a_{ij}^{(1)})_{2\leqslant i\leqslant m,\ 1+r_1\leqslant j\leqslant n}$, of $A^{(1)}$. If this sub-matrix is zero or of order 1, the process stops otherwise we apply the same procedure as that done for $A^{(0)}$.

At the end of the algorithm, we end up with a matrix $A^{(s)}$ verifying only one of the following statements:

- $r_1 + r_2 + \ldots + r_s = n$.

- $r_1 + r_2 + \ldots + r_s < n$ and $C_j^{(s)} = 0$ for every $j = 1 + \sum_{k=1}^{s} r_k,\ \ldots\ ,n$

where $C_j^{(s)}$ is the jth column of the matrix $A^{(s)}$.

Finally, any encountered zero column has to be moved to the rightmost of the processed matrix. This column permutation should be included in the matrix $P^{(k)}$ of the treated step.

The algorithm produces a matrix U in column echelon rational form. Actually, we have:

THEOREM 2.2. *Given a matrix $A \in \mathbb{R}^{m\times n}$ then there exists $Q \in GL(n,\mathbb{Q})$ such that $U := AQ \in \mathbb{R}^{m\times n}$ is in column echelon rational form.*

The fact that the matrix Q has rational entries is very important as we will see later. Moreover such a matrix is not unique.

REMARK 2.3. Our reduction can be treated in an algorithmic process. Indeed, let $A = (a_{i,j})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$ and set

$$i_1 = \min\{1 \leqslant i \leqslant m : i^{th} \text{ row of } A \text{ is nonzero}\}.$$

For $k = 1,\dots,n$ we let $V_{i_1}^{(k)} = \mathbb{Q}$-span of $(a_{i_1,j})_{1\leqslant j\leqslant k}$, the sequence $(V_{i_1}^{(k)})_{1\leqslant k\leqslant n}$ is nondecreasing. Then we define inductively the sequence $(j_k^{(1)})_{1\leqslant k\leqslant n}$ by

$$j_1^{(1)} = \min\{1 \leqslant j \leqslant n : a_{i_1,j} \neq 0\},$$

$$j_k^{(1)} = \min\{j_{k-1}^{(1)} \leqslant j \leqslant n : \dim_{\mathbb{Q}} V_{i_1}^{(j)} > \dim_{\mathbb{Q}} V_{i_1}^{(j_{k-1}^{(1)})}\},$$

Set $r_1 = \dim_{\mathbb{Q}} V_{i_1}^{(n)}$ and $J_1 = \{j_1^{(1)},\dots,j_{r_1}^{(1)}\}$, then for each $j \notin J_1$ one has a unique decomposition

$$a_{i_1,j} = \sum_{k=1}^{r_1} \gamma_{j,k}^{(1)} a_{i_1,j_k^{(1)}}, \quad (\gamma_{j,k}^{(1)})_{1\leqslant k\leqslant r_1} \subset \mathbb{Q},$$

and then we apply the following column operation

$$C_j \leftarrow C_j - \sum_{k=1}^{r_1} \gamma_{j,k}^{(1)} C_{j_k}, \quad j \notin J_1.$$

Finally we consider a permutation by which the columns $C_{j_1},\dots,C_{j_{r_1}}$ are moved to the left. Let $A^{(1)}$ be the resulting matrix (obtained after this column reduction and permutation on A). Then the matrix $A^{(1)}$ satisfies in particular:

$$(a_{i_1,j}^{(1)})_{1\leqslant j\leqslant r_1} \quad \text{are} \quad \mathbb{Q}\text{-linearly independent and for } j \geqslant 1 + r_1, \quad a_{i_1,j}^{(1)} = 0.$$

If $(a_{i,j}^{(1)})_{1+i_1\leqslant i\leqslant m, 1+r_1\leqslant j\leqslant n}$ is the zero matrix or of order 1, then the algorithm stops, otherwise we move to the next step which is analog to the preceding one except that the modifications should be done on the sub-matrix $(a_{i,j}^{(1)})_{1+i_1\leqslant i\leqslant m\,,1+r_1\leqslant j\leqslant n}$ of $A^{(1)}$.

REMARK 2.4. Note that if the entries of the matrix A are all rational then our process is the usual Gauss column reduction with $r_i = 1$.

3. The generalized elimination algorithm

In this section we will present a generalization of the algorithm introduced in the previous section.

Let α be a real algebraic number of order p *i.e*; p is the largest positive integer such that $1,\alpha,\dots,\alpha^{p-1}$ are rationally independent, then $\mathbb{Q}[\alpha]$ is a subfield of $\mathbb{R}$.

The following lemma characterises the independence of a finite set of real numbers over $Q[\alpha]$.

LEMMA 3.1. *Let* $(x_i)_{1\leqslant i\leqslant N}$ *be a set of* N *real numbers. Then* $(x_i)_{1\leqslant i\leqslant N}$ *are independent over* $\mathbb{Q}[\alpha]$ *if and only if* $(x_i\alpha^j)_{1\leqslant i\leqslant N, 0\leqslant j\leqslant p-1}$ *are rationally independent.*

To define a matrix $A\in\mathbb{R}^{m\times n}$ as being in $Q[\alpha]$-echelon form, we generalize Definition (2.1) by replacing the field $\mathbb{Q}$ by $\mathbb{Q}[\alpha]$. The steps of the algorithm of reduction over $\mathbb{Q}$ is then extended to $\mathbb{Q}[\alpha]$. For instance the entries of the leading sequence are independent over $\mathbb{Q}[\alpha]$. An analog of Theorem (2.2) is now stated:

THEOREM 3.2. *Given a matrix* $A\in\mathbb{R}^{m\times n}$ *then there exists* $Q\in GL(n,\mathbb{Q}[\alpha])$ *such that* $U := AQ\in\mathbb{R}^{m\times n}$ *is in* $\mathbb{Q}[\alpha]$*-echelon form.*

EXAMPLE 3.3. Let $A=\begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} & \sqrt{2}+\sqrt{3} \\ \sqrt{6} & \sqrt{3} & -\sqrt{2} & \sqrt{3}\end{pmatrix}$.
If the reduction is done in $\mathbb{Q}$ then:

$$U=\begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} & 0 \\ \sqrt{6} & \sqrt{3} & -\sqrt{2} & \sqrt{2}\end{pmatrix} \quad \text{and} \quad Q=\begin{pmatrix} 1&0&0&0\\ 0&1&0&-1\\ 0&0&1&-1\\ 0&0&0&1\end{pmatrix}.$$

However in $\mathbb{Q}[\sqrt{3}]$, we get:

$$U=\begin{pmatrix} 1 & \sqrt{2} & 0 & 0 \\ \sqrt{6} & \sqrt{3} & -4\sqrt{2} & 0\end{pmatrix} \quad \text{and} \quad Q=\begin{pmatrix} 1&0&-\sqrt{3}&-\frac{\sqrt{3}}{4}\\ 0&1&0&-1\\ 0&0&1&-\frac{3}{4}\\ 0&0&0&1\end{pmatrix}.$$

And finally, in $\mathbb{Q}[\sqrt{2}+\sqrt{3}]$:

$$U=\begin{pmatrix} 1 & 0 & 0 & 0 \\ \sqrt{6} & -\sqrt{3} & 0 & 0\end{pmatrix} \quad \text{and} \quad Q=\begin{pmatrix} 1&-\sqrt{2}&\frac{5\sqrt{3}}{3}&\sqrt{3}\\ 0&1&-\frac{4\sqrt{6}}{3}&-1-\sqrt{6}\\ 0&0&1&0\\ 0&0&0&1\end{pmatrix}.$$

We can easily show that $\sqrt{2},\sqrt{3}\in\mathbb{Q}[\sqrt{2}+\sqrt{3}]$.

4. Applications

4.1. Rank of a matrix over $\mathbb{Q}$

Let us first define the rank of a matrix over $\mathbb{Q}$:

DEFINITION 4.1. The rank of a matrix A over $\mathbb{Q}$ denoted by $\operatorname{rank}_{\mathbb{Q}}(A)$ is the maximal number of rationally linearly independent columns of A, *i.e.*

$$\operatorname{rank}_{\mathbb{Q}}(A)=\dim_{\mathbb{Q}}(\operatorname{colsp}(A)).$$

The reason behind choosing the column definition for the rank is that, unlike the real case, the dimension over $\mathbb{Q}$ of the vector space spanned by the rows of A may be different from the dimension over $\mathbb{Q}$ of the vector space spanned by the columns of A. Take for example the matrix $A = \begin{pmatrix} 1 & \sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix}$; where $\dim_{\mathbb{Q}}(\text{rowsp}(A)) = 1$ whereas $\dim_{\mathbb{Q}}(\text{colsp}(A)) = 2$.

Let $A = (a_{i,j}) \in \mathbb{R}^{m\times n}$ be a matrix in column echelon rational form having s leading sequences carried by the rows $i_1, i_2, \ldots, i_s$ and respectively of length $r_1, r_2, \ldots, r_s$. Let's denote $e_j = (\delta_{ij})_{1\leqslant i\leqslant n} \in \mathbb{R}^n$ so the j^{th} column of A is equal to Ae_j.

With these notations, we have:

LEMMA 4.2. *$colsp_{\mathbb{Q}}(A) = \bigoplus_{j=1}^{\ell} \mathbb{Q}Ae_j$ where $\ell = r_1 + r_2 + \ldots + r_s$.*

Proof. First it is obvious that $\text{colsp}_{\mathbb{Q}}(A) = \sum_{j=1}^{\ell} \mathbb{Q}Ae_j$.

Now if $(\star)$: $\sum_{j=1}^{\ell} p_j Ae_j = 0$ with $p_j \in \mathbb{Q}$, then by identifying the i_1^{th} component of each term, we get $\sum_{j=1}^{r_1} p_j a_{i_1,j} = 0$. As $(a_{i_1,j})_{1\leqslant j\leqslant r_1}$ are the terms of a leading sequence so they are rationally independent. Therefore $p_j = 0$ for every $j = 1, \ldots, r_1$ and $(\star)$ becomes $\sum_{j=1+r_1}^{\ell} p_j Ae_j = 0$.

If we keep on identifying each time the i^{th} component of each term in $(\star)$ for $i = i_2, \ldots, i_s$ while bearing in mind the stairwise property of the leading sequences and observing that the part of columns above any given leading sequence are zero, then we end up having $p_j = 0$ for every $j = 1, \ldots, \ell$ so that $(Ae_j)_{1\leqslant j\leqslant \ell}$ is a $\mathbb{Q}$-basis of $\text{colsp}_{\mathbb{Q}}(A)$. $\square$

COROLLARY 4.3. *For any matrix $A \in \mathbb{R}^{m\times n}$:*

$$rank_{\mathbb{Q}}(A) = \sum_{i=1}^{s} r_i$$

where s is the number of the leading sequences of the matrix obtained after transforming A to column echelon rational form and r_i is the length of i^{th} leading sequence.

Proof. There exists a matrix $Q \in GL(n,\mathbb{Q})$ such that $U = AQ$ is in column echelon rational form, so $colsp_{\mathbb{Q}}(A) = colsp_{\mathbb{Q}}(U)$ and hence $\text{rank}_{\mathbb{Q}}(A) = \text{rank}_{\mathbb{Q}}(U)$. By Lemma (4.2), $\text{rank}_{\mathbb{Q}}(U) = \sum_{i=1}^{s} r_i$. Therefore $\text{rank}_{\mathbb{Q}}(A) = \sum_{i=1}^{s} r_i$. $\square$

A final result is now presented which will be used in the last application concerning the regularity of orbits under linear action. But first, we need the following lemma:

LEMMA 4.4. *Let* $A \in \mathbb{R}^{m\times n}$ *and* $R \in GL(m,\mathbb{R})$ *then*

$$\dim colsp_{\mathbb{Q}}(A) = \dim colsp_{\mathbb{Q}}(RA).$$

Proof. The i^{th} column of RA is $(RA)e_i = R(Ae_i)$ so that $colsp_{\mathbb{Q}}(RA) = R\,colsp_{\mathbb{Q}}(A)$. Hence $\text{rank}_{\mathbb{Q}}(A) = \text{rank}_{\mathbb{Q}}(RA)$ since $R \in GL(m,\mathbb{R})$ *i.e.* R induces a $\mathbb{Q}$-isomorphism between $colsp_{\mathbb{Q}}(A)$ and $colsp_{\mathbb{Q}}(RA)$. □

PROPOSITION 4.5. *Let* $A \in \mathbb{R}^{m\times n}$ *then* $rank_{\mathbb{R}}(A) = rank_{\mathbb{Q}}(A) := r$ *if and only if there exist* $P \in GL(m,\mathbb{R})$ *and* $R \in GL(n,\mathbb{Q})$ *such that:*

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} R.$$

Proof. There exists a matrix $Q \in GL(n,\mathbb{Q})$ such that $U = AQ$ is in column echelon rational form. If $\text{rank}_{\mathbb{R}}(A) = \text{rank}_{\mathbb{Q}}(A) := r$ then the first r columns of U are also independent over $\mathbb{R}$ (the remaining columns if any are zero). Hence there exists $P \in GL(m,\mathbb{R})$ so that $U = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ and so $A = P\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}$.

Conversely, suppose there exist $P \in GL(m,\mathbb{R})$ and $R \in GL(n,\mathbb{Q})$ such that:

$$A = P \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} R.$$

By Lemma (4.4) and using the fact that $P^{-1}A \in \mathbb{Q}^{m\times n}$, we can write

$$\text{rank}_{\mathbb{Q}}(A) = \text{rank}_{\mathbb{Q}}(P^{-1}A) = \text{rank}_{\mathbb{R}}(P^{-1}A) = \text{rank}_{\mathbb{R}}(A) = r. \quad \square$$

4.2. Solving a real linear system over $\mathbb{Q}$

Let $AX = B$ be a linear system where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^m$. We present an algorithm to solve this system over the rationals, *i.e.* $X \in \mathbb{Q}^n$.

4.2.1. *Case when the system is homogeneous*

Let us first consider the case when A is already in echelon rational form and set $\ell = \text{rank}_{\mathbb{Q}}(A)$. By Lemma (4.2), the first ℓ columns are rationally independent (the rest of the columns if any are zero) so the system $AX = 0$ leads to $x_i = 0$ for $i = 1,2,\dots,\ell$.

Therefore the solution is:

1. $X = 0$ if $\ell = n$.
2. $X = [0,\dots,0,t_{\ell+1},\dots,t_n]^T$ with $t_{\ell+1},\dots,t_n \in \mathbb{Q}$ if $\ell < n$.

In the case when the matrix A is not in column echelon rational form, then there exists a matrix $Q \in GL(n,\mathbb{Q})$ such that $AQ := U$ is in echelon rational form. So if we set $Y = Q^{-1}X \in \mathbb{Q}^n$ then the system $AX = 0$ is transformed to the system $UY = 0$ which can be solved as described in the last paragraph. More precisely the rational solutions set of the system $AX = 0$ is $\sum_{j=\ell+1}^{n} \mathbb{Q}(Qe_j)$.

LEMMA 4.6. *Let $AX = 0$ be a homogeneous linear system with $A \in \mathbb{R}^{m\times n}$ then the system admits the trivial solution over $\mathbb{Q}$ if and only if:*

$$rank_{\mathbb{Q}}(A) = n$$

4.2.2. The general case

As before, there exists a matrix $Q \in GL(n,\mathbb{Q})$ such that $AQ = U$ is in echelon rational form. So if we set $Y = Q^{-1}X \in \mathbb{Q}^n$ then the system $AX = B$ is transformed to $UY = B$.

Assume that the matrix U have s leading sequences carried by the rows $i_1, i_2, \ldots, i_s$ and respectively of length $r_1, r_2, \ldots, r_s$ and define the integers $\tilde{r}_i$ $(1 \leqslant i \leqslant n)$ such that:

$$\begin{cases} \tilde{r}_{i_k} = r_k & \text{for every} \quad k = 1,2,\ldots,s \\ \tilde{r}_i \ = 0 & \text{if} \quad i \notin \{i_1, i_2, \ldots, i_s\} \end{cases}$$

Then the first equation of the system $UY = B$ reads:

$$\sum_{j=1}^{\tilde{r}_1} u_{1,j} y_j = b_1$$

If $\tilde{r}_1 = 0$ then either $b_1 = 0$ and so we proceed to the next equation, or $b_1 \neq 0$ and the resolution stops with no solution found (inconsistent system over $\mathbb{Q}$).

If $\tilde{r}_1 \neq 0$ then either b_1 is in the $\text{span}_{\mathbb{Q}}(u_{1,1}, \ldots, u_{1,\tilde{r}_1})$ and so we choose $y_1, \ldots, y_{\tilde{r}_1}$ are the coordinates of b_1 in the $\mathbb{Q}$-basis $(u_{1,1}, \ldots, u_{1,\tilde{r}_1})$ of this span (these numbers are independent over $\mathbb{Q}$ by construction) or otherwise the resolution stops with no solution found. This kind of inconsistency is not observed in the case the solutions are sought in $\mathbb{R}$.

Now the i^{th} equation once it is reached during the resolution can be written as:

$$\sum_{j=1+s_{i-1}}^{s_i} u_{i,j} y_j = b_i - \sum_{j=1}^{s_{i-1}} u_{i,j} y_j$$

where $s_i = \tilde{r}_1 + \tilde{r}_2 + \ldots + \tilde{r}_i$.

At this stage of the resolution, the sum in the right hand side is known and so we can apply the same argument as before.

As expected, three cases can be encountered when we solve a linear system of equations over $\mathbb{Q}$:

Case 1: The system has infinitely many solutions of the form $X_p + \mathscr{V}$; where

$$\mathscr{V} \in \{X \in \mathbb{Q}^n \mid AX = 0\}$$

and X_p is a particular rational solution to $AX = B$.

Case 2: The system has a unique solution when it is consistent and $\text{rank}_{\mathbb{Q}}(A) = n$.

Case 3: The system has no solution.

4.3. Hypercyclicity of finitely generated abelian subgroups of $GL(n,\mathbb{C})$

4.3.1. Density of additive subgroups of $\mathbb{R}^n$ and $\mathbb{C}^n$

The following proposition characterizes the density of additive subgroups $\mathbb{Z}u_1 + \ldots + \mathbb{Z}u_p$ of $\mathbb{R}^n$:

PROPOSITION 4.7. ([7], Proposition (4.3), Chapter II) *Let $H = \mathbb{Z}u_1 + \ldots + \mathbb{Z}u_p$ with $u_k \in \mathbb{R}^n$, $k = 1,\ldots,p$. Then H is dense in $\mathbb{R}^n$ if and only if for every $(s_1,\ldots,s_p) \in \mathbb{Z}^p \setminus \{0\}$:*

$$\mathrm{rank}_{\mathbb{R}} \begin{pmatrix} u_1 & \ldots & \ldots & u_p \\ s_1 & \ldots & \ldots & s_p \end{pmatrix} = n+1.$$

As explained in [4], we assume that $p \geqslant n+1$ and $\sum_{k=1}^{p} \mathbb{R}u_k = \mathbb{R}^n$, otherwise H is not dense in $\mathbb{R}^n$. To be able to apply the results of this paper, we need to proceed differently from [4].

THEOREM 4.8. *Let $H = \mathbb{Z}u_1 + \ldots + \mathbb{Z}u_p$ and assume that $p \geqslant n+1$ and $(u_1,\ldots,u_n)$ is a basis of $\mathbb{R}^n$. If $u_k = \sum_{i=1}^{n} \alpha_{k,i} u_i$, for every $k \geqslant n+1$, then H is dense in $\mathbb{R}^n$ if and only if*

$$rank_{\mathbb{Q}} \left(I_{p-n} \quad \begin{pmatrix} \alpha_{n+1,1} & \ldots & \alpha_{n+1,n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \alpha_{p,1} & \ldots & \alpha_{p,n} \end{pmatrix} \right) = p.$$

Proof. The rank condition in Proposition (4.7) can be written as:

$$\mathrm{rank}_{\mathbb{R}} \begin{pmatrix} 1 & 0 & \ldots & 0 & \alpha_{n+1,1} & \ldots & \alpha_{p,1} \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & \alpha_{n+1,n} & \ldots & \alpha_{p,n} \\ s_1 & \ldots & \ldots & s_n & s_{n+1} & \ldots & s_p \end{pmatrix} = n+1, \tag{4.1}$$

which, by elementary row operations, simplifies to:

$$\mathrm{rank}_{\mathbb{R}} \begin{pmatrix} 1 & 0 & \ldots & 0 & \alpha_{n+1,1} & \ldots & \alpha_{p,1} \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & \alpha_{n+1,n} & \ldots & \alpha_{p,n} \\ 0 & \ldots & \ldots & 0 & s_{n+1} - \sum_{i=1}^{n} s_i \alpha_{n+1,i} & \ldots & s_p - \sum_{i=1}^{n} s_i \alpha_{p,i} \end{pmatrix} = n+1,$$

but this condition is true for every $(s_1,\dots,s_p)\in\mathbb{Z}^p\setminus\{0\}$ if and only if the linear system of equations in the variables $s_1,\dots,s_p$:

$$s_k-\sum_{i=1}^{n}s_i\alpha_{k,i}=0;\quad k=n+1,\dots,p$$

also written in matrix form as

$$\begin{pmatrix} -\alpha_{n+1,1} & \dots & -\alpha_{n+1,n} & 1 & 0 & \dots & 0\\ \vdots & & \vdots & 0 & \ddots & \ddots & \vdots\\ \vdots & & \vdots & \vdots & \ddots & \ddots & 0\\ -\alpha_{p,1} & \dots & -\alpha_{p,n} & 0 & \dots & 0 & 1 \end{pmatrix}\begin{pmatrix} s_1\\ \vdots\\ \vdots\\ s_p\end{pmatrix}=0$$

has no solution in $\mathbb{Z}^p$ other than the trivial solution. As the system is homogeneous, the last claim is also true (equivalent) when we replace $\mathbb{Z}^p$ by $\mathbb{Q}^p$. Then by applying Lemma(4.6) and swapping the columns, we get the required result (once we scale the columns containing the α's terms by -1). $\square$

REMARK 4.9. We note that when $p=n+1$, we recover the well-known result (see for instance [5, 4, 7]): $H=\mathbb{Z}u_1+\dots+\mathbb{Z}u_{n+1}$, with $(u_1,\dots,u_n)$ a basis of $\mathbb{R}^n$ and $u_{n+1}=\sum\limits_{i=1}^{n}\theta_i u_i$, is dense in $\mathbb{R}^n$ if and only if $(1,\theta_1,\dots,\theta_n)$ are rationally independent or equivalently $\operatorname{rank}_{\mathbb{Q}}\begin{pmatrix}1 & \theta_1 & \dots & \theta_n\end{pmatrix} = n+1=p$.

REMARK 4.10. In the same way, when $n=1$ and $p\geqslant 2$, we get the following classical result: $H=a_1\mathbb{Z}+\dots+a_p\mathbb{Z}$ with $a_1\neq 0$, is dense in $\mathbb{R}$ if and only if there exists $k\in\{2,\dots,p\}$ such that $\dfrac{a_k}{a_1}$ is irrational.

Indeed, in this case $\alpha_{k,1}=a_k/a_1$ for every $k=2,\dots,p$ and the matrix in Theorem (4.8) becomes:

$$A:=\begin{pmatrix} 1 & 0 & \dots & 0 & \frac{a_2}{a_1}\\ 0 & \ddots & \ddots & \vdots & \vdots\\ \vdots & \ddots & \ddots & 0 & \vdots\\ 0 & \dots & 0 & 1 & \frac{a_p}{a_1}\end{pmatrix}.$$

It is clear that $r_i\in\{1,2\}$ for every $i\in\{1,\dots,p-1\}$, so $\operatorname{rank}_{\mathbb{Q}}(A)=p$ if and only if there exists $i_0\in\{1,\dots,p-1\}$ such that $r_{i_0}=2$ or equivalently $\dfrac{a_{1+i_0}}{a_1}$ is irrational.

REMARK 4.11. Let $H=\mathbb{Z}u_1+\dots+\mathbb{Z}u_p$ with $u_k\in\mathbb{C}^n$.

To test the density of H in $\mathbb{C}^n$, we proceed as follows: we define (by realification) the real additive subgroup $\widetilde{H}\subset\mathbb{R}^{2n}$ as $\widetilde{H}=\mathbb{Z}\tilde{u_1}+\dots+\mathbb{Z}\tilde{u_p}$ where $\tilde{u_k}=(\Re(u_k),\,\Im(u_k))^T$ for every $k=1,\dots,p$. Then H is dense in $\mathbb{C}^n$ if and only if $\widetilde{H}$ is dense in $\mathbb{R}^{2n}$. Finally, we use Theorem (4.8) to test the density of $\widetilde{H}$ in $\mathbb{R}^{2n}$.

EXAMPLE 4.12. The following example is taken from ([4], Example 3.6.).

Let $H = \mathbb{Z}u_1 + \ldots + \mathbb{Z}u_7$, where $u_1 = (1,0,0)^T$, $u_2 = (0,1,0)^T$, $u_3 = (0,0,1)^T$, $u_4 = \left(1,\sqrt{2},1\right)^T$, $u_5 = \left(0,1,\sqrt{3}\right)^T$, $u_6 = \left(\sqrt{2},\sqrt{3},1\right)^T$, $u_7 = \left(1,\sqrt{2},\sqrt{2}\right)^T$.

Here $n = 3$ and $p = 7$ and the matrix in Theorem (4.8) is given by:

$$A := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & \sqrt{2} & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & \sqrt{3} \\ 0 & 0 & 1 & 0 & \sqrt{2} & \sqrt{3} & 1 \\ 0 & 0 & 0 & 1 & 1 & \sqrt{2} & \sqrt{2} \end{pmatrix}.$$

Transforming A to column rational echelon form:

$$\widetilde{A} = \begin{pmatrix} 1 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 1 & 1 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{2} & 0 & 1 & 1 \end{pmatrix}.$$

We have:

$$r_1 = r_2 = r_3 = 2 \quad \text{and} \quad r_4 = 1.$$

So $\mathrm{rank}_{\mathbb{Q}}(A) = 7$ and by Theorem (4.8), H is dense in $\mathbb{R}^3$.

4.3.2. Hypercyclicity test

Let $r \in \mathbb{N}_0 := \mathbb{N} \setminus \{0\}$ and $\eta = (n_1, \ldots, n_r) \in \mathbb{N}_0^r$ such that $\sum_{i=1}^{r} n_i = n$. Then denote

- $\mathcal{K}^*_{\eta,r}(\mathbb{C}) = \left\{\mathrm{diag}(T_1, \ldots, T_r) \in M_n(\mathbb{C}) : \ T_k \in \mathbb{T}_{n_k}(\mathbb{C}),\ k = 1, \ldots, r\right\} \cap GL(n,\mathbb{C})$, where $\mathbb{T}_n(\mathbb{C})$ is the set of all lower-triangular matrices over $\mathbb{C}$, of order n and with only one eigenvalue.
- $u_0 = (e_{1,1}, \ldots, e_{r,1})^T \in \mathbb{C}^n$, where $e_{k,1} = (1,0,\ldots,0)^T \in \mathbb{C}^{n_k}$, $\quad 1 \leqslant k \leqslant r$.
- $e^{(k)} = \left(0_{\mathbb{C}^{n_1}}, \ldots, 0_{\mathbb{C}^{n_{k-1}}}, e^T_{k,1}, 0_{\mathbb{C}^{n_{k+1}}}, \ldots, 0_{\mathbb{C}^{n_r}}\right)^T \in \mathbb{C}^n, \quad 1 \leqslant k \leqslant r.$

In this section, we shall study the notion of hyperciclicity, to this end we first recall the following

DEFINITION 4.13. Let G be an abelian subgroup of $GL(n,\mathbb{C})$. We say that G is hypercyclic if there exists a vector $x \in \mathbb{C}^n$ such that the orbit of x under the action of G is dense in $\mathbb{C}^n$.

In [1], the authors proved the following

PROPOSITION 4.14. ([1], Proposition 6.1.) *Let G be an abelian subgroup of $GL(n,\mathbb{C})$, then there exists $P \in GL(n,\mathbb{C})$ such that $\widetilde{G} = P^{-1}GP$ is a subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$, for some $1 \leqslant r \leqslant n$ and $\eta \in \mathbb{N}_0^r$.*

THEOREM 4.15. ([1], equivalent version of Theorem 1.3) *Let G be an abelian subgroup of $GL(n,\mathbb{C})$ and $P \in GL(n,\mathbb{C})$ such that $P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$. Assume that G is generated by $A_1,\dots,A_p$. Then G is hypercyclic if and only if $\sum_{k=1}^{p}\mathbb{Z}B_k u_0 + 2i\pi\sum_{k=1}^{r}\mathbb{Z}e^{(k)}$ is a dense additive subgroup of $\mathbb{C}^n$, where $P^{-1}A_1P = e^{B_1},\dots,P^{-1}A_pP = e^{B_p}$ with $B_1,\dots,B_p \in \mathcal{K}^*_{\eta,r}(\mathbb{C})$.*

COROLLARY 4.16. *Let G be an abelian subgroup of $GL(n,\mathbb{C})$, generated by $A_1,\dots,A_p$ and $P \in GL(n,\mathbb{C})$ such that $P^{-1}GP \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$. If $p+r \leqslant 2n$, then G has no dense orbit.*

So let G be an abelian subgroup of $GL(n,\mathbb{C})$ generated by $A_1,\dots,A_p$.

In [5], the authors gave a complete algorithm to test the hypercyclicity of finitely generated abelian subgroups of $GL(n,\mathbb{C})$. In this paper, this algorithm is only used to get the expression of the complex additive subgroup $H(G) := \sum_{k=1}^{p}\mathbb{Z}B_k u_0 + 2i\pi\sum_{k=1}^{r}\mathbb{Z}e^{(k)}$ stated in Theorem (4.15). To test the density of $H(G)$ in $\mathbb{C}^n$ and subsequently the hypercyclicity of G, we refer to Remark (4.11) and Theorem (4.8).

The following example is taken from [5].

EXAMPLE 4.17. Let G be the subgroup of $GL(3,\mathbb{C})$ generated by A_1, A_2, A_3, A_4, A_5 and A_6, where:

$$A_1 = \begin{pmatrix} e & 3-2e+i & -2+e-i \\ 0 & 2+i & -1-i \\ 0 & 1+i & -i \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & -2+2e^{\sqrt{2}} & 1-e^{\sqrt{2}} \\ 0 & e^{\sqrt{2}} & 0 \\ 0 & 0 & e^{\sqrt{2}} \end{pmatrix}$$

$$A_3 = \begin{pmatrix} e^{\sqrt{3}} & -2e^{\sqrt{3}}+2e^{i} & e^{\sqrt{3}}-e^{i} \\ 0 & e^{i} & 0 \\ 0 & 0 & e^{i} \end{pmatrix}$$

$$A_4 = \begin{pmatrix} e^{i\sqrt{5}} & \sqrt{2}(\sqrt{2}+i)e-2e^{i\sqrt{5}} & e^{i\sqrt{5}}-(1+i\sqrt{2})e \\ 0 & (1+i\sqrt{2})e & -i\sqrt{2}e \\ 0 & i\sqrt{2}e & (1-i\sqrt{2})e \end{pmatrix}$$

$$A_5 = \begin{pmatrix} e & 2-2e+\sqrt{7}+i\sqrt{2} & e-1-\sqrt{7}-i\sqrt{2} \\ 0 & 1+\sqrt{7}+i\sqrt{2} & -\sqrt{7}-i\sqrt{2} \\ 0 & \sqrt{7}+i\sqrt{2} & 1-\sqrt{7}-i\sqrt{2} \end{pmatrix}$$

$$A_6 = \begin{pmatrix} 1 & i\sqrt{2} & -i\sqrt{2} \\ 0 & 1+i\sqrt{2} & -i\sqrt{2} \\ 0 & i\sqrt{2} & 1-i\sqrt{2} \end{pmatrix}.$$

Then the complex additive group $H(G) = \sum_{k=1}^{6}\mathbb{Z}B_k u_0 + 2\pi i\,\mathbb{Z}e_1 + 2\pi i\,\mathbb{Z}e_2$ where e_i is the i^{th} vector of the $\mathbb{C}^n$-standard basis and $u_0 = e_1+e_2$.

As shown in [5], $H(G) = \sum_{k=1}^{8} \mathbb{Z}u_k$ where

$$u_1 = \left(1, 0, \tfrac{1}{2} + \tfrac{1}{2}i\right)^T, \quad u_2 = \left(0, \sqrt{2}, 0\right)^T, \ u_3 = \left(\sqrt{3}, i, 0\right)^T, \ u_4 = \left(i\sqrt{5}, 1, \tfrac{\sqrt{2}}{2}i\right)^T,$$
$$u_5 = \left(1, 0, \tfrac{\sqrt{7}}{2} + \tfrac{\sqrt{2}}{2}i\right)^T, \ u_6 = \left(0, 0, \tfrac{\sqrt{2}}{2}i\right)^T, \ u_7 = (2\pi i, 0, 0)^T, \ u_8 = (0, 2\pi i, 0)^T$$

Therefore, the real additive group $\widetilde{H}(G) = \sum_{k=1}^{8} \mathbb{Z}u_k$ where:

$$\tilde{u}_1 = \left(1, 0, \tfrac{1}{2}, 0, 0, \tfrac{1}{2}\right)^T, \quad \tilde{u}_2 = \left(0, \sqrt{2}, 0, 0, 0, 0\right)^T, \quad \tilde{u}_3 = \left(\sqrt{3}, 0, 0, 0, 1, 0\right)^T,$$
$$\tilde{u}_4 = \left(0, 1, 0, \sqrt{5}, 0, \tfrac{\sqrt{2}}{2}\right)^T, \ \tilde{u}_5 = \left(1, 0, \tfrac{\sqrt{7}}{2}, 0, 0, \tfrac{\sqrt{2}}{2}\right)^T, \ \tilde{u}_6 = \left(0, 0, 0, 0, 0, \tfrac{\sqrt{2}}{2}\right)^T,$$
$$\tilde{u}_7 = (0, 0, 0, 2\pi, 0, 0)^T, \quad \tilde{u}_8 = (0, 0, 0, 0, 2\pi, 0)^T$$

The vectors $\tilde{u}_7$ and $\tilde{u}_8$ can be expressed in the basis $(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_6)$ as

$$\tilde{u}_7 = -\pi\frac{\sqrt{10}}{5}\tilde{u}_2 + 2\pi\frac{\sqrt{5}}{5}\tilde{u}_4 - 2\pi\frac{\sqrt{5}}{5}\tilde{u}_6$$
$$\tilde{u}_8 = -\pi\frac{7\sqrt{3}+\sqrt{21}}{3}\tilde{u}_1 + 2\pi\tilde{u}_3 + \pi\frac{\sqrt{3}+\sqrt{21}}{3}\tilde{u}_5 + \pi\frac{\sqrt{42}-2\sqrt{21}+7\sqrt{6}-2\sqrt{3}}{6}\tilde{u}_6$$

To test the density of $\widetilde{H}(G)$ in $\mathbb{R}^6$, we evaluate first the matrix in Theorem (4.8):

$$A := \begin{pmatrix} 1 & 0 & 0 & -\pi\frac{\sqrt{10}}{5} & 0 & 2\pi\frac{\sqrt{5}}{5} & 0 & -2\pi\frac{\sqrt{5}}{5} \\ 0 & 1 & -\pi\frac{7\sqrt{3}+\sqrt{21}}{3} & 0 & 2\pi & 0 & \pi\frac{\sqrt{3}+\sqrt{21}}{3} & \pi\frac{\sqrt{42}-2\sqrt{21}+7\sqrt{6}-2\sqrt{3}}{6} \end{pmatrix}.$$

Transforming A to column rational echelon form:

$$\widetilde{A} = \begin{pmatrix} 1 & -\pi\frac{\sqrt{10}}{5} & 2\pi\frac{\sqrt{5}}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\pi\frac{7\sqrt{3}+\sqrt{21}}{3} & 2\pi & \pi\frac{\sqrt{3}+\sqrt{21}}{3} & \pi\frac{\sqrt{42}-2\sqrt{21}+7\sqrt{6}-2\sqrt{3}}{6} \end{pmatrix}.$$

We have:

$$r_1 = 3 \quad \text{and} \quad r_2 = 5.$$

So $\operatorname{rank}_{\mathbb{Q}}(A) = 8$ and by Theorem (4.8), $\widetilde{H}(G)$ is dense in $\mathbb{R}^6$. Then we conclude that $H(G)$ is dense in $\mathbb{C}^3$ and finally G is hypercyclic.

4.4. Regularity of orbits under linear actions

DEFINITION 4.18. ([2]) Let G be a topological group acting continuously on a topological space, the orbit of a point is said to be regular if the relative topology of the orbit coincides with the quotient topology that the orbit carries as a homogeneous space.

As in [2, 3], we let

(i) V a finite dimensional real vector space with $\dim V = n$,

(ii) $M_1, \dots, M_k$ linearly independent pairwise commuting square matrices of order n and $G = \exp\left(\sum_{j=1}^{k} \mathbb{R} M_j\right)$, where $\exp$ denotes the matrix exponential mapping.

The group G is acting on V via

$$G \times V \longrightarrow V, \quad (g = \exp \sum_{j=1}^{k} t_j M_j, v) \longmapsto g.v = e^{\sum_{j=1}^{k} t_j M_j} v.$$

If Gv denotes the G-orbit of $v \in V$, Gv is in bijection with G/G_v, where

$$G_v = \{g \in G : g \cdot v = v\}$$

is the stability subgroup of v in G which is closed in G. On other hand, the quotient topology on G/G_v is Hausdorff and the orbit Gv is regular if the natural bijection $G/G_v \to Gv$ is a homeomorphism. Following [3], let $\mathfrak{g} = \sum_{j=1}^{k} \mathbb{R} M_j$ which is realized as a space of lower triangular complex matrices, simultaneously in block form, with the j-th block having scalar diagonal part given by a linear form

$$\lambda_j : \mathfrak{g} \to \mathbb{C}, \quad M \mapsto \lambda_j(M) = \alpha_j(M) + i\beta_j(M).$$

The functionnals $(\lambda_j)_{1 \leqslant j \leqslant k}$ are called the roots of $\mathfrak{g}$. For each matrix $M \in \mathfrak{g}$, let $n(M)$ be the nilpotent (strictly lower triangular) part of M, so we write the decomposition $M = d(M) + n(M)$ where $d(M)$ is diagonal and $n(M)$ is nilpotent (with $d(M)n(M) = n(M)d(M)$). Put

$$\mathfrak{g}_0 = \bigcap_{j=1}^{k} \ker \alpha_j.$$

That is $\mathfrak{g}_0$ is the subspace of $\mathfrak{g}$ consisting of those M whose eigenvalues have vanishing real part. Next, for each $v \in V$, let

$$\mathfrak{n}(v) = \{M \in \mathfrak{g}_0 : n(M)v = 0\}.$$

The subgroup $\exp \mathfrak{n}(v)$ acts on v by rotations, and regularity of the orbit Gv is reduced to regularity of $\exp \mathfrak{n}(v)v$ ([3], Theorem 3.6 p.8). It follows that Gv is regular if and only if

$$\dim_{\mathbb{Q}} span_{\mathbb{Q}}\{\lambda_{j_{|n(v)}}\} = \dim_{\mathbb{R}} span_{\mathbb{R}}\{\lambda_{j_{|n(v)}}\}. \tag{4.2}$$

For simplicity of notation, let $\{N_1, \dots, N_p\}$ be a basis in the $\mathbb{R}$-space $\mathfrak{n}(v)$ and $\{\lambda_1, \dots, \lambda_q\}$ be the set of roots of $\mathfrak{g}$, then one has

$$\lambda_r(N_s) = i\beta_r(N_s) \in i\mathbb{R} \quad (i^2 = -1), \;\; r = 1, \dots, q, \;\; s = 1, \dots, p.$$

Let $A = (a_{i,j})$ be the real matrix whose entries

$$a_{i,j} = \beta_j(N_i), \quad i = 1, \dots p, \;\; j = 1, \dots, n.$$

Then the regularity condition (4.2) is traduced as follows:

$$Gv \text{ is regular if and only if } \operatorname{rank}_{\mathbb{Q}}(A) = \operatorname{rank}_{\mathbb{R}}(A).$$

EXAMPLE 4.19. Let $V = \mathbb{R}^6 = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ and $\mathfrak{g} = \mathbb{R}M$ with

$$M = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -c \\ 0 & 0 & 0 & 0 & c & 0 \end{pmatrix}; \quad a,b,c \in \mathbb{R}.$$

The eigenvalues of M are $a, \pm ib, \pm ic$. We can easily check, that there is a non singular (complex) matrix P such that

$$PMP^{-1} = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \\ 1 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & ib & 0 & 0 & 0 \\ 0 & 0 & 0 & -ib & 0 & 0 \\ 0 & 0 & 0 & 0 & ic & 0 \\ 0 & 0 & 0 & 0 & 0 & -ic \end{pmatrix}.$$

The roots of $\mathfrak{g}$ are $\lambda_1, \lambda_2, \lambda_3 = \overline{\lambda_2}, \lambda_4, \lambda_5 = \overline{\lambda_4}$, with

$$\lambda_1(M) = a, \quad \lambda_2(M) = ib, \quad \lambda_4(M) = ic, \quad (i^2 = -1).$$

The only real root of $\mathfrak{g}$ is $\alpha_1(M) = a$. For any $v = (v_1, \dots, v_6) \in V$ the orbit Gv of v is given by

$$Gv = \{e^{tM}v, \quad t \in \mathbb{R}\} = \{(e^{at}v_1, e^{at}(tv_1 + v_2), e^{ibt}v_3, e^{-ibt}v_4, e^{ict}v_5, e^{-ict}v_6),\ t \in \mathbb{R}\}$$

Now assume that $a = 0$, then $\mathfrak{g}_0 = \mathfrak{g}$ and let $v = (0, v_2, v_3, v_4, v_5, v_6)$, then

$$\mathfrak{n}(v) = \{M \in \mathfrak{g}_0 : n(M)v = 0\} = \mathfrak{g} = \mathfrak{g}_0,$$

In this case the matrix A is

$$A = \begin{pmatrix} b & -b & c & -c \end{pmatrix}.$$

Consequently, Gv is regular if and only if b and c are rationally dependent.

EXAMPLE 4.20. In this example we shall consider the space $V = \mathbb{R}^7 = \mathbb{R}^3 \times \mathbb{C}^2$ and $\mathfrak{g} = \mathbb{R}M_1 \oplus \mathbb{R}M_2 \oplus \mathbb{R}M_3$ with

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ia & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -ia & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ib & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -ib \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +ic & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -ic & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & id & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -id \end{pmatrix},$$

and

$$M_3=\begin{pmatrix} 0\,0\,0 & 0 & 0 & 0 & 0\\ 0\,0\,0 & 0 & 0 & 0 & 0\\ 0\,0\,0 & 0 & 0 & 0 & 0\\ 0\,0\,0 & +ie & 0 & 0 & 0\\ 0\,0\,0 & 0 & -ie & 0 & 0\\ 0\,0\,0 & 0 & 0 & if & 0\\ 0\,0\,0 & 0 & 0 & 0 & -if \end{pmatrix}; \quad a,b,c,d,e,f\in\mathbb{R}.$$

It is clear that $\dim\mathfrak{g}=3$ and the roots of $\mathfrak{g}$ are given by

$$\lambda_1=0,\quad \begin{cases}\lambda_2(M_1)=ia\\ \lambda_2(M_2)=ic\\ \lambda_2(M_3)=ie\end{cases},\quad \lambda_3=\overline{\lambda_2},\quad \begin{cases}\lambda_4(M_1)=ib\\ \lambda_4(M_2)=id\\ \lambda_4(M_3)=if\end{cases},\quad \lambda_5=\overline{\lambda_4}.$$

On the other hand, for any $v=(v_1,\dots,v_7)\in V$ the orbit Gv is given by

$$Gv=\{e^{t_1M_1+t_2M_2+t_3M_3}v,\ t_1,t_2,t_3\in\mathbb{R}\}.$$

Now let $v=(0,0,v_3,v_4,v_5,v_6,v_7)$, hence

$$\mathfrak{n}(v)=\{M\in\mathfrak{g}_0 : n(M)v=0\}=\mathfrak{g}=\mathfrak{g}_0,$$

In this case the matrix A is

$$A=\begin{pmatrix} 0 & a & -a & b & -b\\ 0 & c & -c & d & -d\\ 0 & e & -e & f & -f\end{pmatrix}.$$

As $r:=\mathrm{rank}_{\mathbb{R}}(A)\in\{1,2\}$ then Gv is regular if and only if:

- either $r=\mathrm{rank}_{\mathbb{Q}}(A)=2$. In this case $(ad-bc,cf-ed,af-be)\neq(0,0,0)$. Assume for example that $ad-bc\neq 0$ then

$$A=\begin{pmatrix} a & b & 0\\ c & d & 0\\ e & f & 1\end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0\end{pmatrix}\begin{pmatrix} 0&1&-1&0&0\\ 0&0&0&1&-1\\ 0&0&1&0&0\\ 1&0&0&0&0\\ 0&0&0&0&1\end{pmatrix}.$$

  By Proposition (4.5), no other condition is needed.

- or $r=\mathrm{rank}_{\mathbb{Q}}(A)=1$. In this case $(ad-bc,cf-ed,af-be)=(0,0,0)$ and $\frac{b}{a}\in\mathbb{Q}$ where we assume that $a\neq 0$ so that $(a,c,e)\neq(0,0,0)$ then

$$A=\begin{pmatrix} a & 0 & 0\\ c & 1 & 0\\ e & 0 & 1\end{pmatrix}\begin{pmatrix} 1&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{pmatrix}\begin{pmatrix} 0&1&-1&\frac{b}{a}&-\frac{b}{a}\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\end{pmatrix}.$$

REFERENCES

[1] A. AYADI AND H. MARZOUGUI, *Dense orbits for abelian subgroups of* $GL(n,\mathbb{C})$, Foliations 2005: World Scientific, Hackensack, NJ, (2006), 47–69.

[2] D. ARNAL, B. DALI, B. CURREY, V. OUSSA, *Regularity of abelian linear actions*, Commutative and noncommutative harmonic analysis and applications, 89–109, Contemp. Math., 603, Amer. Math. Soc., Providence, RI, 2013.

[3] DIDIER ARNAL, BRADLEY CURREY, VIGNON OUSSA, *Characterization of regularity for a connected Abelian action*, Monatsh. Math. Volume 180. Issue 1, pp. 1–37, doi:10.1007/s00605-015-0811-y.

[4] ELGHAOUI, MOHAMED, AND ADLENE AYADI, *Rational Criterion Testing the Density of Additive Subgroups of* $\mathbb{R}^n$ *and* $\mathbb{C}^n$, Applied General Topology, vol. 16, no. 2, Oct. 2015, pp. 127–139, doi:10.4995/agt.2015.3257.

[5] MOHAMED ELGHAOUI AND ADLENE AYADI, *Algorithm testing for the hypercyclicity of finitely abelian subgroups of* $GL(n,\mathbb{C})$, Operators and Matrices Volume 10, Number 3 (2016), 669–678, doi:10.7153/oam-10-40.

[6] O. L. MANGASARIAN, AND M. C. FERRIS, *Uniqueness of Integer Solution of Linear Equations*, Optimization Letters, vol. 4, no. 4, Nov. 2010, pp. 559–65, doi:10.1007/s11590-010-0183-0.

[7] M. WALDSCHMIDT, *Topologie des points rationnels*, Cours de troisième Cycle, Université P. et M. Curie (Paris VI), (1994/95).

[8] MARTIN, RICHARD KIPP, *Large Scale Linear and Integer Optimization: A Unified Approach*, Springer US, 1999, doi:10.1007/978-1-4615-4975-8.

(Received March 16, 2020)

Béchir Dali
University of Carthage
Faculty of Sciences of Bizerte
7021 Jarzouna, Tunisia
e-mail: bechir.dali@fsb.u-carthage.tn;
bechir.dali@yahoo.fr

Mohamed Elghaoui
University of Carthage
Faculty of Sciences of Bizerte
7021 Jarzouna, Tunisia
e-mail: elghaoui.mohamed@yahoo.fr