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THE ALGEBRA GENERATED BY SIMPLE ELEMENTS OF A MATRIX CENTRALIZER

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Abstract. Let $\mathscr{C}(S)$ denote the centralizer of an arbitrary square matrix S. An element $A \in \mathscr{C}(S)$ is simple if $A - I$ is of rank 1. Let $\mathscr{A}_S$ denote the subalgebra generated by the simple elements of $\mathscr{C}(S)$. We use the Weyr canonical form to describe the subalgebra $\mathscr{A}_S$, and we show that if $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of S, and l is the number of defective eigenvalues of S, then $\mathscr{A}_S$ is of dimension $l + \sum_{i=1}^{k} \operatorname{nullity}(S - \lambda_i I)^2$.

1. Introduction

We consider matrices over an algebraically closed field $\mathbb{F}$ with zero characteristic. Let $\mathscr{S}$ be any set of n-by-n matrices. We call an element $A \in \mathscr{S}$ *simple* if $A - I$ is of rank 1. The following are known matrix decompositions with simple elements as factors.

- Every n-by-n matrix with determinant ± 1 is a product of $2n-1$ involutions which are simple [10].
- Every n-by-n orthogonal matrix is a product of $n+1$ simple orthogonal matrices [11, 14].
- Every $2n$-by-$2n$ symplectic matrix is a product of $2n+1$ simple symplectic matrices [2, 5].

For nonsingular matrices A and S, we say that A is ϕ_S*-orthogonal* if $SA^TS^{-1} = A^{-1}$, or equivalently,

$$AS = SA^{-T}.$$

Notice that if $S = I$, then a ϕ_S-orthogonal matrix is an orthogonal matrix, and that if $S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, then a ϕ_S-orthogonal matrix is a symplectic matrix. Thus the ϕ_S-orthogonal matrices may be viewed as generalizations of symplectic and orthogonal matrices. Let $\mathscr{O}_S$ be the set of ϕ_S-orthogonal matrices. If $\mathbb{F} = \mathbb{C}$, then every element in $\mathscr{O}_S$ is a product of simple elements in $\mathscr{O}_S$ if and only if $S^{-T}S$ is an involution [4]. A

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related study [1] has been recently done for the set of ψ_S-orthogonal matrices, where given nonsingular matrices S and A, we say that A is *ψ_S-orthogonal* if $S\overline{A^{-1}}S^{-1} = A^{-1}$, or equivalently,

$$AS = S\overline{A}.$$

It is shown in [1] that if $\mathbb{F} = \mathbb{C}$, then every ψ_S-orthogonal matrix is a product of simple ψ_S-orthogonal matrices if and only if S is consimilar to a diagonal matrix.

For an arbitrary square matrix S the *centralizer* $\mathscr{C}(S)$ of S is the set of all A such that

$$AS = SA.$$

If S is a nontrivial involution, and oftentimes assumed to be also Hermitian, the elements of $\mathscr{C}(S)$ are also called S-symmetric, and has been characterized in [12]. Generalizations and eigenvalue problems relating to S-symmetric matrices have also been considered [3, 7, 8, 13]. In this paper, we consider an arbitrary square matrix S and we use the Weyr canonical form to describe the subalgebra generated by the simple elements of $\mathscr{C}(S)$. We use the preceding to prove our main result, which we state in the following theorem. Recall that a *defective eigenvalue* is an eigenvalue whose geometric multiplicity is strictly less than its algebraic multiplicity.

THEOREM 1. *Let S be an arbitrary square matrix over an algebraically closed field of zero characteristic, and $\mathscr{A}_S$ be the subalgebra generated by the elements X that satisfy $XS = SX$ and $\operatorname{rank}(X - I) = 1$. Suppose that $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of S and l is the number of defective eigenvalues of S. Then*

$$dim\,\mathscr{A}_S = l + \sum_{i=1}^{k} nullity(S - \lambda_i I)^2.$$

The following is immediate from Theorem 1 as we will see in our discussions.

COROLLARY 1. *Following the assumptions and notations in Theorem 1, we have that $\mathscr{C}(S) = \mathscr{A}_S$ if and only if the Jordan structure of S corresponding to each eigenvalue is of the form $(2, 1, \ldots, 1)$ or $(1, 1, \ldots, 1)$.*

In other words, the simple elements of $\mathscr{C}(S)$ generate the whole algebra if and only if S is *almost diagonalizable*.

2. Proof of the main result

If $XS_1X^{-1} = S_2$ for some nonsingular matrix X, then

$$\mathscr{C}(S_1) = X^{-1}\mathscr{C}(S_2)X = \{X^{-1}AX \mid A \in \mathscr{C}(S_2)\},$$

that is, there is an isomorphism between the algebras $\mathscr{C}(S_1)$ and $\mathscr{C}(S_2)$. Thus, to count the dimension of $\mathscr{A}_S$, we can assume without loss of generality that S is in a canonical form under similarity. Both the Jordan Canonical form and Weyr Canonical form imply that if $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of S, then we can assume $S = W_1 \oplus \cdots \oplus W_k$

where λ_i is the only eigenvalue of W_i. Due to Sylvester's Theorem [6, Theorem 4.4.6], if X commutes with S, then $X = X_1 \oplus \cdots \oplus X_k$, where X_i and W_i have the same sizes for $i = 1,\dots,k$. Thus, $\dim \mathscr{A}_S = \sum_{i=1}^{k} \dim \mathscr{A}_{W_i}$. Since $\mathscr{A}_{W_i} \leqslant \mathscr{C}(W_i)$, we have that $\mathscr{A}_S = \mathscr{C}(S)$ if and only if $\dim \mathscr{A}_{W_i} = \dim \mathscr{C}(W_i)$ for $i = 1,\dots,k$.

PROPOSITION 1. Theorem 1 and Corollary 1 are true if they are true for the case when S has only one eigenvalue.

If $S = \lambda I_n$ for some $\lambda \in \mathbb{F}$, then $\mathscr{C}(S) = \mathbb{F}^{n\times n}$. Define $E_{i,j}$ to be the matrix whose (i,j)-entry is 1 and whose other entries are 0. Observe that if $i \neq j$, then $E_{i,j} = (I + (2E_{i,j})) - (I + E_{i,j})$ is a difference of simple elements in $\mathscr{C}(S)$. If $i = j$, then $E_{i,i} = \mathrm{diag}(I_{i-1}, 3, I_{n-i}) - \mathrm{diag}(I_{i-1}, 2, I_{n-i})$ is a difference of simple elements in $\mathscr{C}(S)$. Since the $E_{i,j}$'s form a basis for $\mathbb{F}^{n\times n}$, we have $\mathscr{A}_S = \mathbb{F}^{n\times n}$ and so

$$\dim \mathscr{A}_S = \dim \mathbb{F}^{n\times n} = \mathrm{nullity}(S - \lambda I)^2.$$

We are left to prove Theorem 1 for defective eigenvalues.

Let J be an n-by-n Jordan matrix with only one eigenvalue and suppose $(m_1,\dots,m_s)$ is the Jordan structure of J, where $m_1 > m_2 > \cdots > m_s$. Then an n-by-n blocked matrix $K = [K_{i,j}]$, where each $K_{i,j}$ is m_i-by-m_j, commutes with J if and only if $K_{i,j} = \begin{bmatrix} 0 & T \end{bmatrix}$ for $i \geqslant j$, and $K_{i,j} = \begin{bmatrix} T \\ 0 \end{bmatrix}$ for $i \leqslant j$, where T is a matrix of the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & \dots & a_{m_j} \\ 0 & a_1 & a_2 & a_3 & \dots & \\ 0 & 0 & a_1 & a_2 & & \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \dots & & a_1 \end{bmatrix},$$

see [9, Proposition 3.1.2]. For instance, if

$$J = \left[\begin{array}{ccc|ccc|c} 1&1&0&0&0&0&0 \\ 0&1&1&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ \hline 0&0&0&1&1&0&0 \\ 0&0&0&0&1&1&0 \\ 0&0&0&0&0&1&0 \\ \hline 0&0&0&0&0&0&1 \end{array}\right],$$

then K commutes with J if and only if

$$K = \left[\begin{array}{ccc|ccc|c} a&b&c&d&e&f&g \\ 0&a&b&0&d&e&0 \\ 0&0&a&0&0&d&0 \\ \hline h&i&j&k&l&m&n \\ 0&h&i&0&k&l&0 \\ 0&0&h&0&0&k&0 \\ \hline 0&0&o&0&0&p&q \end{array}\right].$$

The Frobenius formula [9, Proposition 3.1.3] gives the dimension of $\mathscr{C}(J)$ as $m_1 + 3m_2 + \cdots + (2s-1)m_s$, which justifies the use of 17 variables in the above matrix K. We note that the rank of $K - I$ is not immediately obtained from this form, and so we turn to the Weyr canonical form.

A very good reference material for properties and applications of the Weyr canonical form is [9]. We recall some concepts and adapt notations from this book.

DEFINITION 1. (Definition 2.1.1 in [9]) A basic Weyr matrix with eigenvalue λ is an n-by-n matrix W of the following form: There is a partition $n_1 + \cdots + n_r = n$ of n with $n_1 \geqslant \cdots \geqslant n_r \geqslant 1$ such that, when W is viewed as an r-by-r blocked matrix $(W_{i,j})$, where the (i,j) block $W_{i,j}$ is an n_i-by-n_j matrix, the following three features are present:

1. The main diagonal blocks $W_{i,i}$ are the n_i-by-n_i scalar matrices λI for $i = 1, \ldots, r$.
2. The first superdiagonal blocks $W_{i,i+1}$ are full column-rank n_i-by-n_{i+1} matrices in reduced row echelon form (that is, an identity matrix followed by zero rows) for $i = 1, \ldots, r-1$.
3. All other blocks of W are zero.

In this case, we say that W has Weyr structure $(n_1, \ldots, n_r)$. A matrix W is a Weyr matrix, or is in Weyr form if it is a direct sum of basic Weyr matrices with distinct eigenvalues.

We also have that the number n_1 is the nullity of $W - \lambda I_n$. For example,

$$W = \left[\begin{array}{cccc|ccc|ccc|c}
\lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda
\end{array}\right] \tag{1}$$

is a basic Weyr matrix with Weyr structure (4,3,3,1) such that nullity $(W - \lambda I) = 4$.

THEOREM 2. (Theorem 2.2.4 in [9]) *To within permutation of basic Weyr blocks, each square matrix A over an algebraically closed field is similar to a unique Weyr matrix W. The matrix W is called the Weyr canonical form of A.*

With Proposition 1 in mind, we prove Theorem 1 only for the case when $S = W$ is a nonscalar basic Weyr matrix. The following completely describes the elements in $\mathscr{C}(W)$.

LEMMA 1. (Proposition 2.3.3 in [9]) *Let W be an n-by-n basic Weyr matrix with Weyr structure $(n_1,\ldots,n_r)$, $r \geqslant 2$. Let K be an n-by-n matrix, blocked according to the partition $n = n_1 + \cdots + n_r$, and let $K_{i,j}$ denote its (i,j) block (an n_i-by-n_j matrix) for $i,j = 1,\ldots,r$. Then W and K commute if and only if K is a block upper triangular matrix for which*

$$K_{i,j} = \begin{bmatrix} K_{i+1,j+1} & * \\ 0 & * \end{bmatrix} \text{ for } 1 \leqslant i \leqslant j \leqslant r-1. \tag{2}$$

Here, we have written $K_{i,j}$ as a blocked matrix where the zero block is $(n_i - n_{i+1})$-by-n_{j+1}.

In particular, a matrix in $C(W)$ is completely determined by its top row of blocks. For illustration, suppose a basic Weyr matrix W has Weyr structure $(3,3,2,1)$. When we write $K \in \mathscr{C}(W)$ as

$$K = \left[\begin{array}{ccc|ccc|cc|c} 1 & 0 & 1 & 1 & -1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & -2 & 1 & 1 & 0 \end{array}\right]$$

we basically mean that

$$K = \left[\begin{array}{ccc|ccc|cc|c} 1 & 0 & 1 & 1 & -1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & -2 & 1 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

For convenience, a linear combination of elements in $\mathscr{C}(W)$ will also be written simply as a linear combination of the first row of these elements.

THEOREM 3. *Let W be a basic Weyr matrix with Weyr structure $(n_1,\ldots,n_r)$. Let $K = [K_{i,j}] \in \mathscr{C}(W)$ such that $K_{i,j}$ is n_i-by-n_j. Then K is simple if and only if the following hold:*

(a) $K_{1,1} = \begin{bmatrix} I_{n_2} & B_1 \\ 0 & C_1 \end{bmatrix}$.

(b) $K_{1,i} = \begin{bmatrix} 0_{n_2 \times n_{i+1}} & B_i \\ 0 & C_i \end{bmatrix}$ *for* $i = 2,\ldots,r-1$.

(c) $K_{1,r} = \begin{bmatrix} B_r \\ C_r \end{bmatrix}$, *where B_r has n_2 rows.*

(d) $\begin{bmatrix} B_1 & B_2 & \ldots & B_r \\ C_1 - I & C_2 & \ldots & C_r \end{bmatrix}$ *has rank 1.*

Proof. Let $K \in \mathscr{C}(W)$. It is straightforward to verify that if K satisfies the set of conditions above, then K is simple. Now, suppose K is simple. In view of Lemma 1, if $K_{i,i} \neq I$ for some $i = 2,\ldots,r$, then $K_{1,1} = \begin{bmatrix} K_{i,i} & * \\ 0 & * \end{bmatrix} \neq I$ and we have

$$\begin{aligned}\operatorname{rank}(K-I) &= \operatorname{rank}\left(\begin{bmatrix} K_{1,1}-I & & & & * \\ & \ddots & & & \\ & & K_{i,i}-I & & \\ & & & \ddots & \\ 0 & & & & K_{r,r}-I \end{bmatrix}\right) \\ &\geqslant \operatorname{rank}(K_{i,i}-I) + \operatorname{rank}(K_{1,1}-I) \\ &\geqslant 2.\end{aligned}$$

This is a contradiction since $K-I$ is of rank 1, and so $K_{i,i} = I$ for $i = 2,\ldots,r$, that is,

$$K = \begin{bmatrix} K_{1,1} & \ldots & & & \\ & I_{n_2} & K_{2,3} & & \\ & & \ddots & \ddots & \\ & & & I_{n_{r-1}} & K_{r-1,r} \\ 0 & & & & I_{n_r} \end{bmatrix}.$$

Suppose that $K_{i,i+j} \neq 0$ for some $2 \leqslant i \leqslant r-1$ and $j = 1,\ldots,r-i$. We assume without loss of generality that $K_{i,i+j}$ is the immediate nonzero superdiagonal block. This gives us,

$$\begin{aligned}\operatorname{rank}(K-I) &= \operatorname{rank}\left(\begin{bmatrix} K_{1,1}-I & \cdots & K_{1,1+j} & & & \\ & 0 & 0 & \ddots & & \\ & & \ddots & 0 & K_{i,i+j} & \\ & & & 0 & \ddots & \\ & & & & \ddots & 0 \\ 0 & & & & & 0 \end{bmatrix}\right) \\ &\geqslant \operatorname{rank}(K_{1,1+j}) + \operatorname{rank}(K_{i,i+j}) \\ &\geqslant 2,\end{aligned}$$

which is again a contradiction to the assumption that K is simple. This implies that $K = \begin{bmatrix} K_{1,1} & * \\ 0 & I \end{bmatrix}$ and so the rank of $K-I$ is equal to the rank of the first row of blocks of $K-I$. The previous discussion and Lemma 1 give us the desired form of K. □

The characterization in Theorem 3 of simple elements in $\mathscr{C}(W)$ implies that $\mathscr{A}_W$ is contained in the algebra $\mathscr{B}_W$ of matrices in $\mathscr{C}(W)$ whose first row of blocks is of the form

$$K = \left[\begin{bmatrix} \alpha I_{n_2} & B_1 \\ 0 & C_1 \end{bmatrix} \begin{bmatrix} 0_{n_2 \times n_{i+1}} & B_2 \\ 0 & C_2 \end{bmatrix} \cdots \begin{bmatrix} B_r \\ C_r \end{bmatrix}\right], \tag{3}$$

where each $B_i \in \mathbb{F}^{n_2 \times (n_i - n_{i+1})}$ and each $C_i \in \mathbb{F}^{(n_1-n_2)\times(n_i-n_{i+1})}$ for $i = 1,\ldots,r-1$, $B_r \in \mathbb{F}^{n_2 \times n_r}$, and $C_r \in \mathbb{F}^{(n_1-n_2)\times n_r}$. It is then straightforward to count that the dimension of $\mathscr{B}_W$ is

$$1 + n_1((n_1 - n_2) + (n_2 - n_3) + \cdots + (n_{r-1} - n_r) + n_r) = 1 + n_1^2.$$

Hence, since $n_1 = \text{nullity}(W - I)$, we are done if we show that $\mathscr{B}_W = \mathscr{A}_W$. It is enough to prove that the matrix K in Equation 3 is in $\mathscr{A}_W$, for the cases when

- $K = \alpha I$ for some $\alpha \in \mathbb{F}$, or
- $\alpha = 0$ and all other blocks are zero except for one which is a matrix of the form $E_{k,l}$.

Indeed, $I \in \mathscr{A}_W$ since it is the linear combination

$$2\begin{bmatrix}\begin{bmatrix} I_{n_2} & E_{i,j} \\ 0 & I_s \end{bmatrix} & 0\end{bmatrix} - \begin{bmatrix}\begin{bmatrix} I_{n_2} & 2E_{i,j} \\ 0 & I_s \end{bmatrix} & 0\end{bmatrix} \tag{4}$$

of simple elements in $\mathscr{C}(W)$, and so $K = \alpha I \in \mathscr{A}_W$ for all α. Also if K satisfies the other condition, one checks that $K + I$ is a simple element, and thus $K = (K+I) - I \in \mathscr{A}_W$. This proves Theorem 1.

REMARK 1. In both the defective and nondefective case, we have shown that a basis containing only simple elements exists for $\mathscr{A}_S$.

We now consider Corollary 1 for the case when S has only one eigenvalue. If $S = \lambda I$, that is, when the Jordan structure of S is $(1,\ldots,1)$, we have shown that $\mathscr{A}_S = \mathscr{C}(S)$. Assume that S is a nonscalar basic Weyr matrix with eigenvalue λ. We use Proposition 3.2.2 in [9] which implies that if S has Weyr structure $(n_1,\ldots,n_r)$, the dimension of $\mathscr{C}(S)$ is $n_1^2 + \cdots + n_r^2$. Thus, Theorem 1 and the fact that $n_1 = \text{nullity}(S - \lambda I)$ imply that $\mathscr{A}_S = \mathscr{C}(S)$ if and only if $1 = n_2^2 + \cdots + n_r^2$, and this happens only when $r = 2$ and $n_2 = 1$. In summary, if S is a nonscalar basic Weyr matrix with eigenvalue λ, $\mathscr{A}_S = \mathscr{C}(S)$ if and only if the Weyr structure of S is $(n_1, 1)$. One easily checks that a basic Weyr matrix with this Weyr structure has Jordan structure $(2,1,\ldots,1)$ (or one may consult Corollary 2.4.3 in [9] for the relation of the Weyr structure and Jordan structure).

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