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SOME REMARKS IN C^*– AND K–THEORY

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Abstract. This note consists of three unrelated remarks. First, we demonstrate how roughly speaking $*$-homomorphisms between matrix stable C^*-algebras are exactly the uniformly continuous $*$-preserving group homomorphisms between their general linear groups. Second, using the Cuntz picture in KK-theory we bring morphisms in KK-theory represented by generators and relations to a particular simple form. Third, we show that for an inverse semigroup its associated groupoid is Hausdorff if and only if the inverse semigroup is E-continuous.

1. Introduction

In this note we present three unrelated results in C^*-theory and K-theory. The first result is demonstrated in Section 2 and shows that for all unital C^*-algebras A and B, every uniformly continuous, $*$-preserving group homomorphism $\varphi : GL(A \otimes M_2) \to GL(B)$ can be extended to a $*$-homomorphism $A \otimes M_2 \to B$, provided a very light additional technical condition for the restriction of φ to the complex numbers is satisfied, see Corollary 2.3 and Section 2. Actually, we have demonstrated a similar result already in [6], but the improvement, thanks to some trick by L. Molnár [14], is that the additional technical condition is here subjectively somewhat easier, even if not strictly logically comparable with the one in [6].

In the next Section 3, we make a turn to KK-theory [11]. J. Cuntz [7] and N. Higson [10] found out that Kasparov's KK-theory is the universal stable, homotopy invariant, split-exact functor from the C^*-category to an additive category. This makes it possible to describe KK-theory as a localization of the category of C^*-algebras, or expressed in less technical terms, by adding certain synthetical inverses to the category of C^*-algebras and moding out certain relations to form KK-theory. We slightly simplify the representation of KK-elements in this picture at first, but make the most dramatical simplification by using the Cuntz-picture [7, 8] of KK-theory elements. This picture of KK-theory may also be analogously and readily defined equivariantly for other equivariant structures than groups, say semigroups, categories and so on, and even the category of C^*-algebras may be changed to other (topological) algebras.

In the last Section 4 we observe that a discrete inverse semigroup induces a Hausdorff groupoid if and only if the inverse semigroup is E-continuous. We also note

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that both equivalent technical conditions appear necessary to define a non-degenerate, $C_0(X)$-compatible $C_0(X)$-valued $L^2(G)$-module, see Definition 4.4 and Example 4.14 for more on this. Such a module is a useful tool for the computation of the K-theory of inverse semigroup crossed products. However, the lack of such a module in the non-Hausdorff case hinders the computation of beformentioned K-theory groups of crossed products by non-applicability of parallel methods successful in the group case. The difficulty of computation has been already observed by Tu [19] for the more general setting of non-Hausdorff groupoids in the context of Baum–Connes theory.

All chapters in this note can be read completely independently.

2. Group and algebra homomorphisms

In this section we show how certain group homomorphisms between the group of invertible elements of C^*-algebras can be extended to $*$-homomorphisms. A map $\varphi : A \to B$ between C^*-algebras A and B is called a *$*$-semigroup homomorphism* if it is multiplicative (i.e. $\varphi(ab) = \varphi(a)\varphi(b)$) and *$*$-preserving* (i.e. $\varphi(a^*) = \varphi(a)^*$). As usual, M_n denotes the C^*-algebra of all complex-valued $n \times n$-matrices, and $GL(A)$ the general linear group of A.

PROPOSITION 2.1. *Let $\varphi : GL(A \otimes M_2) \to B$ be an arbitrary function where A and B are C^*-algebras and A is unital. Then the following are equivalent:*

(a) φ extends to a $*$-homomorphism $A \otimes M_2 \to B$.

(b) φ is a uniformly continuous, $*$-semigroup homomorphism with

$$\|\varphi(1/2)\| < 1, \quad \varphi(i1) = i\varphi(1). \tag{1}$$

REMARK 2.2. Alternatively, instead of requiring $\|\varphi(1/2)\| < 1$ in Proposition 2.1.(b), we may equivalently require that $\|\varphi(z)\| < 1$ for any single fixed $z \in GL(A \otimes M_n)$ with $\|z\| < 1$.

Proof. (a) to (b) is clear. To show (b) to (a), we are going to apply [6, Proposition 2.6]. At first we continuously extend φ to an equally denoted function $\varphi : \overline{GL(A \otimes M_2)} \to B$ (norm closure) by using Cauchy sequences and the uniform continuity of φ. Then φ is a $*$-semigroup homomorphism. Notice that $\varphi(0) = \lim_n \varphi(z^n) = 0$ by Remark 2.2. By applying Proposition 2.6 of [6] we are done when showing the ortho-additivity relation $\varphi(e_{11} + e_{22}) = \varphi(e_{11}) + \varphi(e_{22})$, where e_{ii} are the standard matrix corners. To this end, we use the following trick by L. Molnár [14] by means of the exponential function, which we are going to recall for convenience of the reader.

Consider the C^*-subalgebra B' of B generated by the image of φ. It is unital with unit $\varphi(1)$. Represent B' faithfully on a Hilbert space H such that $1_{B(H)}$ is the unit of B'. In the following, identify now B' as a subalgebra of $B(H)$.

Let P be a projection in $M_2(A)$. Clearly $e^{\lambda P}$ is invertible for every $\lambda \in \mathbb{R}$ and so in the domain of φ. Consider the map $\lambda \mapsto \varphi(e^{\lambda P}) = \varphi(1 - P + e^{\lambda}P)$ from $\mathbb{R}$ into

$GL(B(H))$. This is a one-parameter group. Thus there exists an operator $T \in B(H)$ such that

$$\varphi(1-P+e^{\lambda}P) = e^{\lambda T}.$$

Since φ is $*$-preserving, $e^{\lambda T}$ is self-adjoint for all $\lambda \in \mathbb{R}$. This implies that T is also self-adjoint. By the uniform continuity of φ, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|e^{\lambda T} - e^{\mu T}\| = \sup_{t\in\sigma(T)} |e^{\lambda t} - e^{\mu t}| < \varepsilon$$

if $|e^{\lambda} - e^{\mu}| < \delta$. The last identity is by standard functional calculus. Therefore, the function $x \mapsto x^t$ is uniformly continuous on the positive half-line for all $t \in \sigma(T)$. Hence $\sigma(T) \subseteq \{0,1\}$ and so T is a projection.

Consequently,

$$\varphi(1-P+e^{\lambda}P) = 1-T+e^{\lambda}T.$$

For $\lambda \to -\infty$ we get $\varphi(1-P) = 1-T$. Setting $P=1$ and using $\varphi(0)=0$ this implies $T=1$, and consequently $\varphi(e^{\lambda}1) = e^{\lambda}1$. In particular, φ is $\mathbb{R}_+$-homogeneous.

Hence the above equality divided by e^{λ} and letting $\lambda \to \infty$ yields $\varphi(P) = T$. Thus, putting $\lambda = 1$,

$$\varphi(1) = \varphi(1-P) + \varphi(P).$$

Now set $P = e_{11}$. $\square$

We remark that in Proposition 2.1.(b) φ is obviously actually a *group* homomorphism into the image of φ. So let us also state the following variant to emphasize this fact:

COROLLARY 2.3. *Let $\varphi : GL(A \otimes M_2) \to GL(B)$ be an arbitrary function where A and B are unital C^*-algebras. Then the following are equivalent:*

(a) φ extends to a unital $*$-homomorphism $A \otimes M_2 \to B$.

(b) φ is a uniformly continuous, $*$-preserving group homomorphism satisfying (1).

EXAMPLE 2.4.

(a) The determinant $\det : GL(M_n(\mathbb{C}))) \to GL(\mathbb{C})$, though a continuous $*$-preserving group homomorphism, cannot be extended to a $*$-homomorphism because $\det(\lambda 1) = \lambda^n$, which is not uniformly continuous.

(b) The trivial group homomorphism $\varphi : GL(M_n(A)) \to GL(B)$, $\varphi(x) = 1$, though a uniformly continuous $*$-preserving group homomorphism, cannot be extended to a $*$-homomorphism because $\|\varphi(1/2)\| = 1$.

3. KK-theory and generators

In this section we deal with the Kasparov category KK. This is the category with object class being the C^*-algebras, and morphism class from C^*-algebra A to C^*-algebra B being the Kasparov group $KK(A,B)$. Composition of morphisms is defined to be the Kasparov product $KK(A,B) \times KK(B,C) \to KK(A,C) : (f,g) \mapsto fg := f \otimes_B g$. Analogously, we have the Kasparov category KK^G in the group equivariant setting with respect to a given second-countable locally compact group G.

By the work of J. Cuntz [7] and N. Higson [10] it became clear that Kasparov's KK-theory allows a very elegant characterization when restricted to the class of ungraded separable C^*-algebras. Cuntz noted that if F is a stable, homotopy invariant, split-exact functor F from the C^*-category C^* to the abelian groups Ab, then each KK-theory element of $KK(A,B)$ induces a map $F(A) \to F(B)$. Higson brought these findings to its final form by showing that the Kasparov category KK is universal in this respect in the sense that every such functor F factorizes over the Kasparov category KK. This fact is called the *universal property of KK-theory*. K. Thomsen has generalized this result to the group equivariant setting, that is, to the category KK^G.

Quite straightforward, in [2] we described KK^G-theory by generators and relations based on Cuntz and Higsons's findings. We denoted it by GK-theory ('generators K-theory', the group G is not indicated) for better clearity. One advantage of this basic construction is that it may be straightforwardly generalized to other modes of equivariance, that is, to other objects than groups G, for example semigroups G, categories G and so on. Also, one may change the category C^* to another category of (topological) algebras under adaption of the stability property, say. Another advantage is that it is more elementary than Kasparov's original definition. Its definition is also clearer motivated by its relative naturality, whereas the definition of the original KK-theory appears highly unmotivated at first (without further background like the Atyiah–Singer index theory). Also Cuntz's picture of KK-theory by quasi isomorphisms in [8] appears still rather technical and difficult.

A disadvantage of GK-theory is that the Kasparov product is not computed. It remains a formal, uncomputed product $f \cdot g$. On the other hand, this makes GK-theory also easy, again. Also, the general construction of the Kasparov product in KK-theory uses the indirect, unexplicit axiom of choice. In concrete computations the product has to be guessed, which is rather difficult.

We are going to briefly recall GK-theory. For more details see [2].

DEFINITION 3.1. (C^*-category C^*) Let G be a second-countable locally compact group, or a discrete countable inverse semigroup. Denote by C^* the category with objects being the C^*-algebras equipped with an action by G, and morphisms being the G-equivariant $*$-homomorphisms.

If nothing else is said, we could also allow that G is another equivariance-inducing object like a general topological group, or a groupoid, or a category, or a semigroup and so on.

DEFINITION 3.2. (Synthetical morphisms) We introduce two types of *synthetical morphisms*.

(a) For each *corner embedding* $c \in C^*(A, A \otimes \mathscr{K})$, that is a map defined by $c(a) = a \otimes e$ for a one-dimensional projection $e \in \mathscr{K}$ (where the G-action on $A \otimes \mathscr{K}$ need not be diagonal but may be any) introduce one synthetical morphism (inverse map, localization) $c^{-1} : A \otimes \mathscr{K} \to A$.

(b) For each short split exact sequence

$$\mathscr{S}: \; 0 \longrightarrow A \xrightarrow{\;i\;} D \underset{s}{\overset{f}{\rightleftarrows}} B \longrightarrow 0 \tag{2}$$

in C^* introduce one synthetical morphism $P_{\mathscr{S}}^{-1} : D \to A \oplus B$ (inverse map, localization).

DEFINITION 3.3. (Preadditive Category W) Let W be the preadditive category with object class $\mathrm{Obj}(C^*)$. The morphism class $W(A,B)$ from object A to object B let be the collection of all formal expressions

$$\pm a_{11}a_{12}\cdots a_{1n_1} \pm \cdots\cdots \pm a_{k,1}a_{k,2}\cdots a_{k,n_k}, \tag{3}$$

where each *letter* a_{ij} is either a morphism in C^* or one of the synthetical morphisms c^{-1} or $P_{\mathscr{S}}^{-1}$ of Definition 3.2. Each $\pm$ stands here either for a single $+$-sign or a single $-$-sign.

We think of a *word* $a_{i1}\cdots a_{i,n_i}$ as a composition of morphisms (=arrows) a_{ij} going from the left to the right with start point A and end point B, that is, as a picture

$$A = A_{i1} \xrightarrow{a_{i1}} A_{i2} \xrightarrow{a_{i2}} A_{i3} \xrightarrow{a_{i3}} \cdots \qquad \xrightarrow{a_{i,n_i}} A_{i,n_i} = B$$

for objects A_{ij}. We require here that the range object $A_{i,j+1}$ of the morphism a_{ij} coincides with the source object of the morphism $a_{i,j+1}$ for all ij.

Composition and addition of morphisms in W is given formally (i.e. freely). That is, we add and multiply morphisms of the from (3) like in a ring by using the distributive law.

DEFINITION 3.4. (GK-theory) The category GK is defined to be additive category which comes out when dividing the preadditive category W by the following relations:

(a) The canonical assignment $C^* \to GK$ is a *functor*, i.e. we require $fg = g \circ f$ in $GK(A,C)$ for all elements $f \in C^*(A,B)$ and $g \in C^*(B,C)$.

(b) The category GK is *additive*, i.e. we require $p_A i_A + p_B i_B = 1_{A\oplus B}$ in $GK(A \oplus B, A \oplus B)$ for all natural diagrams $A \underset{p_A}{\overset{i_A}{\rightleftarrows}} A \oplus B \underset{i_B}{\overset{p_B}{\rightleftarrows}} B$ (canonical injections and projections) in C^*.

(c) The category GK is *homotopy invariant*, that is, every pair of homotopic G-equivariant $*$-homomorphisms $f_0, f_1 : A \to B$ (homotopic within C^*) satisfies the identity $f_0 = f_1$ in GK.

(d) The category GK is *stable*, that is, every corner embedding c is invertible in GK with inverse c^{-1} as introduced in Definition 3.2.(a).

(e) The category GK is *split exact*, that is, for every split exact sequence (2) in C^* the morphism $P_{\mathscr{S}} := p_A i + p_B s$ in the following diagramm

$$\begin{tikzcd}
& A \oplus B \arrow[dl, shift right, "p_A"'] \arrow[d, shift left, "P_{\mathscr{S}}"] \arrow[dr, shift left, "p_B"] & \\
A \arrow[ur, shift right, "i_A"'] \arrow[r, shift left, "i"] & D \arrow[l, dotted, shift left, "t_{\mathscr{S}}"] \arrow[u, dotted, shift left, "{P_{\mathscr{S}}}^{-1}"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "s"] \arrow[ul, shift left, "i_B"]
\end{tikzcd} \tag{4}$$

is invertible in GK with inverse $P_{\mathscr{S}}^{-1}$ as introduced in Definition 3.2.(b). (Here, p_A, p_B, i_A, i_B are the canonical projections and injections, and the dotted arrow $t_{\mathscr{S}}$ may be ignored here.)

The category GK is just another model for Kasparov's KK^G-theory:

PROPOSITION 3.5. ([2]) *Let G be a locally compact second-countable group, or a discrete countable inverse semigroup. Let C^* be restricted to the subcategory of separable C^*-algebras.*

Then, the categories KK^G and GK are isomorphic.

Proof. Almost evident as KK^G-theory and GK-theory are characterized by the same universal property. See [2, Theorem 5.1] for more details. □

In this section we are going to show that expression (3) of a morphism in GK may be considerably simplified. A first simplification will be reduction of sum, where the notion word is defined in Definition 3.3:

LEMMA 3.6. *In GK we may rewrite any plus-signed sum $x_1 + \ldots + x_n$ of words x_i as a single word x. In particular, any morphism in GK is presentable as a difference $x - y$ of some words $x, y \in GK$.*

Proof. By induction, it clearly suffices to show that any sum $x + y$ of two words $x, y \in GK$ is presentable as a single word.

Assume that we have given a split exact sequence $\mathscr{S}$, see (2), for which we consider $\vartheta := P_{\mathscr{S}} = p_A i + p_B s \in GK(X,Y)$ of Definition 3.4. Define

$$(\vartheta \oplus \mathrm{id}_X) : X \oplus X \to Y \oplus X :$$

$$\vartheta \oplus \mathrm{id}_X := p_A i \oplus \mathrm{id}_X + p_B s \oplus 0_X = (p_A \oplus \mathrm{id}_X)(i \oplus \mathrm{id}_X) + (p_B \oplus 0_X)(s \oplus 0_X).$$

Notice that $\vartheta \oplus \mathrm{id}_X$ is just $P_{\mathscr{T}}$ for the split exact sequence

$$\mathscr{T}:\ 0 \longrightarrow A \oplus X \xrightarrow{i \oplus \mathrm{id}_X} D \oplus X \underset{s \oplus 0}{\overset{f \oplus 0}{\rightleftarrows}} B \longrightarrow 0\ .$$

Consider the canonical projections and embeddings

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} X \oplus X \underset{i_2}{\overset{p_2}{\rightleftarrows}} X, \qquad Y \xleftarrow{p_1'} Y \oplus X \xrightarrow{p_2'} X\ .$$

Set $\vartheta^{-1} := P_{\mathscr{S}}^{-1}$. Then observe that

$$p_1 i_1 = (\vartheta \oplus \mathrm{id}_X) p_1' \vartheta^{-1} i_1, \qquad p_2 i_2 = (\vartheta \oplus \mathrm{id}_X) p_2' i_2,$$

so that with $p_1 i_1 + p_2 i_2 = \mathrm{id}_{X \oplus X}$ we get

$$(\vartheta \oplus \mathrm{id}_X)^{-1} = p_1' \vartheta^{-1} i_1 + p_2' i_2, \tag{5}$$

$$(\vartheta \oplus \mathrm{id}_X) = (\mathrm{id}_{X \oplus X})(\vartheta \oplus \mathrm{id}_X) = p_1 \vartheta i_1 + p_2 i_2'. \tag{6}$$

If we have given a corner embedding $\vartheta := c \in C^*(X := A, Y := A \otimes \mathscr{K})$ then we set $(\vartheta \oplus \mathrm{id}_X) : X \oplus X \to Y \oplus X$ obvious and get again relations (5) and (6). Notice that in this case $(\vartheta \oplus \mathrm{id}_X)^{-1}$ is just the word $(\mathrm{id}_{A \otimes \mathscr{K}} \oplus e) d^{-1}$ for the corner embeddings $d \in C^*(A \oplus X, A \otimes \mathscr{K} \oplus X \otimes \mathscr{K})$ and $e \in C^*(X, X \otimes \mathscr{K})$.

By some abuse of notation, in the sequel we shall omit notating the primes in p_1' and p_2' and simply write p_1 and p_2 instead. In other words, we shall not indicate the involved spaces X and Y in our notation, even when we are going to have different spaces. As already above, the index 1 will mean projection or embedding on the first (left hand sided) coordinate, and 2 on the second (right hand sided) coordinate.

Let us be given two words $x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$ and $y_1^{\varepsilon_1} \ldots y_m^{\varepsilon_m}$ in $GK(X,Y)$, where $x_i \in GK(X_i, X_{i+1})$ and $y_j \in GK(Y_j, Y_{j+1})$ are either morphisms in C^* or morphisms $P_{\mathscr{S}}$, and let $\varepsilon_i, \varepsilon_j \in \{1, -1\}$ present exponents in case letters are invertible by synthetical inverses as defined in Definition 3.2. The expression $x_i^1 = P_{\mathscr{S}}^1$ is not allowed, because $P_{\mathscr{S}}$ can be expressed by morphisms in C^*.

Let $j : X \to X \oplus X$ be defined by $j(x) = (x,x)$. Let $d : Y \oplus Y \to M_2(Y)$ be the diagonal embedding $d(x,y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ and $k : B \to M_2(Y)$ the corner embedding $k(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$. Using the identities (5) and (6) and their analogs, and the orthogonality relations $i_2 p_1 = 0$ and $i_1 p_2 = 0$, the following computation shows our claim. Simply consider the word

$$\begin{aligned}
& j(x_1 \oplus \mathrm{id}_X)^{\varepsilon_1} \cdots (x_n \oplus \mathrm{id}_X)^{\varepsilon_n} (\mathrm{id}_X \oplus y_1)^{\varepsilon_1} \cdots (\mathrm{id}_X \oplus y_m)^{\varepsilon_m} d k^{-1} \\
= {} & j(p_1 x^{\varepsilon_1} i_1 + p_2 i_2) \cdots (p_1 x_n^{\varepsilon_n} i_1 + p_2 i_2) \\
& \cdot (p_1 i_1 + p_2 y_1^{\varepsilon_1} i_2) \cdots (p_1 i_1 + p_2 y_m^{\varepsilon_m} i_2) d k^{-1} \\
= {} & j(p_1 x^{\varepsilon_1} \cdots x_n^{\varepsilon_n} i_1 + p_2 y_1^{\varepsilon_1} \cdots y_m^{\varepsilon_m} i_2) d k^{-1} \\
= {} & x^{\varepsilon_1} \cdots x_n^{\varepsilon_n} + y_1^{\varepsilon_1} \cdots y_m^{\varepsilon_m},
\end{aligned}$$

where for the last identity we have used that the $*$-homomorphism $i_2 d$ is homotopic to the $*$-homomorphism $i_1 d$ by rotation, and $i_1 d k^{-1} = \mathrm{id}_Y$. $\square$

Instead of the split exactness axiom in the definition of GK we may use alternatively the following axiom without difference.

LEMMA 3.7. *Instead of introducing the synthetical arrows $P_{\mathscr{S}}^{-1}$ in Definition 3.2.(b) and using axiom 3.4.(e) we may alternatively introduce the dotted arrow $t_{\mathscr{S}}$ for each split exact sequence (2) and the axiomatic relations*

$$i t_{\mathscr{S}} = 1_A, \qquad t_{\mathscr{S}} i + f s = 1_D$$

(as a replacement of Definition 3.4.(e)) without changing the definition of GK.

It would not make any difference in the definition of GK if we added both $P_{\mathscr{S}}^{-1}$ and $t_{\mathscr{S}}$ simultaneously, because they automatically define each other as follows in GK:

LEMMA 3.8. *$P_{\mathscr{S}}^{-1}$ and $t_{\mathscr{S}}$ of diagram (4) define each other as follows:*

$$t_{\mathscr{S}} = P_{\mathscr{S}}^{-1} p_A, \qquad P_{\mathscr{S}}^{-1} = t_{\mathscr{S}} i_A + f i_B$$

Proof of Lemmas 3.7 and 3.8. Let GK be the category with the usual split exactness axiom involving $P_{\mathscr{S}}$, and GK' the category with the alternative split exactness axiom involving $t_{\mathscr{S}}$. Let $\Phi : GK \to GK'$ and $\Psi : GK' \to GK$ be the functors which are identical on C^* and on the synthetical inverses of corner embeddings, and according to the 'transformation' rules defined to be

$$\Phi(P_{\mathscr{S}}^{-1}) = t_{\mathscr{S}} i_A + f i_B, \quad \Psi(t_{\mathscr{S}}) = P_{\mathscr{S}}^{-1} p_A$$

for each split exact sequence $\mathscr{S}$.

We remark that $s t_{\mathscr{S}} = 0$ because $s t_{\mathscr{S}} = s t_{\mathscr{S}} i t_{\mathscr{S}} = s(1 - fs) t_{\mathscr{S}} = 0$. To see that Φ is well-defined we compute

$$\Phi(P_{\mathscr{S}})\Phi(P_{\mathscr{S}}^{-1}) = (p_A i + p_B s)(t_{\mathscr{S}} i_A + f i_B) = 1_{A \oplus B}, \quad \Phi(P_{\mathscr{S}}^{-1})\Phi(P_{\mathscr{S}}) = 1_D.$$

To show that Ψ is well-defined we calculate

$$\Psi(i)\Psi(t_{\mathscr{S}}) = i P_{\mathscr{S}}^{-1} p_A = i_A p_A i P_{\mathscr{S}}^{-1} p_A = i_A (P_{\mathscr{S}} - p_B s) P_{\mathscr{S}}^{-1} p_A = i_A p_A = 1_A,$$

$$\Psi(t_{\mathscr{S}})\Psi(i) + \Psi(f)\Psi(s) = P_{\mathscr{S}}^{-1} p_A i + fs = P_{\mathscr{S}}^{-1}(P_{\mathscr{S}} - p_B s + P_{\mathscr{S}} f s) = 1_D.$$

That Ψ and Φ are inverses to each other follows then from the observation

$$\Psi \circ \Phi(P_{\mathscr{S}}^{-1}) = P_{\mathscr{S}}^{-1}(p_A i_A + P_{\mathscr{S}} f i_B) = P_{\mathscr{S}}^{-1}, \quad \Phi \circ \Psi(t_{\mathscr{S}}) = t_{\mathscr{S}}. \quad \square$$

We remark that we have also shown in the last proof that $s t_{\mathscr{S}} = 0$. (That shows even more more clearly that $D \cong A \oplus B$ in GK.) Again, the element $t_{\mathscr{S}}$ is uniquely defined by its defining relations. Also, Lemma 3.6 would hold if we had introduced $t_{\mathscr{S}}$ instead of $P_{\mathscr{S}}^{-1}$. All these follows immediately as a corollary from the formula $t_{\mathscr{S}} = P_{\mathscr{S}}^{-1} p_A$ of Lemma 3.8.

We can always move the inverse c^{-1} of a corner embedding $c \in C^*$ to the right in a word if the stabilization is equipped with a diagonal action:

LEMMA 3.9. *Let (A,α) and (B,β) be G-algebras. If $f : A \to B$ is a morphism in C^*, $c : (A,\alpha) \to (A \otimes \mathscr{K}, \alpha \otimes \gamma)$ (the diagonal action is essential here) a corner embedding, then there exists a corner embedding $c' : B \to B \otimes \mathscr{K}$ and a morphism $f' : A \otimes \mathscr{K} \to B \otimes \mathscr{K}$ in C^* such that $c^{-1} f = f' c'^{-1}$.*

Analogously, we have $c^{-1} t_{\mathscr{S}} = t_{\mathscr{S}'} c'^{-1}$. Similarly, $c^{-1} P_{\mathscr{S}}^{-1} = P_{\mathscr{S}'}^{-1} \varphi^{-1} c'^{-1}$, where φ is the canonical isomorphism $(A \oplus B) \otimes \mathscr{K} \to A \otimes \mathscr{K} \oplus B \otimes \mathscr{K}$.

Proof. This follows from the commutation relation $c(f \otimes \mathrm{id}_{\mathscr{K}}) = f c'$ for the morphism $f' = f \otimes \mathrm{id}_{\mathscr{K}}$ and the diagonal action $\beta \otimes \gamma$ on $B \otimes \mathscr{K}$. The case $P_{\mathscr{S}}^{-1}$ is analog: since $\mathscr{K}$ is an exact C^*-algebra we can tensor the diagrams (2) and (4) with $\mathscr{K}$, then check $P_{\mathscr{S}} c = c' \varphi P_{\mathscr{S}'}$, where $c : D \to D \otimes \mathscr{K}$, $c' : A \oplus B \to (A \oplus B) \otimes \mathscr{K}$ and $\mathscr{S}' = \mathscr{S} \otimes \mathscr{K}$ (also with additivity, Definition 3.4.(b)). The case $t_{\mathscr{S}}$ follows from that and $t_{\mathscr{S}} = P_{\mathscr{S}}^{-1} p_A$ of Lemma 3.8. $\square$

A drastical simplification of morphisms in GK goes by the Cuntz picture:

PROPOSITION 3.10. *Let G be a locally compact second-countable group or a countable inverse semigroup and the category C^* be restricted to separable C^*-algebras.*

Every morphism z in GK may be written in the form

$$z = a d^{-1} \cdot t_{\mathscr{S}} \cdot e f^{-1} \cdot t_{\mathscr{T}} \cdot c^{-1}$$

for some homomorphism $a \in C^*$, some split exact sequences $\mathscr{S}$ and $\mathscr{T}$, and some corner embeddings $c, d, e, f \in C^*$.

If the morphism z is in $GK(A,B)$ and B is unital we can omit $t_{\mathscr{T}}$ (i.e. $t_{\mathscr{T}} = 1$). If G is the trivial group then d^{-1}, f^{-1} and e can be omitted (i.e. $d^{-1} = f^{-1} = e = 1$). Both simplifications can be combined simultaneously.

Proof. By the universal property of KK^G and GK there is an isomorphism of categories $\hat{G} : KK^G \to GK$, see Proposition 3.5. The idea is now to keep track of the formulas appearing in the proof of this fact and see how a morphism $z \in KK^G$ is presented as $\hat{G}(z)$ in GK. The original proof of the universal property of KK is by Cuntz [7] and Higson [10], and by Thomsen [18] in the group equivariant setting for KK^G. We shall refer here to our exposition in the inverse semigroup equivariant setting [3]. All we shall do here may be read verbatim topological group equivariantly.

Let us be given fixed objects $A, B \in C^*$. Assume at first that B is stable, i.e. $B \cong B \otimes \mathscr{K}$ in C^* ($\mathscr{K}$ equipped with the trivial G-action).

In [3, Theorem 8.5], there is stated an isomorphism

$$\Phi : \mathbb{F}^G(A,B) \to KK^G(A,B).$$

Here, $\mathbb{F}^G(A,B)$ is just the Cuntz-picture of G-equivariant KK-theory by quasi homomorphisms and G-cocycles, see [3, Def. 7.1 and Def. 7.8]. To recall it, an element $x = [\varphi_+, \varphi_-, u_+, u_-] \in \mathbb{F}^G(A,B)$ is given by two G-equivariant $*$-homomorphisms $\varphi_\pm : A \to \mathscr{M}(B)$ and two α-cocycles $u_\pm : G \to \mathscr{M}(B)$, see [3, Def. 5.1].

One has two split-exact sequences (for $+$ and $-$)

$$\mathscr{S}_\pm:\ 0 \longrightarrow (B,\Gamma^\pm) \xrightarrow{j} (A_x,\Gamma^\pm) \underset{s^\pm}{\overset{p}{\rightleftarrows}} (A,\alpha) \longrightarrow 0$$

for $A_x := \{A \oplus \mathscr{M}(B) | \varphi_+(a) = m \mod B\}$ by [3, Def. 9.1 and 9.4].

Define the split-exact, homotopy invariant, stable functor F from C^* to the abelian groups by

$$F(B) = GK(A,B) \text{ and } F(f : B \to C) : GK(A,B) \to GK(A,C) : z \mapsto zf.$$

For an α-cocycle $u \in \mathscr{M}(A)$, recall [3, Def. 5.4, 6.1 and 6.2] for the definition of an abelian group isomorphism

$$u_\# = F(T_{u,A})^{-1} \circ F(S_{u,A}) : F(A,\alpha) \to F(A, u\alpha u^*)$$

and corner embeddings $S_{u,A}, T_{u,A} : A \to M_2(A,\delta_u)$.

As in [3, Def. 9.4], define an abelian group homomorphism

$$\Psi_x : F(A) \to F(B) : \Psi_x = u_{-\#}^{-1} \circ F(j)^{-1} \circ \big(u_\# \circ F(s_+) - F(s_-)\big) \tag{7}$$

(here u is the cocycle for A_x of [3, Def. 9.1]!).

Now assume that B is not necessarily stable. In [3, Def. 10.2] there appears a similar variant

$$\Psi'_z : F(A) \to F(B) : \Psi'_z = F(c_B)^{-1} \circ F(j_B)^{-1} \circ \Psi_{j_{B*}c_{B*}(\Phi^{-1}(z))}$$

of Ψ_x, where $z \in KK^G(A,B)$. Here $c_B : B \to B \otimes \mathscr{K}$ is the corner embedding, see [3, Def. 10.1], and j_B appears in some split exact sequence

$$\mathscr{T}:\ 0 \longrightarrow B \otimes \mathscr{K} \xrightarrow{j_B} B^+ \otimes \mathscr{K} \xrightarrow{p_B} C^*(E) \otimes \mathscr{K} \longrightarrow 0$$

in [3, Def. 10.2]. The stars in j_{B*} and c_{B*} are defined in [3, Def. 8.6].

By [3, Def. 11.1] there is a natural transformation

$$\xi : KK(A,-) \to F(-) : \xi_B(z) = \Psi'_z(1_{GK(A,A)}).$$

We are now applying [3, Thm. 1.3] (= [3, Thm. 12.4]) to the canonical quotient functor $G : C^* \to GK$, which is split-exact, homotopy invariant and stable. The claim and proof of [3, Thm. 12.4] show that there is a functor $\hat{G} : KK^G \to GK$ defined by

$$\hat{G}(z) = \xi_B(z)$$

for all $z \in KK^G(A,B)$ such that G factorizes over $\hat{G}$ (i.e. $G = \hat{G} \circ G_2$ for the canonical quotient functor $G_2 : C^* \to KK^G$). This functor is an isomorphism, since GK itself has the universal properties of KK^G, confer [2, 5.1].

In details we get

$$\hat{G}(z) = \xi_B(z) = \Psi'_z(1_{GK(A,A)}) = F(c_B)^{-1} \circ F(j_B)^{-1} \circ \Psi_{j_{B*}c_{B*}(\Phi^{-1}(z))}(1_{GK(A,A)}).$$

Now observe that for the corner embedding c_B, the inverse map $F(c_B)^{-1}$ is just realized by right multiplication with the synthetical inverse c_B^{-1} in GK. Similarly, according to the split-exactness of GK the (one-sided) inverse map $F(j_B)^{-1}$ is just right multiplication with the synthetical (one-sided) inverse $t_{\mathscr{T}}$. Indeed, by the definition of F, if z in GK is of the form $z = F(j)(w) = wj$ then $w = zt_{\mathscr{T}}$.

We choose now the x from above as $x := j_{B*}c_{B*}(\Phi^{-1}(z)) \in \mathbb{F}^G(A, B^+ \otimes \mathscr{K})$ and put formula (7) into the formula of $\hat{G}(z)$. Here, z is the given morphism in KK^G that we want to present in GK via $\hat{G}$. Then we have

$$\begin{aligned} \hat{G}(z) &= F(c_B)^{-1} \circ F(j_B)^{-1} \circ u_{-\#}^{-1} \circ F(j)^{-1} \circ \big(u_\# \circ F(s_+) - F(s_-)\big)\,(1_{GK(A,A)}) \\ &= 1_{GK(A,A)} \cdot (s_+ S_{u,A} T_{u,A}^{-1} - s_-) \cdot t_{\mathscr{S}_-} T_{u_-,A} S_{u_-,A}^{-1} \cdot t_{\mathscr{T}} c_B^{-1} \\ &= ad^{-1} \cdot t_{\mathscr{S}_-} ef^{-1} t_{\mathscr{T}} \cdot c^{-1} \end{aligned}$$

in $GK(A,B)$ by Lemma 3.9 for a suitable homomorphism $a \in C^*$ and corner embeddings c,d,e,f. We used here the fact that $s_- t_{\mathscr{S}_-} = 0$ by the remark after Lemma 3.8.

If B is unital we can omit j_B in the definition of Ψ'_z. If G is trivial all cocycles satisfy $u = 1$ and thus all $u_\# = 1$. $\square$

It is however rather difficult to bring a product of such standardized elements as in Proposition 3.10 again to such a standard form, see Cuntz [7]. It is not really easier than forming the Kasparov product of Kasparov cycles.

REMARK 3.11. A further slight simplification of the split exactness axiom could be done by observing that the split exact sequence (2) is isomorphic in C^* to an idempotent $*$-homomorphisms $P : D \to D$ (translation is $P = fs$). Then split exactness just says that every idempotent $P \in C^*$ has an orthogonal split $t_{\mathscr{S}} : D \to \ker(P)$ in GK (orthogonal projection: $t_{\mathscr{S}} i = 1_D - P$).

4. E-continuity and Hausdorff property

In this section we shall see that the groupoid associated to an inverse semigroup is Hausdorff if and only if the inverse semigroup is E-continuous. This condition is technically easier and more intrinsic to the inverse semigroup. We shall see that E-continuity is a necessary and sufficient condition to define a non-degenerate $C_0(X)$-compatible $C_0(X)$-valued $L^2(G)$-module.

Let G be a discrete inverse semigroup.

DEFINITION 4.1. (E and X) Let E denote the subset of idempotent elements of G. The free universal *abelian* C^*-algebra $C^*(E)$ generated by the commuting self-adjoint projections of E has a totally disconnected Gelfand spectrum X. That is we have $C^*(E) \cong C_0(X)$. Under this isomorphism we identify E as a subset of $C_0(X)$

(under the formula $e(x) = x(e)$). To this end, we also use the suggestive notation $1_e \in C_0(X)$ for the corresponding element of $e \in E$ in $C_0(X)$. We write "$x \in e$" for $x \in X$ and $e \in E$ iff x is an element of the support of $1_e \in C_0(X)$ (also denoted by carrier(1_e)). For $e, f \in E$ we use the usual order $e \leqslant f$ in a C^*-algebra. This *order* can be extended to G by saying that $g \leqslant h$ for $g, h \in G$ iff $g = hg^*g$ (or equivalently iff $g = gg^*h$).

DEFINITION 4.2. (G-action) In this note we understand under a *G-action* on a C^*-algebra A a semigroup homomorphism $\alpha : G \to \mathrm{End(A)}$ such that $\alpha_e(a)b = a\alpha_e(b)$ (compatibility) for all e in E. In this case, A is called a *G-algebra*. A G-action on a Hilbert A-module $\mathscr{E}$ is a semigroup homomorphism $U : G \to \mathrm{LinMaps}(\mathscr{E})$ (linear maps) such that U_e is an adjointable operator for all $e \in E$, and

$$\langle U_g(\xi), U_g(\eta)\rangle = g(\langle \xi, \eta\rangle), \;\; U_g(\xi a) = U_g(\xi)\alpha_g(a), \;\; U_e(\xi)a = \xi\alpha_e(a)$$

(the last identity being called *compatibility* or $C_0(X)$*-compatibility* of U) for all $\xi, \eta \in \mathscr{E}, a \in A, g \in G$ and $e \in E$. Then $\mathscr{E}$ is called a (compatible) *G-Hilbert A-module*. Often we write the G-action in the form $g(\xi) := U_g(\xi)$ and $g(a) := \alpha_g(a)$.

DEFINITION 4.3. (G-action on X) The C^*-algebra $C_0(X)$ is equipped with the G-action $g(1_e) := 1_{geg^*}$ for $e \in E, g \in G$. This G-action may be extended to the bigger C^*-algebra $\ell^\infty(X,\mu)$, where μ is the discrete counting measure, by setting $(g(f))(x) := 1_{\{x\cdot g \neq 0\}} f(x \cdot g)$ for $g \in G, f \in \ell^\infty(X,\mu)$ and characters $x \in X$, where the (possibly zero) character $x \cdot g : C^*(E) \to \mathbb{C}$ is defined by $(x \cdot g)(e) = x(geg^*)$ for all $e \in E$.

We are going to recall the E-continuity property of an inverse semigroup. For more details see [4] or [5].

Immediately after publishing a preprint of this paper, Benjamin Steinberg and Ruy Exel came in contact with us per email and kindly pointed out to us that they have already considered this condition. Steinberg refers to [16] and [17], and Exel to [9]. We thank them for informing us about their work.

In the next few paragraphs (until Lemma 4.7) we shall identify elements $e \in E$ with their corresponding characteristic functions 1_e in $C_0(X)$. Write $\mathrm{Alg}^*(E)$ for the dense $*$-subalgebra of $C_0(X)$ generated by the characteristic functions 1_e for all $e \in E$. Moreover, write $\bigvee_i f_i : X \to \mathbb{C}$ for the pointwise supremum of a family of functions $f_i : X \to \mathbb{C}$.

DEFINITION 4.4. An inverse semigroup G is called *E-continuous* if the function $\bigvee\{e \in E | e \leqslant g\} \in \mathbb{C}^X$ (in precise notation: $\bigvee\{1_e \in C_0(X) | e \in E, e \leqslant g\} \in \mathbb{C}^X$) is a *continuous* function in $C_0(X)$ for all $g \in G$.

A simple compactness argument shows the following, see [4] or [5]:

LEMMA 4.5. *An inverse semigroup G is E-continuous if and only if for every $g \in G$ there exists a finite subset $F \subseteq E$ such that* $\bigvee\{e \in E | e \leqslant g\} = \bigvee\{e \in F | e \leqslant g\}$.

DEFINITION 4.6. (Compatible $C_0(X)$-valued $L^2(G)$-module, [4] or [5]) Let G be an E-continuous inverse semigroup. Write c for the linear span of all functions $\varphi_g : G \to \mathbb{C}$ (in the linear space $\mathbb{C}^G$) defined by

$$\varphi_g(t) := 1_{\{t \leqslant g\}}$$

(characteristic function) for all $g,t \in G$. Endow c with the G-action $g(\varphi_h) := \varphi_{gh}$ for all $g,h \in G$. Turn c to an $\mathrm{Alg}^*(E)$-module by setting $\xi e := e(\xi)$ for all $\xi \in c$ and $e \in E$. Define an $\mathrm{Alg}^*(E)$-valued inner product on c by

$$\langle \varphi_g, \varphi_h \rangle := \bigvee \{ e \in E \,|\, eg = eh,\, e \leqslant gg^*hh^* \}. \tag{8}$$

The norm completion of c is a G-Hilbert $C_0(X)$-module denoted by $\widehat{\ell^2}(G)$.

LEMMA 4.7. ([4] or [5]) *The vectors* $(\varphi_g)_{g \in G} \subseteq \widehat{\ell^2}(G)$ *are linearly independent.*

We recall the well-known topological groupoid associated to an inverse semigroup by Paterson [15]:

DEFINITION 4.8. (Groupoid associated to an inverse semigroup) Let G be a discrete inverse semigroup and X the Gelfand spectrum of $C^*(E)$. Consider the topological subspace $G * X = \{(g,x) \in G \times X \,|\, g \in G,\, x \in g^*g\}$ of the topological space $G \times X$ (product topology with G having the discrete topology). Two points $(g,x),(h,y)$ in $G * X$ are called *equivalent*, also denoted $(g,x) \equiv (h,y)$, iff $x = y$ and $ge = he$ for some $e \in E$ with $x \in e$. Let $\pi : G * X \to G * X/\equiv$ denote the set-theoretical quotient map. The quotient is a groupoid under the multiplication: $\pi(g,x)\pi(h,y) = \pi(gh,y)$ if and only if for all $e \in E$ such that $y \in e$ one has $x \in (he)(he)^*$. Otherwise the composition is declared to be undefined.

We now regard the quotient $G * X/\equiv$ as a topological groupoid under the quotient topology and call it the *groupoid asscociated to the inverse semigroup* G. (Recall that a subset $Y \subseteq G * X/\equiv$ is declared to be open if and only if $\pi^{-1}(Y)$ is open.)

Usually the groupoid associated to G is a non-Hausdorff topological space. We are going to prove that the Hausdorff condition is equivalent to E-continuity of G.

LEMMA 4.9. *The sets of the form* $\pi(g \times U)$*, where* $g \in G$ *and* $U \subseteq X$ *is an open subset of* X *with* $U \subseteq \mathrm{carrier}(g^*g)$*, are open and generate the topology of* $G * X/\equiv$*. (Here* $g \times U := \{g\} \times U$*.)*

Proof. We claim that the inverse $\pi^{-1}(\pi(g \times U))$ is open. Indeed if $(h,x) \in \pi^{-1}(\pi(g \times U))$ then it is equivalent to some $(g,x) \in g \times U$. Hence there exists some $e \in E$ with $x \in e$ and $he = ge$. Let $V = \mathrm{carrier}(e) \cap U \cap \mathrm{carrier}(h^*h)$. Then $h \times V$ is an open subset of $\pi^{-1}(\pi(g \times U))$ containing (h,x).

If $\pi^{-1}(O)$ is open and contains the point (g,x) together with its open neighborhood $g \times U$ then $\pi^{-1}(\pi(g \times U)) \subseteq \pi^{-1}(O)$. Thus $\pi(g \times U) \subseteq O$. Hence such sets generate the topology. $\square$

We call $\pi(g \times U)$ the *open set in* $G * X/\equiv$ *generated by* $g \times U$.

LEMMA 4.10. *If G is E-continuous then its associated groupoid is Hausdorff.*

Proof. Let $(g,x),(h,x) \in G*X$ be two points such that $(g,x) \not\equiv (h,x)$. Then for all $e \in E$ with $x \in e$ and $e \leqslant g^*gh^*h$ one has $ge \neq he$, and so $e \nleqslant h^*g$. Since G is E-continuous the function $F := \bigvee_{f \in E, f \leqslant h^*g} f$ is continuous. Note that $x \notin F$.

Let $t \subseteq X$ be the (open!) complement of the carrier of F. Consider $U_g := \{g\} \times t \cap \text{carrier}(g^*g)$ and $U_h := \{h\} \times t \cap \text{carrier}(h^*h)$. Clearly $x \in t$ and so $(g,x) \in U_g$ and $(h,x) \in U_h$.

Consider the open subsets W_g and W_h that U_g and U_h generate in $G*X/\equiv$. Assume W_g and W_h would intersect. Then there are $(g,y) \in U_g$, $(h,z) \in U_h$ such that $(g,y) \equiv (h,z)$. That is, there is a $e \in E$ such that $y = z \in e$, $e \leqslant g^*gh^*h$ and $ge = he$. Hence $y \in e \leqslant F$. By definition of U_g one has also certainly $y \in t$. A contradiction. This shows that W_g and W_h are disjoint neighborhoods which separate (g,x) and (h,x). $\square$

LEMMA 4.11. *If its associated groupoid is Hausdorff then G is E-continuous.*

Proof. Let $g \in G$. Assume the projection $F := \bigvee_{f \in E, f \leqslant g} f$ would be discontinuous, say in the point $x \in X$.

Then for any neighborhoods U of x there is at least one $f \leqslant g$ ($f \in E$) such that U has nonempty intersection with the carrier of f. On the other hand x is not in the carrier of any $f \in E$ with $f \leqslant g$, because there F is continuous.

Consider the points (g,x) and (g^*g,x) in $G*X$. They must be distinct in the quotient $G*X/\equiv$ because assuming to the contrary the existence of some $e \in E$ with $x \in e$ and $g^*ge = ge$ would imply $g^*ge \leqslant g$; a contradiction to what we said above.

Let $U \subseteq \text{carrier}(g^*g) \subseteq X$ be an open neighborhood of x. Consider the open neighborhoods W_g and W_{g^*g} in $G*X/\equiv$ generated by $\{g\} \times U$ and $\{g^*g\} \times U$. As remarked above we may choose $y \in U, f \in E$ such that $y \in f$ and $f \leqslant g$. Then (g,y) and (g^*g,y) are equivalent because $y \in f$ and $g^*gf = gf$.

Hence W_g, W_{g^*g} intersect. Hence (g,x) and (g^*g,x) cannot be separated. Contradiction. $\square$

COROLLARY 4.12. *An inverse semigroup is E-continuous if and only if its associated groupoid is Hausdorff.*

We have seen in Definition 4.6 that for E-continuous inverse semigroups there exist non-degenerate compatible $L^2(G)$-modules with coefficients in $C_0(X)$. The next example indicates that we cannot construct such $L^2(G)$-modules for E-discontinuous inverse semigroups.

Before that, for the discussion of another $L^2(G)$-module, we recall the following discretized coefficient algebra $\varepsilon(E)$ of $C_0(X)$, see [1].

DEFINITION 4.13. (Discretized coefficient algebra $\varepsilon(E)$ of $C_0(X)$, [1]) Recall that there exists a map $\varepsilon : E \to X$ assigning to each $e \in E$ the character ε_e on $C^*(E)$ determined by the formula $\varepsilon_e(f) = 1_{\{f \geqslant e\}}$ for every $f \in E$. The image $\varepsilon(E)$ is dense in

X, see [15] or [12, 3.2]. We have a G-invariant sub-C^*-algebra

$$\varepsilon(E) := c_0\big(\varepsilon(E)\big) \subseteq \ell^\infty(X)$$

(complex-valued functions on the image of ε vanishing at infinity). Given $e \in E$, we write ε_e for the characteristic one-point supported function $1_{\{\varepsilon_e\}} \in \varepsilon(E) \subseteq \ell^\infty(X)$. One checks that G acts through $g(\varepsilon_e) = \varepsilon_{geg^*}$ if $e \leqslant g^*g$, and $g(\varepsilon_e) = 0$ otherwise.

EXAMPLE 4.14. (Elementary abelian E-discontinuous example) Let us discuss one of the most simplest examples of an (even abelian) inverse semigroup G which is not E-continuous. Let $G = \{1, S, e_1, e_2, e_3, \ldots\}$ consist of an identity element 1, a strictly increasing sequence of projections $e_1 < e_2 < e_3 < \ldots < 1$, and a symmetry $S \neq 1$ (i.e. $S^2 = 1, S^* = S$) such that $Se_n = e_nS = e_n$ for all $n \geqslant 1$. (A concrete representation of G on a direct sum Hilbert space $H \oplus H$ may be given as $1 = \mathrm{id}_H \oplus \mathrm{id}_H$, $S = s \oplus \mathrm{id}_H$ with s a symmetry and $e_n \leqslant 0 \oplus \mathrm{id}_H$.)

The associated C^*-algebra $C^*(G)$ is an AF-algebra. Indeed it is the union of its finite-dimensional sub-C^*-algebras A_n generated by $\{1, S, e_1, \ldots, e_n\}$. One has $A_n \cong \mathbb{C}^{n+2}$ for all $n \geqslant 0$. The two generating projections of $A_0 \cong \mathbb{C}^2$ are $(1 \pm S)/2$. The projection $(1-S)/2$ is orthogonal to all projections e_n, and $e_n < (1+S)/2$. Hence $K_0(C^*(G)) = \bigoplus_{\mathbb{N}} \mathbb{Z} \sqcup \{1\}$ (here 1 denotes an adjoint unit). As an abelian group, this is again isomorphic to $K_0(C^*(G)) \cong \bigoplus_{\mathbb{N}} \mathbb{Z}$.

If we compare this with $\varepsilon(E) \rtimes G$ then we have that it is the union of the sub-C^*-algebras B_n generated by $\varepsilon_1 \rtimes 1, \varepsilon_1 \rtimes S, \varepsilon_{e_1} \rtimes e_1, \ldots, \varepsilon_{e_n} \rtimes e_n$. Again $B_n \cong \mathbb{C}^{n+2}$. Thus $K_0(\varepsilon(E) \rtimes G) = \bigoplus_{\mathbb{N}} \mathbb{Z}$ as the projection $\varepsilon_1 \rtimes (1+S)/2$ is orthogonal to all projections $\varepsilon_{e_n} \rtimes e_n$.

Hence $C^*(G)$ and its "discretized" version $\varepsilon(E) \rtimes G$ have the same K-theory.

We are now coming to the most important point, namely that it appears not possible to construct a non-degenerate $C_0(X)$-compatible $C_0(X)$-valued $L^2(G)$-module. Somehow we should have some sort of generators $\delta_1, \delta_S, \delta_{e_1}, \ldots, \delta_{e_n}, \ldots$ of the module. The G-action should be $g(\delta_h) = \delta_{gh}$ to be regarded as an $L^2(G)$-module. By compatibility of the module product we naturally have $\delta_g e = \delta_g \cdot e(1) = e(\delta_g) \cdot 1 = \delta_{eg}$ for $e \in E \subseteq C_0(X)$. Naturally we should choose $\langle \delta_{e_n}, \delta_{e_n} \rangle = e_n$ for the inner product. By compatibility of the inner product we have $\langle \delta_S, \delta_S \rangle e_n = \langle \delta_S e_n, \delta_S e_n \rangle = e_n$ for all $n \geqslant 1$. Consequently $C_0(X) \ni \langle \delta_S, \delta_S \rangle = 1$ (because the carriers of the elements $e \in E$ generate the topology of X) and similarly $\langle \delta_S, \delta_1 \rangle = \langle \delta_1, \delta_1 \rangle = 1$. But then $\|\delta_1 - \delta_S\| = 0$ and the module degenerates.

Let us discuss another module. We may construct the non-degenerate $\varepsilon(E)$-valued $L^2(G)$-module of [1, Def. 5.5]. The generators are the characteristic functions $\delta_g : G \to \mathbb{C}$ with $\delta_g(h) = 1_{\{g=h\}}$ for $g, h \in G$. The G-action is given by $h(\delta_g) = 1_{\{h^*h \geqslant gg^*\}} \delta_{hg}$. The inner product is determined by $\langle \delta_g, \delta_h \rangle = 1_{\{g=h\}}$, $\langle \delta_p, \delta_p \rangle = \varepsilon_p$ for the projections p in G, and $\langle \delta_S, \delta_S \rangle = \varepsilon_1$. The module product computes as $\delta_p \varepsilon_q = 1_{\{p=q\}}$ for projections p, q, and $\delta_S \varepsilon_1 = \delta_S$ and $\delta_S \varepsilon_{e_n} = 0$.

EXAMPLE 4.15. (Dense E-discontinuity example) In Example 4.14 we had some kind of E-discontinuity only at S (or we may say at 1). We may construct such an E-discontinuity at every e in E by the same method. Start with a given inverse semigroup

$G = E$ consisting only of projections. Adjoin to G for every e in E a symmetry S_e such that $S_e f = f S_e = f$ for all $f < e$. Other relations we do not add. The resulting inverse semigroup G is E-discontinuous in those S_e in the sense that $\bigvee_{f \in E, f \leqslant S_e} f$ is discontinuous where e has no precursor $f < e$. If no element of E has a precursor in E then the E-discontinuity points are dense in X (at the points ε_e we may say, which form a dense subset of X).

EXAMPLE 4.16. (Finitely presented E-discontinuous inverse semigroup) A finitely presented E-discontinuous inverse semigroup may be defined as follows. Consider the finitely presented inverse semigroup

$$G = \langle t, l, e \,|\, tl = lt, t^* l = lt^*, te = e, t^* e = e \rangle.$$

That is t and t^* commute with l and l^*, and e absorbs t and t^*.

Between l and e we have no relations, they are free in G, so that we get infinitely many distinct projections

$$p_0 := e,\ p_1 := lee^* l^*,\ p_2 := llee^* l^* l^*, \ldots,\ p_n := l^n ee^* l^{*n}, \ldots$$

in G. Now $tp_n = p_n$ by the defining relations of G. The projections p_n cannot be compared among each other, i.e. $p_n \leqslant p_m$ implies $n = m$. Hence the criterion for E-continuity of Lemma 4.5 fails for t, as the supremum of $\{e \in E | e \leqslant t\}$ will not be attained at a finite set of projections of E. To see this, let us first note that we have no single projection $q \in E$ such that $t \geqslant q \geqslant p_0, p_1, p_2, \ldots$. Indeed, every such projection q would require to include the letter e to obtain $t \geqslant q$, and consequently any letter t or t^* in q would be absorbed by e. So q would allow a presentation with letters l and e and their adjoints only, and such a $q \geqslant p_1, p_2, p_3, \ldots$ as required does not exist. One can similarly argue that we also cannot choose $q_1, \ldots, q_n \in E$ such that $t \geqslant q_1 \vee \ldots \vee q_n \geqslant p_0, p_1, p_2, \ldots$. So by Lemma 4.5 we get that G is *not* E-continuous.

REMARK 4.17. (Baum–Connes map for inverse semigroups) In [4] we have tried to define a Baum–Connes map for inverse semigroup crossed products parallel to the method of Meyer and Nest in [13] for group crossed products, which automatically would include some theoretical method to compute the left hand side of the Baum–Connes map. On that way, $C_0(X)$-compatible Hilbert modules and their KK-theory appeared the better choice than the corresponding, $C_0(X)$-structure ignoring incompatible tools. Thus $C_0(X)$ is the natural coefficient algebra. But since $L^2(G)$-spaces are in the center and the core of any Baum–Connes theory, and Example 4.14 shows that a compatible $C_0(X)$-valued $L^2(G)$-module requires E-continuity of G, it appears not possible to overcome the E-discontinuity barrier when defining a Baum–Connes map, at least not with the known (group) $L^2(G)$-space methods. That is, as soon as the associated groupoid of G is non-Hausdorff the method fails. More generally, Tu [19] has tried to develop a Baum–Connes theory for non-Hausdorff groupoids, and came to the same conclusion that for non-Hausdorff groupoids the known methods fail, even one may be able to formally write down the Baum–Connes map also for non-Hausdorff groupoids.

REMARK 4.18. (Baum–Connes theory for discretized crossed products) Where as we have no approach to handle the K-theory of a crossed product $A \rtimes G$ for an inverse semigroup G, we have a Baum–Connes map and additionally at least theoretically an approach to treat the K-theory of $(\varepsilon(E) \otimes_{C_0(X)} A) \rtimes G$ by [1]. Even though the K-theories of the latter two crossed products are different in general, they might have some aspects in common in certain good interesting cases as the latter two crossed products are also similar. For example, if $G = E$ consists only of projections then both $C_0(X) \rtimes E$ and $\varepsilon(E) \rtimes E$ are the direct limit of canonically $*$-isomorphic finite dimensional sub-C^*-algebras. Only the direct limit embedding maps are different in both cases. Anyway, the K_0-group of both algebras $C_0(X) \rtimes E$ and $\varepsilon(E) \rtimes E$ are free abelian groups with cardinality $\mathrm{card}(E)$. Hence both algebras have the same K-theory. In Example 4.14 we have already observed this on a concrete example.

EXAMPLE 4.19. That being said, let us remark that the discretized crossed product and the usual crossed product may however also be rather distinct. Write for example the Cuntz algebra $\mathcal{O}_n$ as the inverse semigroup crossed product $\mathcal{O}_n \cong A \rtimes G$ (Sieben's crossed product, which is the universal crossed product subject to the relations $e(a) \rtimes g \equiv a \rtimes eg$ for all $a \in A, e \in E, g \in G$), where G is defined to be the inverse semigroup $G \subseteq \mathcal{O}_n$ generated by the standard generators $S_1, \ldots, S_n$ of the Cuntz algebra, and $A \subseteq \mathcal{O}_n$ denotes the smallest G-invariant C^*-subalgebra of the Cuntz algebra generated by the identity $1 \in \mathcal{O}_n$ under the (incompatible) G-action $g(a) = gag^*$ for $a \in \mathcal{O}_n, g \in G$. Note that A is the commutative G-algebra (in the sense of Definition 4.2) generated by the elements of the form gg^* for $g \in G$. The isomorphism is

$$\varphi : \mathcal{O}_n \to A \rtimes G : \varphi(S_i) = 1 \rtimes S_i.$$

Then we have that

$$0 = (\varepsilon(E) \otimes_{C_0(X)} A) \rtimes G \neq A \rtimes G = \mathcal{O}_n,$$

because in the left hand sided crossed product we have

$$(\varepsilon_1 \otimes 1) \rtimes 1 = \big(\varepsilon_1 \otimes (S_1 S_1^* + \ldots + S_n S_n^*)\big) \rtimes 1 = 0$$

as $S_i S_i^*(\varepsilon_1) = 0$ (action of $S_i S_i^*$ on ε_1) for all i, and by similar reasoning $(\varepsilon_e \otimes 1) \rtimes 1 = 0$ for all $e \in E$. We see thus that the discretized crossed product is not an approximation of the crossed product $A \rtimes G$ at all as it collapses to zero. (As already the discretized coefficient algebra $\varepsilon(E) \otimes_{C_0(X)} A$ is zero.) Still the K-theory of both crossed products is finitely generated. But this need not be in general true, as we may replace A by an infinite sum of copies of A, and so

$$K_0((\bigoplus_{\mathbb{N}} A) \rtimes G) = \bigoplus_{\mathbb{N}} K_0(A \rtimes G)$$

is infinitely generated whereas the K-theory of the discretized crossed product is an infinite sum of zeros, so zero and thus finitely generated.

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