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AN OPPENHEIM TYPE DETERMINANTAL INEQUALITY FOR THE KHATRI–RAO PRODUCT

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Abstract. The Khatri-Rao product is a generalization of the classical Hadamard product for block matrices. In this paper, we give an Oppenheim type determinantal inequality for the Khatri-Rao product of two block positive semidefinite matrices, and then we extend our result to multiple block matrices.

1. Introduction

We use the following standard notation. The set of $m\times n$ complex matrices is denoted by $\mathbb{M}_{m\times n}(\mathbb{C})$, or simply by $\mathbb{M}_{m\times n}$, when $m=n$, we put $\mathbb{M}_n$ for $\mathbb{M}_{n\times n}$. The identity matrix of order n by I_n, or I for short. If $A=[a_{ij}]$ is of order $p\times q$ and B is of order $r\times s$, the *Kronecker product* (*tensor product*) of A with B, denoted by $A\otimes B$, is a $pr\times qs$ matrix, partitioned into $p\times q$ block matrix with the (i,j)-block the $r\times s$ matrix $a_{ij}B$, i.e., $A\otimes B=[a_{ij}B]_{i,j=1}^{p,q}$. Given two matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ with the same order, the *Hadamard product* of A and B is defined as $A\circ B=[a_{ij}b_{ij}]$. It is easy to see that $A\circ B$ is a principal submatrix of $A\otimes B$. By convention, the $\mu\times\mu$ leading principal submatrix of A is denoted by A_μ.

Let $A=[a_{ij}]\in\mathbb{M}_n$ be positive semidefinite. The Hadamard inequality says that

$$\prod_{i=1}^n a_{ii}\geqslant \det A. \tag{1}$$

If $B=[b_{ij}]\in\mathbb{M}_n$ is positive semidefinite, it is well-known that $A\circ B$ is positive semidefinite. Moreover, the celebrated Oppenheim inequality (see [15] or [7, p. 509]) states that

$$\det(A\circ B)\geqslant \det A\cdot\prod_{i=1}^n b_{ii}\geqslant \det(AB). \tag{2}$$

Setting $B=I_n$, then (2) reduces to (1). Note that $A\circ B=B\circ A$, thus we also have

$$\det(A\circ B)\geqslant \det B\cdot\prod_{i=1}^n a_{ii}\geqslant \det(AB). \tag{3}$$

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The following inequality (4) not only generalized Oppenheim's result, but also presented a well connection between (2) and (3); see [16, Theorem 3.7] for more details.

$$\det(A\circ B)+\det(AB)\geqslant \det A\cdot\prod_{i=1}^{n} b_{ii}+\det B\cdot\prod_{i=1}^{n} a_{ii}. \tag{4}$$

Inequality (4) is usually called Oppenheim-Schur's inequality. Furthermore, Chen [2] generalized (4) and proved an implicit improvement, i.e., if A and B are $n\times n$ positive definite matrices, then

$$\det(A\circ B)\geqslant \det(AB)\prod_{\mu=2}^{n}\left(\frac{a_{\mu\mu}\det A_{\mu-1}}{\det A_{\mu}}+\frac{b_{\mu\mu}\det B_{\mu-1}}{\det B_{\mu}}-1\right), \tag{5}$$

where A_μ and B_μ denote the $\mu\times\mu$ leading principal submatrices of A and B, respectively.

Over the past years, various generalizations and extensions of (4) and (5) have been obtained in the literature. For instance, see [18, 19] for the equality cases; see [1, 10, 17, 3] for the extensions of M-matrices; see [6, 14, 4] for the extensions of block Hadamard product.

In this paper, we mainly concentrate on block positive semidefinite matrices. Let $\mathbb{M}_n(\mathbb{M}_{p\times q})$ be the set of complex matrices partitioned into $n\times n$ blocks with each block being a $p\times q$ matrix. The element of $\mathbb{M}_n(\mathbb{M}_{p\times q})$ is usually written as the bold letter $\boldsymbol{A}=[A_{ij}]_{i,j=1}^{n}$, where $A_{ij}\in\mathbb{M}_{p\times q}$ for all $1\leqslant i,j\leqslant n$. For $\boldsymbol{A}=[A_{ij}]\in\mathbb{M}_n(\mathbb{M}_{p\times q})$ and $\boldsymbol{B}=[B_{ij}]\in\mathbb{M}_n(\mathbb{M}_{r\times s})$, the *Khatri-Rao product* $\boldsymbol{A}*\boldsymbol{B}$, first introduced in [8], is given as $\boldsymbol{A}*\boldsymbol{B}:=[A_{ij}\otimes B_{ij}]_{i,j=1}^{n}$, where $A_{ij}\otimes B_{ij}$ denotes the Kronecker product of A_{ij} and B_{ij}. Clearly, when $p=q=r=s=1$, that is, $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n\times n$ matrices with complex entries, the Khatri-Rao product coincides with the classical Hadamard product; when $n=1$, it is identical with the usual Kronecker product. It is easy to verify that $(\boldsymbol{A}*\boldsymbol{B})*\boldsymbol{C}=\boldsymbol{A}*(\boldsymbol{B}*\boldsymbol{C})$, so the Khatri-Rao product of $\boldsymbol{A}^{(1)},\ldots,\boldsymbol{A}^{(m)}$ could be denoted by $\prod_{i=1}^{m}*\boldsymbol{A}^{(i)}$. We refer to [11, 12, 13] for more properties of Khatri-Rao product.

Recently, motivated by the breakthrough of Lin [14], Kim et al. [9] gave the following extension of Chen's result (5) for the Khatri-Rao product, if $\boldsymbol{A},\boldsymbol{B}\in\mathbb{M}_n(\mathbb{M}_k)$ are positive definite, then

$$\begin{aligned}\det(\boldsymbol{A}*\boldsymbol{B})\geqslant&(\det\boldsymbol{AB})^k\\&\times\prod_{\mu=2}^{n}\left(\left(\frac{\det A_{\mu\mu}\det\boldsymbol{A}_{\mu-1}}{\det\boldsymbol{A}_{\mu}}\right)^k+\left(\frac{\det B_{\mu\mu}\det\boldsymbol{B}_{\mu-1}}{\det\boldsymbol{B}_{\mu}}\right)^k-1\right),\end{aligned} \tag{6}$$

where $\boldsymbol{A}_\mu=[A_{ij}]_{i,j=1}^{\mu}$ and $\boldsymbol{B}_\mu=[B_{ij}]_{i,j=1}^{\mu}$ stand for the $\mu\times\mu$ leading principal block submatrices of $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively.

Clearly, when $k=1$, Kim's result (6) reduces to Chen's result (5). In this paper, we will prove the following main result, which is a generalization of the above (6).

THEOREM 1.1. *Let* $\mathbf{A}^{(i)} \in \mathbb{M}_n(\mathbb{M}_{q_i})$, $i = 1, 2, \ldots, m$ *be positive definite. Then*

$$\det\left(\prod_{i=1}^{m} *\mathbf{A}^{(i)}\right) \geqslant \prod_{i=1}^{m}\left(\det \mathbf{A}^{(i)}\right)^{\frac{q_1 q_2 \cdots q_m}{q_i}}$$
$$\times \prod_{\mu=2}^{n}\left(\sum_{i=1}^{m}\left(\frac{\det A_{\mu\mu}^{(i)} \det \mathbf{A}_{\mu-1}^{(i)}}{\det \mathbf{A}_{\mu}^{(i)}}\right)^{\frac{q_1 q_2 \cdots q_m}{q_i}} - (m-1)\right),$$

where $\mathbf{A}_{\mu}^{(i)}$ *stands for the* $\mu \times \mu$ *leading principal block submatrix of* $\mathbf{A}^{(i)}$.

The paper is organized as follows. We first modify Kim's result (6) to more general setting where the blocks in each $n \times n$ block matrix are of different order (Theorem 2.1). Motivated by the works in [5] and [4], we then show a proof of our main result (Theorem 1.1), as a byproduct, we will present the second main result (Theorem 2.5). Our results extend the above mentioned results (4), (5) and (6).

2. Auxiliary results and proofs

To review the proof of (6) in [9], we present a slightly more general result (Theorem 2.1). Clearly, when $p = q = k$, Theorem 2.1 reduces to (6). Such a generalization also actuates our cerebration and propels the main result (Theorem 1.1). Because the lines of proof between Theorem 2.1 and (6) are similar, so we leave the details to the interested readers.

THEOREM 2.1. *Let* $\mathbf{A} = [A_{ij}] \in \mathbb{M}_n(\mathbb{M}_p)$, $\mathbf{B} = [B_{ij}] \in \mathbb{M}_n(\mathbb{M}_q)$ *be positive definite. Then*

$$\det(\mathbf{A} * \mathbf{B}) \geqslant (\det \mathbf{A})^q (\det \mathbf{B})^p$$
$$\times \prod_{\mu=2}^{n}\left(\left(\frac{\det A_{\mu\mu} \det \mathbf{A}_{\mu-1}}{\det \mathbf{A}_{\mu}}\right)^q + \left(\frac{\det B_{\mu\mu} \det \mathbf{B}_{\mu-1}}{\det \mathbf{B}_{\mu}}\right)^p - 1\right),$$

where $\mathbf{A}_{\mu} = [A_{ij}]_{i,j=1}^{\mu}$ *and* $\mathbf{B}_{\mu} = [B_{ij}]_{i,j=1}^{\mu}$ *for every* $\mu = 1, 2, \ldots, n$.

The following Lemma 2.2 is called Fischer's inequality, which is an improvement of Hadamard's inequality (1) for block positive semidefinite matrices.

LEMMA 2.2. (see [7, p. 506] or [20, p. 217]) *If* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *is an* $n \times n$ *positive semidefinite matrix with diagonal blocks being square, then*

$$\prod_{i=1}^{n} a_{ii} \geqslant \det A_{11} \det A_{22} \geqslant \det A.$$

Next, we need to introduce a numerical inequality, which could be found in [4]. For completeness, we here include a proof for the convenience.

LEMMA 2.3. *If* $\left(a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)}\right) \in \mathbb{R}^n$, $i = 1, \ldots, m$ *and* $a_\mu^{(i)} \geqslant 1$ *for all* i, μ, *then*

$$\prod_{\mu=1}^{n}\left(\sum_{i=1}^{m} a_\mu^{(i)} - (m-1)\right) \geqslant \sum_{i=1}^{m}\prod_{\mu=1}^{n} a_\mu^{(i)} - (m-1).$$

Proof. We use induction on n. When $n=1$, there is nothing to show. Suppose that the required inequality is true for $n=k$. Then we consider the case $n=k+1$,

$$\begin{aligned}
&\prod_{\mu=1}^{k+1}\left(\sum_{i=1}^{m} a_\mu^{(i)} - (m-1)\right)\\
&= \left(\sum_{i=1}^{m} a_{k+1}^{(i)} - (m-1)\right)\cdot \prod_{\mu=1}^{k}\left(\sum_{i=1}^{m} a_\mu^{(i)} - (m-1)\right)\\
&\geqslant \left(\sum_{i=1}^{m} a_{k+1}^{(i)} - (m-1)\right)\cdot \left(\sum_{i=1}^{m}\prod_{\mu=1}^{k} a_\mu^{(i)} - (m-1)\right)\\
&= \sum_{i=1}^{m}\prod_{\mu=1}^{k+1} a_\mu^{(i)} - (m-1) + \sum_{i=1}^{m}\left(a_{k+1}^{(i)} - 1\right)\left(\sum_{j=1, j\neq i}^{m}\prod_{\mu=1}^{k} a_\mu^{(j)} - (m-1)\right)\\
&\geqslant \sum_{i=1}^{m}\prod_{\mu=1}^{k+1} a_\mu^{(i)} - (m-1).
\end{aligned}$$

Thus, the required inequality holds for $n=k+1$, so the proof of induction step is complete. ☐

REMARK. When $m=2$, Lemma 2.3 implies that for every $a_\mu, b_\mu \geqslant 1$, then

$$\prod_{\mu=1}^{n}(a_\mu + b_\mu - 1) \geqslant \prod_{\mu=1}^{n} a_\mu + \prod_{\mu=1}^{n} b_\mu - 1. \tag{7}$$

This inequality (7) plays an important role in [14] for deriving determinantal inequalities, and we can see from (7) that Chen's result (5) is indeed an improvement of (4). The above proof of Lemma 2.3 is by induction on n. In fact, combining the above (7) and by induction on m, one could get another way to prove Lemma 2.3.

The following Corollary 2.4 is a direct consequence from Lemma 2.3, it will be used to facilitate the proof of Theorem 1.1.

COROLLARY 2.4. *If* $b_1, b_2, \ldots, b_m \in \mathbb{R}$ *and* $b_i \geqslant 1$ *for all* i, *then for positive integer* q

$$\left(\sum_{i=1}^{m} b_i - (m-1)\right)^q \geqslant \sum_{i=1}^{m} b_i^q - (m-1).$$

Now, we are in a position to give the proof of Theorem 1.1.

Proof of Theorem 1.1. We show the proof by induction on m. When $m=2$, the required result degrades into Theorem 2.1. Assume that the required inequality is true for the case $m-1$, that is

$$\det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right) \geqslant \prod_{i=1}^{m-1}\left(\det \boldsymbol{A}^{(i)}\right)^{\frac{q_1q_2\cdots q_{m-1}}{q_i}} \times \prod_{\mu=2}^{n}\left(\sum_{i=1}^{m-1}\left(\frac{\det A_{\mu\mu}^{(i)} \det \boldsymbol{A}_{\mu-1}^{(i)}}{\det \boldsymbol{A}_{\mu}^{(i)}}\right)^{\frac{q_1q_2\cdots q_{m-1}}{q_i}} - (m-2)\right).$$

Now we consider the case $m>2$, we have

$$\begin{aligned}
&\det\left(\prod_{i=1}^{m} * \boldsymbol{A}^{(i)}\right)\\
&= \det\left(\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right) * \boldsymbol{A}^{(m)}\right)\\
&\geqslant \left(\det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)\right)^{q_m} \left(\det \boldsymbol{A}^{(m)}\right)^{q_1q_2\cdots q_{m-1}}\\
&\quad \times \prod_{\mu=2}^{n}\left(\left(\frac{\det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)_{\mu\mu} \det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)_{\mu-1}}{\det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)_{\mu}}\right)^{q_m} + \left(\frac{\det A_{\mu\mu}^{(m)} \det \boldsymbol{A}_{\mu-1}^{(m)}}{\det \boldsymbol{A}_{\mu}^{(m)}}\right)^{q_1q_2\cdots q_{m-1}} - 1\right)\\
&\geqslant \prod_{i=1}^{m}\left(\det \boldsymbol{A}^{(i)}\right)^{\frac{q_1q_2\cdots q_m}{q_i}} \times \prod_{\mu=2}^{n}\left(\sum_{i=1}^{m-1}\left(\frac{\det A_{\mu\mu}^{(i)} \det \boldsymbol{A}_{\mu-1}^{(i)}}{\det \boldsymbol{A}_{\mu}^{(i)}}\right)^{\frac{q_1q_2\cdots q_{m-1}}{q_i}} - (m-2)\right)^{q_m}\\
&\quad \times \prod_{\mu=2}^{n}\left(\left(\frac{\det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)_{\mu\mu} \det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)_{\mu-1}}{\det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)_{\mu}}\right)^{q_m} + \left(\frac{\det A_{\mu\mu}^{(m)} \det \boldsymbol{A}_{\mu-1}^{(m)}}{\det \boldsymbol{A}_{\mu}^{(m)}}\right)^{q_1q_2\cdots q_{m-1}} - 1\right).
\end{aligned}$$

For notational convenience, we denote

$$R_\mu := \left(\sum_{i=1}^{m-1}\left(\frac{\det A_{\mu\mu}^{(i)} \det \boldsymbol{A}_{\mu-1}^{(i)}}{\det \boldsymbol{A}_{\mu}^{(i)}}\right)^{\frac{q_1q_2\cdots q_{m-1}}{q_i}} - (m-2)\right)^{q_m},$$

and

$$S_\mu := \left(\frac{\det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)_{\mu\mu} \det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)_{\mu-1}}{\det\left(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)}\right)_{\mu}}\right)^{q_m} + \left(\frac{\det A_{\mu\mu}^{(m)} \det \boldsymbol{A}_{\mu-1}^{(m)}}{\det \boldsymbol{A}_{\mu}^{(m)}}\right)^{q_1q_2\cdots q_{m-1}} - 1.$$

By Fischer's inequality (Lemma 2.2), we can see that

$$\det A_{\mu\mu}^{(i)} \det \boldsymbol{A}_{\mu-1}^{(i)} \geqslant \det \boldsymbol{A}_{\mu}^{(i)}, \quad i = 1, 2, \ldots, m,$$

which together with Corollary 2.4 yields

$$R_\mu \geqslant \sum_{i=1}^{m-1} \left(\frac{\det A_{\mu\mu}^{(i)} \det \boldsymbol{A}_{\mu-1}^{(i)}}{\det \boldsymbol{A}_{\mu}^{(i)}} \right)^{\frac{q_1 q_2 \cdots q_{m-1} q_m}{q_i}} - (m-2) \geqslant 1. \tag{8}$$

On the other hand, by Fischer's inequality (Lemma 2.2) again, we have

$$\det \Big(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)} \Big)_{\mu\mu} \det \Big(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)} \Big)_{\mu-1} \geqslant \det \Big(\prod_{i=1}^{m-1} * \boldsymbol{A}^{(i)} \Big)_{\mu}.$$

Therefore, we obtain

$$S_\mu \geqslant \left(\frac{\det A_{\mu\mu}^{(m)} \det \boldsymbol{A}_{\mu-1}^{(m)}}{\det \boldsymbol{A}_{\mu}^{(m)}} \right)^{q_1 q_2 \cdots q_{m-1}} \geqslant 1. \tag{9}$$

Since $R_\mu \geqslant 1$ and $S_\mu \geqslant 1$, this leads to

$$R_\mu S_\mu \geqslant R_\mu + S_\mu - 1.$$

Hence, we get from (8) and (9) that

$$\begin{aligned}
\det \left(\prod_{i=1}^{m} * \boldsymbol{A}^{(i)} \right) &\geqslant \prod_{i=1}^{m} \left(\det \boldsymbol{A}^{(i)} \right)^{\frac{q_1 q_2 \cdots q_m}{q_i}} \prod_{\mu=2}^{n} R_\mu \prod_{\mu=2}^{n} S_\mu \\
&\geqslant \prod_{i=1}^{m} \left(\det \boldsymbol{A}^{(i)} \right)^{\frac{q_1 q_2 \cdots q_m}{q_i}} \prod_{\mu=2}^{n} (R_\mu + S_\mu - 1) \\
&\geqslant \prod_{i=1}^{m} \left(\det \boldsymbol{A}^{(i)} \right)^{\frac{q_1 q_2 \cdots q_m}{q_i}} \\
&\quad \times \prod_{\mu=2}^{n} \left(\sum_{i=1}^{m} \left(\frac{\det A_{\mu\mu}^{(i)} \det \boldsymbol{A}_{\mu-1}^{(i)}}{\det \boldsymbol{A}_{\mu}^{(i)}} \right)^{\frac{q_1 q_2 \cdots q_m}{q_i}} - (m-1) \right).
\end{aligned}$$

This completes the proof. □

Next, we will present the extension of Oppenheim type determinantal inequality (4).

THEOREM 2.5. *Let $\boldsymbol{A}^{(i)} \in \mathbb{M}_n(\mathbb{M}_{q_i})$, $i = 1,2,\ldots,m$ be positive demidefinite. Then*

$$\begin{aligned} &\det\left(\prod_{i=1}^{m} {*}\boldsymbol{A}^{(i)}\right) + (m-1)\prod_{i=1}^{m}\left(\det \boldsymbol{A}^{(i)}\right)^{\frac{q_1q_2\cdots q_m}{q_i}} \\ &\geqslant \sum_{i=1}^{m}\prod_{j=1, j\neq i}^{m}\left(\det \boldsymbol{A}^{(j)} \cdot \prod_{\mu=1}^{n}\det A_{\mu\mu}^{(i)}\right)^{\frac{q_1q_2\cdots q_m}{q_i}}. \end{aligned} \tag{10}$$

Proof. If some of $A_{\mu\mu}^{(i)}$ in (10) is singular, then so is $\boldsymbol{A}^{(i)}$. In this case, the right hand side of (10) equals to zero. Indeed, by a standard perturbation argument, we may assume without loss of generality that all $\boldsymbol{A}^{(i)}$ are positive definite. Thus, we may rewrite (10) as

$$\det\left(\prod_{i=1}^{m} {*}\boldsymbol{A}^{(i)}\right) \geqslant \prod_{i=1}^{m}\left(\det \boldsymbol{A}^{(i)}\right)^{\frac{q_1q_2\cdots q_m}{q_i}}\left(\sum_{i=1}^{m}\left(\frac{\prod_{\mu=1}^{n}\det A_{\mu\mu}^{(i)}}{\det \boldsymbol{A}^{(i)}}\right)^{\frac{q_1q_2\cdots q_m}{q_i}} - (m-1)\right). \tag{11}$$

By Fischer's inequality (Lemma 2.2), we have

$$\det A_{\mu\mu}^{(i)} \det \boldsymbol{A}_{\mu-1}^{(i)} \geqslant \det \boldsymbol{A}_{\mu}^{(i)}.$$

Therefore, it follows from Theorem 1.1 and Lemma 2.3 that

$$\begin{aligned} &\det\left(\prod_{i=1}^{m} {*}\boldsymbol{A}^{(i)}\right) \\ &\geqslant \prod_{i=1}^{m}\left(\det \boldsymbol{A}^{(i)}\right)^{\frac{q_1q_2\cdots q_m}{q_i}}\prod_{\mu=2}^{n}\left(\sum_{i=1}^{m}\left(\frac{\det A_{\mu\mu}^{(i)}\det \boldsymbol{A}_{\mu-1}^{(i)}}{\det \boldsymbol{A}_{\mu}^{(i)}}\right)^{\frac{q_1q_2\cdots q_m}{q_i}} - (m-1)\right) \\ &\geqslant \prod_{i=1}^{m}\left(\det \boldsymbol{A}^{(i)}\right)^{\frac{q_1q_2\cdots q_m}{q_i}}\left(\sum_{i=1}^{m}\prod_{\mu=2}^{n}\left(\frac{\det A_{\mu\mu}^{(i)}\det \boldsymbol{A}_{\mu-1}^{(i)}}{\det \boldsymbol{A}_{\mu}^{(i)}}\right)^{\frac{q_1q_2\cdots q_m}{q_i}} - (m-1)\right). \end{aligned}$$

Observe that

$$\prod_{\mu=2}^{n}\frac{\det A_{\mu\mu}^{(i)}\det \boldsymbol{A}_{\mu-1}^{(i)}}{\det \boldsymbol{A}_{\mu}^{(i)}} = \frac{\prod_{\mu=1}^{n}\det A_{\mu\mu}^{(i)}}{\det \boldsymbol{A}^{(i)}}.$$

Hence, the proof of (11) is complete. □

In the sequel, by setting $q_1 = q_2 = \cdots = q_m = 1$ in Theorem 1.1 and Theorem 2.5, we can get the following Corollary 2.6 and Corollary 2.7 for the Hadamard product, respectively. These two corollaries are extensions of Oppenheim-Schur's inequality (4) and Chen's result (5). The first corollary can be found in [5, Theorem 7] and the second one can be seen in [4, Theorem 4].

COROLLARY 2.6. *Let $A^{(i)} \in \mathbb{M}_n(\mathbb{C})$, $i = 1,2,\ldots,m$ be positive definite. Then*

$$\det\left(\prod_{i=1}^{m} \circ A^{(i)}\right) \geqslant \left(\prod_{i=1}^{m} \det A^{(i)}\right) \prod_{\mu=2}^{n} \left(\sum_{i=1}^{m} \frac{a_{\mu\mu}^{(i)} \det A_{\mu-1}^{(i)}}{\det A_{\mu}^{(i)}} - (m-1)\right),$$

where $A_{\mu}^{(i)}$ stands for the $\mu \times \mu$ leading principal submatrix of $A^{(i)}$.

COROLLARY 2.7. *Let $A^{(i)} \in \mathbb{M}_n(\mathbb{C})$, $i = 1,2,\ldots,m$ be positive semidefinite. Then*

$$\det\left(\prod_{i=1}^{m} \circ A^{(i)}\right) + (m-1)\prod_{i=1}^{m} \det A^{(i)} \geqslant \sum_{i=1}^{m} \prod_{j=1, j\neq i}^{m} \det A^{(j)} \prod_{\mu=1}^{n} a_{\mu\mu}^{(i)}.$$

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