Operators
and
Matrices
Volume 15, Number 2 (2021), 721–727 doi:10.7153/oam-2021-15-49

A NOTE ON BLOCK VANDERMONDE MATRICES

CHAO XIA

(*Communicated by B. Jacob*)

Abstract. Motivated by the concept of complete pairs, the concept of an almost complete l tuple of matrices is presented. We show that an almost complete l tuple of matrices can be extended to a complete l tuple of matrices.

1. Introduction

Block Vandermonde matrix, known as a generalisation of classical Vandermonde matrix, which extended numbers to matrices. It is widely used in linear algebras and other fields. For instance, using block Vandermonde matrix, one can obtain explicit formulas for solutions of matrix polynomial equations (see e.g. [3, 4]).

In this paper, the identity matrix is denoted by I. If $S_1,S_2,\ldots,S_l \in M_n(\mathbb{C})$, where l is a positive integer greater than 2, then the $l\times l$ block Vandermonde matrix of $S_1,S_2,\ldots,S_l$ is defined as follows,

$$W(S_1,S_2,\cdots,S_l) := \begin{bmatrix} I & I & \cdots & I \\ S_1 & S_2 & \cdots & S_l \\ S_1^2 & S_2^2 & \cdots & S_l^2 \\ \vdots & \vdots & & \vdots \\ S_1^{l-1} & S_2^{l-1} & \cdots & S_l^{l-1} \end{bmatrix}_{l\times l}. \tag{1}$$

For applications of Vandermonde matrix in solving matrix polynomial equations, we refer to [2, 8]. It is well known that in the scalar case $n=1$,

$$\det W(s_1,s_2,\ldots,s_l) = \prod_{1\leqslant i<j\leqslant l}(s_i-s_j)$$

and, thus, the Vandermonde matrix is nonsingular if the set of $s_i's$ are distinct. However, in the case $n\geqslant 2$, the determinant of the block Vandermonde at two points is

$$\det W(S_1,S_2) = \det(S_2-S_1).$$

Mathematics subject classification (2020): 15A80, 15A03.
Keywords and phrases: Vandermonde matrix, matrix polynomial, almost complete l tuple of matrices.

Even if S_1 and S_2 have no eigenvalues in common, $S_2 - S_1$ may still be singular. For example,

$$S_1 = \begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 4 & 2 \\ 0 & 3 \end{bmatrix}$$

yields $S_2 - S_1$ singular.

In the scalar case $n = 1$, if $W(s_1, s_2, \dots, s_l)$ is singular, then for any $s_{l+1} \in \mathbb{C}$, $W(s_1, s_2, \dots, s_l, s_{l+1})$ is also singular. However, in cases $n > 1$, one might find the above conclusion is not true. For example, let $n = 2$, $l = 2$, and

$$S_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $\det W(S_1, S_2) = \det(S_2 - S_1) = 0$. Choose

$$S_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

it is not difficult to compute $\det W(S_1, S_2, S_3) = -1$, which says $W(S_1, S_2, S_3)$ is nonsingular.

Motivated by above and the results in [7], we are interested in the following question.

QUESTION. *Suppose that $W(S_1, S_2, \dots, S_l)$ is singular. Is it possible to find a matrix S_{l+1} such that $W(S_1, S_2, \dots, S_l, S_{l+1})$ is nonsingular?*

We call $(S_1, S_2, \cdots, S_l)$ a *complete l-tuple of matrices* if the $l \times l$ block Vandermonde matrix $W(S_1, S_2, \cdots, S_l)$ is nonsingular, the concept of complete l-tuple was introduced in [5, 6]. In [7], Lancaster gave some important applications of complete l-tuple to solutions of differential equations.

To present the main results of this paper, we need some definitions and notations. Denoted by $\widehat{W}(S_1, S_2, \dots, S_l)$ the following $(l+1) \times l$ block matrix,

$$\begin{bmatrix} I & I & \cdots & I \\ S_1 & S_2 & \cdots & S_l \\ S_1^2 & S_2^2 & \cdots & S_l^2 \\ \vdots & \vdots & & \vdots \\ S_1^l & S_2^l & \cdots & S_l^l \end{bmatrix},$$

which will play a key role in the this paper.

Here we recall the definition of direct sum of matrices.

DEFINITION 1. Let $A \in M_n(\mathbb{C})$ and $B \in M_m(\mathbb{C})$. Then the direct sum of A and B, denoted by $A \oplus B$, is the following matrix,

$$A \oplus B := \begin{bmatrix} A & O \\ O & B \end{bmatrix}. \tag{2}$$

Now we give the definition of almost complete l-tuple of matrices.

DEFINITION 2. Let $S_1,S_2,\dots,S_l$ be square matrices of orders n. Then $(S_1,S_2,\dots,S_l)$ is called an *almost complete l-tuple of matrices* if $\widehat{W}(S_1,S_2,\dots,S_l)$ is full column rank and

$$\operatorname{rank} W(S_1,S_2,\dots,S_l) = ln-1.$$

The main purpose of this paper is to show that an almost complete l tuple of matrices can be extended to a complete $l+1$-tuple. Here we ask $\widehat{W}(S_1,S_2,\dots,S_l)$ to be full column rank, since if not, for any $S_{l+1}\in M_n(\mathbb{C})$, $W(S_1,S_2,\dots,S_l,S_{l+1})$ is, of course, not full column rank, which means it is singular. The case of $l=2$ is proved in [1]. In this paper, we prove the general case $l\geqslant 2$.

2. Main result

In this section, we show that an almost complete l tuple of matrices can be extended to a complete $l+1$-tuple.

THEOREM 1. *Suppose that $S_1,S_2,\cdots S_l\in M_n(\mathbb{C})$, $n\geqslant 2$ and $(S_1,S_2,\dots,S_l)$ is an almost complete l tuple of matrices. Then we can always find a matrix $S_{l+1}\in M_n(\mathbb{C})$ depends on $S_1,S_2,\dots,S_l$, such that $(S_1,S_2,\dots,S_l,S_{l+1})$ is a complete $l+1$ tuple of matrices. i.e., $W(S_1,S_2,\dots,S_l,S_{l+1})$ is invertible.*

Proof. The $(l+1)\times l+1$ block matrix $W(S_1,S_2,\dots,S_l,S_{l+1})$ is nonsingular if and only if its rank is $(l+1)n$. Take elementary transformation on it, we obtain that the rank of following $(l+1)\times(l+1)$ block matrix

$$\begin{bmatrix} I & O & \cdots & O & O \\ S_1 & S_2-S_1 & \cdots & S_l-S_1 & S_{l+1}-S_1 \\ S_1^2 & S_2^2-S_1^2 & \cdots & S_l^2-S_1^2 & S_{l+1}^2-S_1^2 \\ \vdots & \vdots & & \vdots & \vdots \\ S_1^l & S_2^l-S_1^l & \cdots & S_l^l-S_1^l & S_{l+1}^l-S_1^l \end{bmatrix}$$

is $(l+1)n$, which means that following $l\times l$ block matrix

$$\widetilde{W}_{l+1} = \begin{bmatrix} S_2-S_1 & \cdots & S_l-S_1 & S_{l+1}-S_1 \\ S_2^2-S_1^2 & \cdots & S_l^2-S_1^2 & S_{l+1}^2-S_1^2 \\ \vdots & & \vdots & \vdots \\ S_2^l-S_1^l & \cdots & S_l^l-S_1^l & S_{l+1}^l-S_1^l \end{bmatrix}$$

is nonsingular. According to the definition of almost complete l tuple of matrices, the following matrix has rank $n(l-1)-1$

$$\widetilde{W}_l = \begin{bmatrix} S_2-S_1 & S_3-S_1 & \cdots & S_l-S_1 \\ S_2^2-S_1^2 & S_3^2-S_1^2 & \cdots & S_l^2-S_1^2 \\ \vdots & \vdots & & \vdots \\ S_2^{l-1}-S_1^{l-1} & S_3^{l-1}-S_1^{l-1} & \cdots & S_l^{l-1}-S_1^{l-1} \end{bmatrix} \tag{3}$$

and the matrix $\widehat{W}(S_1,S_2,\ldots,S_l)$ is full column rank. As the rank of $(l-1)\times(l-1)$ block matrix $\widetilde{W}_l$ is $n(l-1)-1$, the Jordan norm of $\widetilde{W}_l$ has a unique 0-Jordan block, which means that $\widetilde{W}_l$ has the Jordan norm $J_{s_0}(0)\oplus J_{s_1}(\lambda_1)\oplus J_{s_2}(\lambda_2)\oplus\cdots\oplus J_{s_k}(\lambda_k)$, here $J_{s_i}(\lambda)$ denote the $s_i\times s_i$ matrix

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}_{s_i\times s_i} \qquad \text{for} \quad 0\leqslant i\leqslant k,$$

and $s_0+s_1+\cdots+s_k=n(l-1)$, $\lambda_1\lambda_2\cdots\lambda_k\neq 0$. Accordingly, we can find an invertible $nl\times nl$ matrix P such that

$$P^{-1}\widetilde{W}_lP=J_{s_0}(0)\oplus J_{s_1}(\lambda_1)\oplus J_{s_2}(\lambda_2)\oplus\cdots\oplus J_{s_k}(\lambda_k).$$

For convenience, we write matrices

$$A=\left[S_2^l-S_1^l\ S_3^l-S_1^l\ \cdots\ S_l^l-S_1^l\right]$$

and

$$B=\begin{bmatrix} S_{l+1}-S_1 \\ S_{l+1}^2-S_1^2 \\ \vdots \\ S_{l+1}^l-S_1^l \end{bmatrix},$$

thus, the matrix $\widetilde{W}_{l+1}$ can be written as

$$\widetilde{W}_{l+1}=\begin{bmatrix} \widetilde{W}_l & B \\ A & S_{l+1}^l-S_1^l \end{bmatrix}.$$

We will divide the proof into three cases, and prove the result case by case.

Case 1. $s_0=1$.

Case 2. $s_0=n(l-1)$.

Case 3. $1<s_0<n(l-1)$.

It is clear that

$$\begin{aligned} &\begin{bmatrix} P^{-1} & O \\ O & I \end{bmatrix}\widetilde{W}_{l+1}\begin{bmatrix} P & O \\ O & I \end{bmatrix} \\ &=\begin{bmatrix} J_{s_0}(0)\oplus J_{s_1}(\lambda_1)\oplus J_{s_2}(\lambda_2)\oplus\cdots\oplus J_{s_k}(\lambda_k) & P^{-1}B \\ AP & S_{l+1}^l-S_1^l \end{bmatrix}, \end{aligned} \tag{4}$$

for convenience, we can assume that $P=I_{n(l-1)}$, which says $\widetilde{W}_l$ itself is in Jordan norm $J_{s_0}(0)\oplus J_{s_1}(\lambda_1)\oplus J_{s_2}(\lambda_2)\oplus\cdots\oplus J_{s_k}(\lambda_k)$. Denoted by $\alpha=[a_1,a_2,\cdots,a_n]^T$ the first column of matrix A. According to the definition of almost complete l tuple, α is a non-zero vector.

Now, we prove the theorem in *Case* 1.

If $a_1 \neq 0$, taking $S_{l+1} = S_l + tI$, then $\det(\widetilde{W}_{l+1})$ is a complex coefficient polynomial in the indeterminate t, and the leading term of this polynomial is

$$(-1)^{nl-l} a_1 \lambda_1^{s_1} \cdots \lambda_k^{s_k} t^{ln-l+1},$$

let $|t|$ be sufficiently large, $\det(\widetilde{W}_{l+1})$ is not zero, we get the conclusion.

If $a_1 = 0$, then there is an $a_i \neq 0, i \neq 1$. Putting $S_{l+1} = S_l + t(I + E_{1i})$, where E_{1i} is a n square matrix with $(1,i)$ entry 1 and other entries zero, then $\det(\widetilde{W}_{l+1})$ is a polynomial in the indeterminate t, and the leading term of this polynomial is

$$(-1)^{nl-l+i-1} a_i \lambda_1^{s_1} \cdots \lambda_k^{s_k} t^{ln-l+1},$$

let $|t|$ be sufficiently large (here $|t|$ denote the module of complex number t), $\det(\widetilde{W}_{l+1})$ is not zero, we also get the conclusion.

The proof in *Case* 1 is finished.

Next, we prove this theorem in *Case* 2, in which, $\widetilde{W}_l$ is just the following matrix

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{n(l-1)\times n(l-1)}.$$

If $a_n \neq 0$, taking $S_{l+1} = S_l + tI$, then $\det(\widetilde{W}_{l+1})$ is a polynomial in the indeterminate t with the leading term

$$(-1)^{nl+n-1} a_n t^{ln-1},$$

let $|t|$ be sufficiently large, $\det(\widetilde{W}_{l+1})$ is not zero, we get the conclusion.

If $a_n = 0$, then there must be an $a_i \neq 0$ with $1 \leqslant i < n$. Taking $S_{l+1} = S_l + t(I + E_{ni})$, where E_{ni} denote the n square matrix with the (n,i) entry 1 and other entries zero. Then $\det(\widetilde{W}_{l+1})$ is a polynomial in the indeterminate t, and the leading term of this polynomial is

$$(-1)^{nl-i+1} a_i t^{ln-1},$$

let $|t|$ be sufficiently large, $\det(\widetilde{W}_{l+1})$ is non-zero, we also get the conclusion.

Then we prove in the remaining *Case* 3, in which, W_{l+1} is just the following Jordan block matrix

$$J_{s_0}(0) \oplus J_{s_1}(\lambda_1) \oplus J_{s_2}(\lambda_2) \oplus \cdots \oplus J_{s_k}(\lambda_k)$$

where $\lambda_i \neq 0, 1 \leqslant i \leqslant k, 1 < s_0 < n(l-1)$. Suppose $s_0 \equiv r(\mod n)$, where $1 \leqslant r \leqslant n$.

If $a_r \neq 0$, taking $S_{l+1} - S_l = tI$, then $\det(\widetilde{W}_{l+1})$ is a polynomial in the indeterminate t, the leading term of this polynomial is

$$(-1)^{nl-l+q} a_1 \lambda_1^{s_1} \cdots \lambda_k^{s_k} t^{ln-l+q+1},$$

for sufficiently large $|t|$, $\det(\widetilde{W}_{l+1})$ is not zero, we get the conclusion.

If $a_r = 0$, then there is an $a_i \neq 0, i \neq r$. Putting $S_{l+1} - S_l = t(I + E_{ri})$, where E_{ri} is an n square matrix with (r,i) entry 1 and other entries zero, then $\det(\widetilde{W}_{l+1})$ is a polynomial in the indeterminate t. The leading term of this polynomial is

$$(-1)^{nl-l+q+i-1} a_i \lambda_1^{s_1} \cdots \lambda_k^{s_k} t^{ln-l+q+1},$$

let $|t|$ be sufficiently large, $\det(\widetilde{W}_{l+1})$ is not zero, we also get the conclusion. Up to now, we have completed the proof of the theorem. $\square$

3. Example

EXAMPLE 1. Let $n = 2$, $l = 2$ and (S_1, S_2) be an almost complete pair of matrices. Then the Jordan norm of $S_2 - S_1$ is either $\begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$ with $\lambda \neq 0$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Suppose that $S_2 - S_1 = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$ with $\lambda \neq 0$ and $S_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\begin{bmatrix} S_2 - S_1 \\ S_2^2 - S_1^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & \lambda b \\ \lambda c & 2\lambda d + \lambda^2 \end{bmatrix},$$

is full column rank, which says $c \neq 0$. Taking $S_3 = S_1 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, thus

$$\begin{bmatrix} S_2 - S_1 & S_3 - S_1 \\ S_2^2 - S_1^2 & S_3^2 - S_1^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \lambda & 0 & 0 \\ 0 & \lambda b & c & a+d \\ \lambda c & 2\lambda d + \lambda^2 & 0 & c \end{bmatrix},$$

hence

$$\det W(S_1, S_2, S_3) = \det \begin{bmatrix} S_2 - S_1 & S_3 - S_1 \\ S_2^2 - S_1^2 & S_3^2 - S_1^2 \end{bmatrix} = -\lambda^2 c^2 \neq 0,$$

which means that (S_1, S_2, S_3) is complete.

Now we assume that $S_2 - S_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$\begin{bmatrix} S_2 - S_1 \\ S_2^2 - S_1^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ c & a+d \\ 0 & c \end{bmatrix},$$

is full column rank, which says $c \neq 0$. Taking $S_3 = S_1 + \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}$, thus $\det W(S_1, S_2, S_3) = \det \begin{bmatrix} S_2 - S_1 & S_3 - S_1 \\ S_2^2 - S_1^2 & S_3^2 - S_1^2 \end{bmatrix} = -c^2 \neq 0$, which means that (S_1, S_2, S_3) is complete.

4. Conclusion

In this paper, we presented the definitions of complete l-tuple of matrices and almost complete l-tuple of matrices, and proved that for an almost complete l-tuple of matrices $S_1, S_2, \cdots S_l \in M_n(\mathbb{C})$, we can always find a matrix $S_{l+1} \in M_n(\mathbb{C})$ depends on $S_1, S_2, \cdots S_l$ such that the almost complete l-tuple of matrices can be extended to a complete $l+1$-tuple of matrices.

Acknowledgements. The author would like to thank the editor and referee for their insightful comments and helpful suggestions.

REFERENCES

[1] S. CHEN AND C. XIA, *Extending an almost complete pair of matrices to a complete triple*, The Scientific World Journal **2014**, 1 (2014), 1–6.
[2] J. E. DENNIS AND R. P. WEBER, *The algebraic theory of matrix polynomials*, SIAM Journal on Numerical Analysis **13**, 1 (1976), 831–845.
[3] I. GOHBERG, M. A. KAASHOEK, L. LERER AND L. RODMAN, *Common multiples and common divisors of matrix polynomials, II, Vandermonde and resultant matrices*, Linear and Multilinear Algebra **12**, 3 (1982), 159–203.
[4] I. GOHBERG, P. LANCASTER AND L. RODMAN, *Matrix Polynomials*, Academic Press, New York, 1982.
[5] M. G. KREIN AND H. LANGER, *On some mathematical principles in linear theory of damped, oscillayions of continua, I*, Integral Equations Operator Theory **1**, 3 (1978), 364–399.
[6] M. G. KREIN AND H. LANGER, *On some mathematical principles in linear theory of damped oscillayions of continua, II*, Integral Equations Operator Theory **1**, 4 (1978), 539–566.
[7] P. LANCASTER, *A fundamental theorem on lambda-matrices with applications, I, Ordinary differential equations with constant coefficients*, Linear Algebra and Its Applications **18**, 3 (1977), 189–211.
[8] F. MARCOS F AND E. PEREIRA, *A fixed point method to compute solvents of matrix polynomials*, Mathematica Bohemica **135**, 1 (2010), 355–362.

(Received April 14, 2020)

Chao Xia
School of Mathematics and Statistics
Changchun University of Technology
Changchun 130012, P. R. China
e-mail: xiachao@ccut.edu.cn