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APPLYING SOLVABILITY THEOREMS FOR MATRIX EQUATIONS

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Abstract. In this paper, using solvability theorems for matrix equations, generally applicable results are proved for the existence of positive semidefinite or asymptotically positive semidefinite solution. In the following, a question about the matrix equation $f(A)X + Xf(A) = AB + BA$ is answered. This question was asked, first by Chan and Kwong [6] and then by Furuta [7].

1. Introduction

It is known that positive semidefiniteness of the matrices A, B does not imply positive semidefiniteness of the $AB + BA$. In [6], Chan and Kwong studied some inequalities involving $AB + BA$ and proved the following theorem.

THEOREM 1.1. *Let A be a positive definite matrix and B a positive semidefinite matrix. The solution X of the following matrix equation is always positive semidefinite.*

$$A^2X + XA^2 = AB + BA. \tag{1.1}$$

At the end of the paper [6], the following problem was posed associated with Theorem 1.1.

PROBLEM. *How can one characterize all the functions f such that the solution of the matrix equation*

$$f(A)X + Xf(A) = AB + BA \tag{1.2}$$

is positive semidefinite?

In order to answer the above question, Furuta proved the existence of the positive semidefinite solution of the following operator equation in the Hilbert space by an order-preserving operator inequality (Furuta's inequality).

$$\sum_{j=1}^{n} A^{n-j}XA^{j-1} = B,$$

where A is a positive definite operator and B is a self-adjoint operator([7]).

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On the other hand, Berman and Ben-Israel have used the special case of *Mazur's theorem* for Lyapunov's characterization of stable matrices (by taking S the cone of positive semi-definite matrices in the real space of Hermitian matrices, $T(X) = A^*X + XA$ and $b = -I$) [4] which means they proved a famous result about Lyapanov equation under the condition that all eigenvalues of A have negative real parts. That was a new method for proving the existence of the positive definite or semidefinite solution for nonlinear matrix equations.

In this paper, using solvability theorems, we study the matrix equations. In section 2, preliminaries are presented. In section 3, a general result to prove the existence of the positive semidefinite solution will be presented which is a kind of Farkas Lemma for nonlinear matrix equations. This method, which can be applied to more nonlinear matrix equation, is used for the equations $f(A)X + Xf(A) = AB + BA$, $\sum_{j=1}^{n} A^{n-j}XA^{j-1} = B$ and $X - A^*XA = B$ in section 4.

2. Preliminaries

Let $\mathbb{C}^n$ be the n-dimensional complex vector space and $\mathbb{C}^{m\times n}$ be the $m \times n$ complex matrices. A^* is used for conjugate of A and if $A = A^*$, A is Hermitian. If A and B are Hermitian matrices and $A - B$ is positive semidefinite (positive definite, resp.), then we write $A \geqslant B$ ($A > B$, resp.). For an arbitrary $n \times n$ complex matrix A, the symbol $\lambda(A)$ stands for the eigenvalue. We denote the $n \times n$ identity matrix by I. The notations $\mathscr{R}(A)$, $\mathscr{N}(A)$, $\sigma(A)$, $\mathrm{tr}(A)$ and A^+ is used for the range of matrix A, the null of A, the spectrum of A, the trace of A and the generalized inverse of A, respectively.

Let A and B be two matrices of order $m \times n$. The Kronecker product of the matrix A and B is denoted by $A \otimes B$ and the vector operator is defined by

$$\mathrm{Vec}(A) = [a_{11}, a_{21}, \ldots, a_{m1}, a_{12}, a_{22}, \ldots, a_{m2}, \ldots, a_{1n}, a_{2n}, \ldots, a_{mn}]^T.$$

We have the following properties:

(i) $\mathrm{Vec}(AB) = (I_n \otimes A)\mathrm{Vec}(B)$.

(ii) Let λ_i and μ_j be the eigenvalues of A and B, respectively. then $\lambda_i\mu_j$ are eigenvalues of $A \otimes B$.

DEFINITION 2.1. A nonempty set S in $\mathbb{C}^n$ is a

(i) convex cone if $S + S \subset S$ and if $\alpha \geqslant 0$ implies $\alpha S \subset S$,

(ii) pointed convex cone if it satisfies (i) and if $S \cap (-S) = \{0\}$.

(iii) The polar set of S is defined as follows:

$$S^p = \{y \in \mathbb{C}^n : x \in S \Rightarrow \mathrm{Re}\langle y, x\rangle \geqslant 0\}.$$

Note that $\langle x, y\rangle = \sum_{i=1}^{n} x_i.\overline{y}_i$, for any $x, y \in \mathbb{C}^n$.

EXAMPLE 2.2. [1]

(a) If S is a subspace of $\mathbb{C}^n$ then $S^P = S^\perp$. Accordingly, $\mathbb{R}$ as a subset of $\mathbb{C}$ has the polar $\mathbb{R}^P = i\mathbb{R}$.

(b) S^P is a closed convex cone.

(c) $S \subset S^{PP}$.
(d) $S^p = (\overline{S})^p$.

Note that $\overline{S}$ is closure of S.

DEFINITION 2.3. The following system

$$Tx = b, \ \ x \in S$$

is consistent if there exists an x satisfying this system.

DEFINITION 2.4. [1] The following system

$$Tx = b, \ \ x \in S$$

is asymptotically consistent if there exists a sequence $\{x_k\} \subset S$ such that

$$\lim_{k\to\infty} Tx_k = b.$$

3. Main results

We recall a few valuable theorems for our discussion calling solvability theorems.

THEOREM 3.1. [1] *Let $T \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$ and S be a nonempty closed convex cone in $\mathbb{C}^n$. Then the following are equivalent:*
(i) The system

$$Tx = b, \quad x \in S$$

is asymptotically consistent.
(ii) $T^*y \in S^p$ implies $\mathrm{Re}\langle b,y\rangle \geqslant 0$.
(iii) $b \in \mathscr{R}(T)$ and $T^+b \in \overline{\mathscr{N}(T)+S}$.

The following solvability theorem is the generalization of the Farkas theorem.

THEOREM 3.2. [1, 3] *Let $T \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$, S be a closed convex cone in $\mathbb{C}^n$, and let $\mathscr{N}(T)+S$ be closed. Then the following are equivalent:*
(i) The system

$$Tx = b, \quad x \in S$$

is consistent.
(ii) $T^*y \in S^p$ implies $\mathrm{Re}\langle b,y\rangle \geqslant 0$.

Comparing Theorems 3.1 and 3.2, it is noticed that being consistent or asymptotically consistent of a solution depends on whether $\mathscr{N}(T)+S$ is closed or not. Thus we are interested in the class of systems which for $\mathscr{N}(T)+S$ is always closed. One of the most known sets of these kinds is the polyhedral cone. It is reminded that a complex finite-dimensional space is a polyhedral cone and if S_1 and S_2 are polyhedral cones then so is S_1+S_2. Since we focus on positive semidefinite matrices, it is worth noting that the cone of positive semidefinite matrices is non-polyhedral.

The following theorem is the geometric form of the Hahn-Banach theorem.

THEOREM 3.3. [3] *Let $T \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$ and S be a closed convex cone with nonempty interior in $\mathbb{C}^n$. Then the following are equivalent:*

(i) The system

$$Tx = b, \quad x \in \operatorname{int} S$$

($\operatorname{int} S$ means the interior of the set S) is consistent.

(ii) $b \in \mathscr{R}(T)$ and $0 \neq T^* y \in S^p$ implies $\operatorname{Re}\langle b, y\rangle > 0$.

In order to use the above theorems, we shall prepare properly the space. Let V be the real space of $n \times n$ Hermitian matrices, S be the cone of positive semidefinite matrices in V and let an inner product on V be defined by

$$\langle X, Y\rangle = \operatorname{tr}(XY^*).$$

For this inner product, it is shown that the cone S of positive semidefinite matrices is self-polar i.e. $S = S^p$ and also the interior of S is

$$\operatorname{int}(S) = \text{ the positive definite matrices in } \mathbb{C}^{n\times n}.$$

We are now ready to present the main result which will be applied for the matrix equations. Note that $TX = B$ is equivalent with $(I_n \otimes T)\operatorname{Vec}(X) = \operatorname{Vec}(B)$ and the eigenvalue of $(I_n \otimes T)$ is eigenvalue of T.

THEOREM 3.4. *Let B be a matrix in $\mathbb{C}^{n\times n}$, $T(X) = Q$ be a linear map on $\mathbb{C}^n$ and Q be a positive semidefinite matrix. Assume that $T^*(Y) = Q$ has a positive semidefinite solution Y. If $\operatorname{Re}(\operatorname{tr}(BY)) \geqslant 0$, then $T(X) = B$ has a asymptotical positive semidefinite solution.*

Proof. Let $T(X) = B$. Considering the second part of Theorem 3.1, we assume $T^*(Y) \in S^p$ which means $T^*(Y)$ is positive semidefinite or $T^*(Y) = Q$ where Q is positive semidefinite. By assumption, the equation $T^*(Y) = Q$ has a positive semidefinite solution Y such that $\operatorname{Re}(\operatorname{tr}(BY)) \geqslant 0$. Therefore, by equivalency of (i) and (ii) in Theorem 3.1, the equation $T(X) = B$ has an asymptotical positive semidefinite solution. □

4. Applications

In order to apply Theorem 3.4, we follow two steps.

Step 1: we prove the existence of the positive semidefinite solution for matrix equation $TX = Q$, where Q is a positive semidefinite matrix.

Step 2: we prove the existence of the positive semidefinite solution for $TX = B$, where B is any matrix.

We are interested in investigating the following equation.

$$f(A)X + Xf(A) = C, \tag{4.1}$$

where A is a positive definite matrix. We recall the following theorem to show that Equation (4.1) has a solution in general. It is necessary to remind that we intend to characterize all the functions f such that the solution of the matrix equation is positive semidefinite.

THEOREM 4.1. [5]*Let A and B be operators whose spectra are contained in the open right half-plane and the open left half-plane, respectively. Then the solution of the equation $AX - XB = C$ can be expressed as*

$$X = \int_0^\infty e^{-tA} C e^{tB} dt.$$

Hence, we conclude the next corollary.

COROLLARY 4.2. *If A is a positive definite matrix and f is a non-negative monotone function, then Equation (4.1) has solution.*

Proof. It is sufficient to prove that the conditions of Theorem 4.1 are satisfied. Since A is positive definite, the spectra of A and $f(A)$ are contained in the open right half-plane. □

We now consider the following equation.

$$f(A)X + Xf(A) = Q, \tag{4.2}$$

where A is a positive definite matrix and Q is a positive semi-definite matrix.

In the following theorem, we are going to use a technique introduced in [6].

THEOREM 4.3. *Let A be a positive definite matrix. Assume that Q is a positive semi-definite (positive definite) matrix. If f is a nonnegative function on $(0,\infty)$, then the solution of the following equation is always positive semi-definite (positive definite).*

$$f(A)X + Xf(A) = Q. \tag{4.3}$$

Proof. Set $Y(t) = (f^2(A) + tQ)^{\frac{1}{2}}$. Since $f^2(A) + tQ \geqslant f^2(A)$ for any $t \geqslant 0$, so $Y^2(t) \geqslant Y^2(0)$ which implies $Y(t) \geqslant Y(0)$ for any $t \geqslant 0$. Hence, $Y'(0) \geqslant 0$. On the other hand, by differentiating the equation $Y^2(t) = f^2(A) + tQ$ and then letting $t = 0$, we get

$$Y(0)Y'(0) + Y'(0)Y(0) = \frac{d}{dt}(f^2(A) + tQ)|_{t=0}.$$

Then for $X = Y'(0) \geqslant 0$, we have

$$f(A)X + Xf(A) = Q. \quad \square$$

We consider the following equation.

$$f(A)X + Xf(A) = AB + BA, \tag{4.4}$$

noting that A, $B \geqslant 0$ does not imply $AB + BA \geqslant 0$. In other words, we want to study this matrix equation in which the right hand side is not positive semidefinite, necessarily.

We assume that $T_A(X) = f(A)X + Xf(A)$ and $b = AB + BA$.

LEMMA 4.4. *If $T_A(X) = f(A)X + Xf(A)$ for any A, then $T_A^* = T_A$.*

Proof. We have

$$\begin{aligned}\langle T_A(X), Y\rangle &= \langle f(A)X + Xf(A), Y\rangle \\ &= \langle f(A)X, Y\rangle + \langle Xf(A), Y\rangle \\ &= \operatorname{tr}(f(A)XY^*) + \operatorname{tr}(Xf(A)Y^*) \\ &= \operatorname{tr}(XY^*f(A)) + \operatorname{tr}(Xf(A)Y^*) \\ &= \langle X, Yf^*(A) + f^*(A)Y\rangle \\ &= \langle X, Yf(A^*) + f(A^*)Y\rangle.\end{aligned}$$

Since A is positive semidefinite, so $T_A = T_{A^*} = T_A^*$. □

By using Theorems 3.1 and 3.4, we have the following theorem for Equation (4.4).

THEOREM 4.5. *Let A, B, Q be positive semidefinite matrices and assume that f is a non-negative function. If Y is a positive semidefinite solution of $f(A)Y + Yf(A) = Q$ and $\operatorname{Re}(\operatorname{tr}((AB+BA)Y)) > 0$, then Equation of (4.4) has a positive definite solution.*

Proof. According to Theorem 4.3, matrix equation $f(A)X + Xf(A) = Q$ has a positive definite solution Y. By assumption, $\operatorname{Re}(\operatorname{tr}((AB+BA)Y)) > 0$. Using Corollary 4.2, the matrix equation $f(A)X + Xf(A) = AB + BA$ has a solution. The second part of Theorem 3.3 is then satisfied. Since the first and the second part of Theorem 3.3 are equivalent, so $f(A)X + Xf(A) = AB + BA$ has a positive definite solution. □

Next, we consider the following equation that Furuta investigated in [7] for spacial type.

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = B,$$

where A is a positive definite operator. Furuta has proved the following theorem.

THEOREM 4.6. [7] *Let A be a positive definite operator and B be a positive semidefinite operator. Let m and n be natural numbers. There exists positive semidefinite operator solution X of the following operator equation:*

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = A^{\frac{nr}{2(m+r)}} \Big(\sum_{j=1}^{m} A^{\frac{n(m-j)}{m+r}} B A^{\frac{n(j-1)}{m+r}}\Big) A^{\frac{nr}{2(m+r)}} \tag{4.5}$$

for r such that $\begin{cases} (i) \quad r \geqslant 0 \quad if \;\; n \geqslant m, \\ (ii) \quad r \geqslant \frac{m-n}{n-1} \quad if m \geqslant n \geqslant 2. \end{cases}$

In the following, we intend to revise and rewrite some results about Equation (4.5) in the matrix space using solvability theorems. First, we consider the following matrix equation.

$$\sum_{j=1}^{n} A^{n-j} X A^{j-1} = Q,$$

where A is a positive definite and Q is a positive semidefinite matrix.

LEMMA 4.7. *If* $T_A(X) = \sum_{j=1}^{n} A^{n-j}XA^{j-1}$ *for any Hermitian matrix* A*, then* $T_A^* = T_A$.

Proof. We have

$$\begin{aligned}\langle T_A(X), Y\rangle &= \langle \sum_{j=1}^{n} A^{n-j}XA^{j-1}, Y\rangle \\ &= \mathrm{tr}(\sum_{j=1}^{n} A^{n-j}XA^{j-1}Y^*) \\ &= \sum_{j=1}^{n} (\mathrm{tr}(A^{n-j}XA^{j-1}Y^*) \\ &= \sum_{j=1}^{n} (\mathrm{tr}(XA^{j-1}Y^*A^{n-j}) \\ &= \mathrm{tr}(\sum_{j=1}^{n} XA^{j-1}Y^*A^{n-j}) \\ &= \langle X, \sum_{j=1}^{n} A^{n-j}YA^{j-1}\rangle. \quad \square\end{aligned}$$

THEOREM 4.8. *Let* A *be a positive definite and* Q *be a positive semidefinite matrix. Then there exists a positive semidefinite solution* X *for the following matrix equation.*

$$\sum_{j=1}^{n} A^{n-j}XA^{j-1} = Q. \tag{4.6}$$

Proof. Let $Y(t) = (A^n + tQ)^{\frac{1}{n}}$ for $n \in \mathbb{N}$. Since $A^n + tQ \geqslant A^n$, so $(A^n + tQ)^{\frac{1}{n}} \geqslant A$. Therefore, $Y(t) \geqslant Y(0)$ which implies $Y'(0) \geqslant 0$. On the other hand, by differentiating the equation $Y^n(t) = A^n + tQ$ and then letting $t = 0$, we get

$$Y'(0)Y^{n-1}(0) + Y(0)Y'(0)Y^{n-2} + \ldots + Y^{n-1}(0)Y'(0) = Q.$$

By substituting $0 \leqslant Y'(0) = X$ and $Y(0) = A$, it is concluded that there exists a positive semidefinite solution X for Equation (4.6). $\square$

THEOREM 4.9. *Let* B *be any matrix. Assume that* A *and* Q *are positive semidefinite matrices. If* Y *is a positive semidefinite solution of* $\sum_{j=1}^{n} A^{n-j}YA^{j-1} = Q$ *and* $\mathrm{Re}(\mathrm{tr}(BY)) \geqslant 0$*, then the equation of*

$$\sum_{j=1}^{n} A^{n-j}XA^{j-1} = B,$$

has a positive semidefinite solution.

Proof. Let $T_A(Y) = \sum_{j=1}^{n} A^{n-j} Y A^{j-1}$ and $b = B$ in Theorem 3.4. □

We consider matrix equation $X - A^*XA = Q$ which is called the Stein equation (Q is positive definite). Stein equation has a unique solution if and only if A is stable ([10], [8]). In addition, this unique solution is positive definite and is given by

$$X = \sum_{n=0}^{\infty} A^{*n} Q A^n.$$

THEOREM 4.10. *Let B be any matrix, A be a stable matrix and Q be a positive definite matrix. If X is a positive definite solution of $X - A^*XA = Q$ and $\mathrm{Re}(\mathrm{tr}(BX)) \geqslant 0$, then the equation of $X - A^*XA = B$ has an asymptotical positive semidefinite solution.*

Proof. Let $T_A(X) = X - A^*XA$ and $b = B$ in Theorem 3.4. □

REFERENCES

[1] A. BEN-ISRAEL, *Linear Equations and Inequalities on Finite Dimensional, Real or Complex, Vector Spaces: A Unified Theory*, J. Math. Anal. Appl. 27, 367–389 (1969).

[2] E. V. BELMEGA, S. LASAULCE AND M. DEBBAH, *A Trace inequality for positive definite matrices*, Journal of inequalities in pure and applied mathematics, Vol. 10 (2009), Issue 1, Article 5, 4 pp.

[3] A. BERMAN AND A. BEN-ISRAEL, *More on linear inequalities with applications to matrix theory*, J. Math. Anal. Appl. **33** (1971), 482–496.

[4] A. BERMAN AND A. BEN-ISRAEL, *Linear Equations over Cones with Interior: A Solvability Theorem with Applications to Matrix Theory*, Linear Algebra Appl. **7** (1973) 139–149.

[5] R. BHATIA, *Matrix Analysis*, Grad. Texts in Math. 169, Springer-Verlag, New York, 1997.

[6] N. N. CHAN AND M. K. KWONG, *Hermitian matrix inequalities and a conjecture*, Amer. Math. Monthly **92** (1985), 533–541.

[7] T. FURUTA, *Positive semidefinite solutions of the operator equation* $\sum_{j=1}^{n} A_{n-j} X A_{j-1} = B$, Linear Algebra Appl. **432** (2010) 949–955.

[8] P. LANCASTER AND L. RODMAN, *Algebraic Riccati Equations*, Oxford University Press, New York, 1995.

[9] T. MORI, *Comments on "A Matrix Inequality Associated with Bounds on Solutions of Algebraic Riccati and Lyapunov Equation"*, IEEE Trans. Automat. Contr., vol. AC-33, p. 1088, Nov. 1988.

[10] A. M. RAN AND M. C. B. REURINGS, *The symmetric linear matrix equation*, The Electronic Journal of Linear Algebra **9** (2002) 93–107.

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