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FURTHER INEQUALITIES FOR SECTOR MATRICES

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Abstract. We mainly generalize a norm inequality of $n \times n$ block accretive-dissipative matrices. This complements the results of Kittaneh [10, Theorem 2.4] and Fu [18, Theorem 2.9]. And then, we present some singular value inequalities for sector matrices.

1. Introduction

Let $\mathbb{M}_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices and I_n be the identity matrix in $\mathbb{M}_n(\mathbb{C})$. For any $T \in \mathbb{M}_n(\mathbb{C})$, T^* stands for the conjugate transpose of T. Every matrix T has its Cartesian (or Toeptliz) decomposition (see [2]),

$$T = \Re T + i\Im T, \tag{1}$$

in which $\Re T = \frac{1}{2}(T + T^*)$, $\Im T = \frac{1}{2i}(T - T^*)$ are Hermitian. A matrix T is said to be accretive (resp. dissipative) if in its cartesian decomposition (1) the matrix $\Re T$ (resp. $\Im T$) is positive defnite. If both $\Re T$ and $\Im T$, in the decomposition (1), are positive defnite, T is called accretive-dissipative. We refer the interested reader to [7, 15, 16, 17] and the references therein for further study of such matrices and their rich applications.

Moreover, if $T \in M_{2n}$, we will consider the partition of T as,

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad \text{where } T_{jk} \in \mathbb{M}_n(\mathbb{C}),\ j,k = 1,2. \tag{2}$$

Recall that a norm $||\cdot||$ on $\mathbb{M}_n(\mathbb{C})$ is unitarily invariant if $||UAV|| = ||A||$ for any $A \in \mathbb{M}_n(\mathbb{C})$ and unitary matrices $U, V \in \mathbb{M}_n(\mathbb{C})$. For $p > 0$ and $A \in \mathbb{M}_n(\mathbb{C})$, let $||A||_p = (\sum_{j=1}^n s_j^p(A))^{\frac{1}{p}}$, where $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of A. This defines the Schatten p-norm (quasinorm) for $p \geqslant 1$ $(0 < p < 1)$. If A is Hermitian, then all eigenvalues of A are real and ordered as $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A)$. We denote $s(A) = (s_1(A), s_2(A), \ldots, s_n(A))$ and $\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$.

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Another important class of matrices, called sectorial matrices, is related to the above classes. First, let us introduce two definitions. The numerical range of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{x^*Ax | x \in \mathbb{C}^n, x^*x = 1\}.$$

For $\alpha \in [0, \frac{\pi}{2})$, S_α denotes the sector in the complex plane given by

$$S_\alpha = \{z \in \mathbb{C} | \Re z > 0, |\Im z| \leqslant (\Re z)\tan(\alpha)\}.$$

Clearly, A is positive definite if and only if $W(A) \subseteq S_0$, and if $W(A), W(B) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$, then $W(A+B) \subseteq S_\alpha$. As $0 \notin S_\alpha$, then A is nonsingular. Recent research interest in this class of matrices starts with a resolution of a problem from numerical analysis [3]. Some research results on sector matrices can be found in [13, 14, 22, 23].

Gumus et al. [7, Theorem 4.2] proved the following norm inequalities.

THEOREM 1. [7, Theorem 4.2] *Let $T \in \mathbb{M}_n(\mathbb{C})$ be accretive-dissipative partitioned as in (2). Then*

$$||T_{12}||_p^P + ||T_{21}||_p^P \leqslant 2^{P-1} ||T_{11}||_p^{\frac{P}{2}} ||T_{22}||_p^{\frac{P}{2}}, \qquad \textit{for } \ p \geqslant 2,$$

and

$$||T_{12}||_p^P + ||T_{21}||_p^P \leqslant 2^{3-P} ||T_{11}||_p^{\frac{P}{2}} ||T_{22}||_p^{\frac{P}{2}}, \qquad \textit{for } \ 0 < p \leqslant 2.$$

Based on Theorem 1, Kittaneh and Sakkijha [10, Theorem 2.4] presented the following norm inequalities, which compares the norms of the off diagonal blocks and the diagonal blocks.

THEOREM 2. [10, Theorem 2.4] *For $i, j = 1, 2, \cdots, n$, let T_{ij} be square matrices of the same size such that the block matrix*

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

is accretive-dissipative. Then

$$\sum_{i \neq j} ||T_{ij}||_p^p \leqslant (n-1)2^{p-2} \sum_{i=1}^n ||T_{ii}||_p^p \qquad \textit{for} \quad p \geqslant 2$$

and

$$\sum_{i \neq j} ||T_{ij}||_p^p \leqslant (n-1)2^{2-p} \sum_{i=1}^n ||T_{ii}||_p^p \qquad \textit{for} \quad 0 < p \leqslant 2.$$

Garg and Aujla [6, Theorem 2.8, 2.10] showed the following inequalities:

$$\prod_{j=1}^{k} s_j(|A+B|^r) \leqslant \prod_{j=1}^{k} s_j(I_n+|A|^r)\prod_{j=1}^{k} s_j(I_n+|B|^r) \qquad 1 \leqslant k \leqslant n, \quad 1 \leqslant r \leqslant 2 \quad (3)$$

and

$$\prod_{j=1}^{k} s_j(I_n+f(|A+B|)) \leqslant \prod_{j=1}^{k} s_j(I_n+f(|A|))\prod_{j=1}^{k} s_j(I_n+f(|B|)), \quad 1 \leqslant k \leqslant n, \quad (4)$$

where $A,B \in \mathbb{M}_n(\mathbb{C})$ and $f:[0,\infty) \to [0,\infty)$ is an operator concave function.

Let $A,B \in \mathbb{M}_n(\mathbb{C})$ be positive semidefinite, $r=1$ and $f(X)=X$ for any $X \in \mathbb{M}_n(\mathbb{C})$ in (3) and (4), we have

$$\prod_{j=1}^{k} s_j(A+B) \leqslant \prod_{j=1}^{k} s_j\,(I_n+A)\prod_{j=1}^{k} s_j\,(I_n+B), \ \ 1 \leqslant k \leqslant n \quad (5)$$

and

$$\prod_{j=1}^{k} s_j\,(I_n+A+B) \leqslant \prod_{j=1}^{k} s_j\,(I_n+A)\prod_{j=1}^{k} s_j\,(I_n+B), \ \ 1 \leqslant k \leqslant n. \quad (6)$$

In this paper, we will extend the results of Theorem 1 and 2 to a larger class of matrices, i.e. sector matrices and give several singular value inequalities based on (5) and (6).

2. Main result

We begin this section with some lemmas which are useful to establish our main results.

LEMMA 1. [1, Theorem 2.6] *Let $T \in \mathbb{M}_n(\mathbb{C})$ be such that $W(T) \subseteq S_\alpha$. Then*

$$s_j(T) \leqslant \sec(\alpha) s_{[(j+1)/2]}(Re(T)) \qquad \text{for } \ j=1,2,\ldots,n, \quad (7)$$

where [x] is the greatest integer $\leqslant x$.

In Lemma 2 and 3, assume that $T \in \mathbb{M}_{2n}(\mathbb{C})$ is partitioned as in (2) and $W(T) \subseteq S_\alpha$.

LEMMA 2. [1, Theorem 3.2] *For $k=1,2,\cdots,n$,*

$$\prod_{l=1}^{k} s_l(T_{ij}) \leqslant \prod_{l=1}^{k} \sec(\alpha) s_l^{1/2}(Re(T_{ii})) s_l^{1/2}(Re(T_{jj})), \quad i,j=1,2. \quad (8)$$

LEMMA 3. [1, Theorem 3.4] *Let r, p and q be positive numbers such that $1/p + 1/q = 1$. Then*

$$\| \, |T_{12}|^r \| \leqslant \sec^r(\alpha) \|(Re(T_{11}))^{rp/2}\|^{1/p} \, \|(Re(T_{22}))^{rq/2}\|^{1/q}$$
$$\leqslant \sec^r(\alpha) \|T_{11}^{rp/2}\|^{1/p} \, \|T_{22}^{rq/2}\|^{1/q},$$

for any unitarily invariant norm $\|\cdot\|$.

LEMMA 4. [2, p.73] *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\lambda_j(\Re A) \leqslant s_j(A), \qquad j = 1, 2, \ldots, n. \tag{9}$$

LEMMA 5. [20, (2.2)] *Let A, $B \in \mathbb{M}_n(\mathbb{C})$ be such that $W(A)$, $W(B) \subseteq S_\alpha$. Then*

$$\Re(A+B)^{-1} \leqslant \frac{\sec^4 \alpha}{4} \Re(A^{-1} + B^{-1}).$$

THEOREM 3. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be partitioned as in (2) and $W(T) \subseteq S_\alpha$ for some $\alpha \in [0, \frac{\pi}{2})$. And let s, r be positive numbers such that $1/s + 1/r = 1$. Then*

$$\max\{\|T_{12}\|_p^p, \|T_{21}\|_p^p\} \leqslant \sec^p(\alpha) \|(Re(T_{11}))^{s/2}\|_p^{p/s} \, \|(Re(T_{22}))^{r/2}\|_p^{p/r} \qquad \textit{for } \; p > 0.$$

Proof. From Lemma 3, we know that, let $r = 1$, the result is true for Schatten p-norms ($p \geqslant 1$). When $0 < p < 1$, from Lemma 2, we know

$$\prod_{l=1}^{k} s_l^p(T_{ij}) \leqslant \prod_{l=1}^{k} \sec^p(\alpha) s_l^{p/2}(Re(T_{ii})) s_l^{p/2}(Re(T_{jj})), \quad i, j = 1, 2.$$

The fact that weak log-majorization implies weak majorization gives

$$\sum_{l=1}^{k} s_l^p(T_{ij}) \leqslant \sum_{l=1}^{k} \sec^p(\alpha) s_l^{p/2}(Re(T_{ii})) s_l^{p/2}(Re(T_{jj}))$$
$$= \sec^p(\alpha) \sum_{l=1}^{k} s_l^{p/2}(Re(T_{ii})) s_l^{p/2}(Re(T_{jj}))$$
$$\leqslant \sec^p(\alpha) (\sum_{l=1}^{k} s_l^{sp/2}(Re(T_{ii})))^{1/s} (\sum_{l=1}^{k} s_l^{rp/2}(Re(T_{jj})))^{1/r}. \text{ (Hölder inequality)}$$

Thus

$$\|T_{ij}\|_p^p \leqslant \sec^p(\alpha) \|(Re(T_{ii}))^{s/2}\|_p^{p/s} \, \|(Re(T_{jj}))^{r/2}\|_p^{p/r}.$$

So

$$\max\{\|T_{12}\|_p^p, \|T_{21}\|_p^p\} \leqslant \sec^p(\alpha) \|(Re(T_{11}))^{s/2}\|_p^{p/s} \, \|(Re(T_{22}))^{r/2}\|_p^{p/r}.$$

This completes the proof. □

COROLLARY 1. *Let $T \in \mathbb{M}_{2n}(\mathbb{C})$ be partitioned as in (2) and assume $W(T) \subseteq S_\alpha$. And let s, r be positive numbers such that $1/s + 1/r = 1$. Then*

$$||T_{12}||_p^p + ||T_{21}||_p^p \leqslant 2\sec^p(\alpha)||(Re(T_{11}))^{s/2}||_p^{p/s}\ ||(Re(T_{22}))^{r/2}||_p^{p/r} \qquad \text{for } p > 0. \tag{10}$$

THEOREM 4. *For $i, j = 1, 2, \cdots, n$, let T_{ij} be square matrices of the same size such that the block matrix*

$$\begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

is a sector matrix. And let s, r be positive numbers such that $1/s + 1/r = 1$. Then

$$\sum_{i \neq j} ||T_{ij}||_p^p \leqslant \frac{n-1}{2}\sec^p(\alpha)\sum_{i=1}^{n}(||(Re(T_{ii}))^{s/2}||_p^{2p/s} + ||(Re(T_{ii}))^{r/2}||_p^{2p/r}), \qquad \text{for } p > 0. \tag{11}$$

Proof. It is easy to obtain that a principal submatrix $\begin{pmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{pmatrix}$ of T is also sector matrix. Now, applying (10) to $\begin{pmatrix} T_{ii} & T_{ij} \\ T_{ji} & T_{jj} \end{pmatrix}$, we get

$$\|T_{ij}\|_p^p + \|T_{ji}\|_p^p \leqslant 2\sec^p(\alpha)||(Re(T_{ii}))^{s/2}||_p^{p/s}\ ||(Re(T_{jj}))^{r/2}||_p^{p/r}$$

for $i \neq j$ and $p > 0$.

Consequently, using the arithmetic-geometric mean inequality, we have

$$\|T_{ij}\|_p^p + \|T_{ji}\|_p^p \leqslant \sec^p(\alpha)(||(Re(T_{ii}))^{s/2}||_p^{2p/s} + ||(Re(T_{jj}))^{r/2}||_p^{2p/r}). \tag{12}$$

Meanwhile, by putting $i := j$, $j := i$,

$$\|T_{ji}\|_p^p + \|T_{ij}\|_p^p \leqslant \sec^p(\alpha)(||(Re(T_{jj}))^{s/2}||_p^{2p/s} + ||(Re(T_{ii}))^{r/2}||_p^{2p/r}) \tag{13}$$

for $i \neq j$ and $p > 0$.

Adding up the previous inequalities (12), (13) for $i, j = 1, 2, \cdots, n$, we get

$$\sum_{i \neq j} \|T_{ij}\|_p^p \leqslant \frac{n-1}{2}\sec^p(\alpha)\sum_{i=1}^{n}(||(Re(T_{ii}))^{s/2}||_p^{2p/s} + ||(Re(T_{ii}))^{r/2}||_p^{2p/r}),$$

which proves the inequality. □

By Lemma 3 and following the same technique which is used in the proof of Theorem 4, we can prove the conclusion for any unitarily invariant norm.

REMARK 1. For $i,j=1,2,\cdots,n$, let T_{ij} be square matrices of the same size such that the block matrix

$$\begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix}$$

is a sector matrix. And let t, s, r be positive numbers such that $1/s+1/r=1$. Then

$$\sum_{i\neq j} || \, |T_{ij}|^t || \leqslant \frac{n-1}{2} \sec^t(\alpha) \sum_{i=1}^{n} (||(Re(T_{ii}))^{ts/2}||^{2/s} + ||(Re(T_{ii}))^{tr/2}||^{2/r}),$$

for any unitarily invariant norm $||\cdot||$.

REMARK 2. Set $r=s=2$ in inequality 11. Then, for $p>0$

$$\begin{aligned} \sum_{i\neq j} ||T_{ij}||_p^p &\leqslant (n-1)\sec^p(\alpha) \sum_{i=1}^{n} ||Re(T_{ii})||_p^p \\ &\leqslant (n-1)\sec^p(\alpha) \sum_{i=1}^{n} ||T_{ii}||_p^p. \end{aligned} \tag{14}$$

The inequality (14) is the result of [18, Theorem 2.9].

If we further set $\alpha = \frac{\pi}{4}$, then we get

$$\begin{aligned} \sum_{i\neq j} ||T_{ij}||_p^p &\leqslant (n-1)2^{p/2} \sum_{i=1}^{n} ||Re(T_{ii})||_p^p \\ &\leqslant (n-1)2^{p/2} \sum_{i=1}^{n} ||T_{ii}||_p^p. \end{aligned} \tag{15}$$

The inequality (15) is the result of [19, Theorem 2.4].

At last, we set $\alpha=0$, then

$$\begin{aligned} \sum_{i\neq j} ||T_{ij}||_p^p &\leqslant (n-1) \sum_{i=1}^{n} ||Re(T_{ii})||_p^p \\ &= (n-1) \sum_{i=1}^{n} ||T_{ii}||_p^p. \end{aligned}$$

Next, we present the singular value inequalities for sector matrices A, B and $A+B$ in $\mathbb{M}_n(\mathbb{C})$.

THEOREM 5. *Let $A,B\in\mathbb{M}_n(\mathbb{C})$ be such that $W(A),W(B)\subset S_\alpha$. Then*

$$\prod_{j=1}^{k} s_j(A+B) \leqslant \sec^k(\alpha) \prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+A) \prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+B) \qquad 1\leqslant k\leqslant n; \tag{16}$$

$$\prod_{j=1}^{k} s_j(I_n+A+B)) \leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+A)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+B), \quad 1\leqslant k\leqslant n; \quad (17)$$

Proof.

$$\begin{aligned}
\prod_{j=1}^{k} s_j(A+B) &\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(A+B)) && \text{(by Lemma 1)}\\
&= \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(A)+\Re(B))\\
&\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+\Re(A))\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+\Re(B)) && \text{(by (5))}\\
&= \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(I_n+A))\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(I_n+B))\\
&\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+A)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+B). && \text{(by Lemma 4)}
\end{aligned}$$

$$\begin{aligned}
\prod_{j=1}^{k} s_j(I_n+A+B) &\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(I_n+A+B)) && \text{(by Lemma 1)}\\
&= \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+\Re(A)+\Re(B))\\
&\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+\Re(A))\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+\Re(B)) && \text{(by (6))}\\
&= \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(I_n+A))\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(I_n+B))\\
&\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+A)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+B). && \text{(by Lemma 4)} \quad \square
\end{aligned}$$

REMARK 3. It's clear that the upper bounds of inequalities (16) and (17) are stronger than that of [21, Theorem 2.7(5,6)], respectively.

Now, we present the singular value inequalities including the inverse of A, B and $A+B$ in $\mathbb{M}_n(\mathbb{C})$, as follows.

THEOREM 6. *Let $A,B \in M_n$ be such that $W(A),W(B) \subset S_\alpha$. Then for $k=1,\dots,n$*

$$\prod_{j=1}^{k} s_j(A+B)^{-1} \leqslant \frac{\sec^{5k}(\alpha)}{4^k}\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+A^{-1})\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+B^{-1}), \quad (18)$$

$$\prod_{j=1}^{k} s_j(I_n+(A+B)^{-1}) \leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}\Big(I_n+\frac{\sec^4(\alpha)}{4}A^{-1}\Big) \times \prod_{j=1}^{k} s_{[\frac{j+1}{2}]}\Big(I_n+\frac{\sec^4(\alpha)}{4}B^{-1}\Big). \tag{19}$$

Proof.

$$\begin{aligned}
&\prod_{j=1}^{k} s_j(A+B)^{-1}\\
&\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(A+B)^{-1}) \qquad \text{(by Lemma 1)}\\
&\leqslant \frac{\sec^{5k}(\alpha)}{4^k}\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(A^{-1})+\Re(B^{-1})) \qquad \text{(by Lemma 5)}\\
&\leqslant \frac{\sec^{5k}(\alpha)}{4^k}\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+\Re(A^{-1}))\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+\Re(B^{-1})) \qquad \text{(by (5))}\\
&= \frac{\sec^{5k}(\alpha)}{4^k}\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(I_n+A^{-1}))\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(I_n+B^{-1}))\\
&\leqslant \frac{\sec^{5k}(\alpha)}{4^k}\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+A^{-1})\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(I_n+B^{-1}). \qquad \text{(by Lemma 4)}
\end{aligned}$$

$$\begin{aligned}
&\prod_{j=1}^{k} s_j(I_n+(A+B)^{-1})\\
&\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re(I_n+(A+B)^{-1}) \qquad \text{(by Lemma 1)}\\
&\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}\Big(I_n+\frac{\sec^4(\alpha)}{4}\Re(A^{-1}+B^{-1})\Big) \qquad \text{(by Lemma 5)}\\
&\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}\Big(I_n+\frac{\sec^4(\alpha)}{4}\Re(A^{-1})\Big)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}\Big(I_n+\frac{\sec^4(\alpha)}{4}\Re(B^{-1})\Big) \quad \text{(by (6))}\\
&= \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}(\Re\Big(I_n+\frac{\sec^4(\alpha)}{4}A^{-1}\Big))\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}\Big(\Re\Big(I_n+\frac{\sec^4(\alpha)}{4}B^{-1}\Big)\Big)\\
&\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}\Big(I_n+\frac{\sec^4(\alpha)}{4}A^{-1}\Big)\prod_{j=1}^{k} s_{[\frac{j+1}{2}]}\Big(I_n+\frac{\sec^4(\alpha)}{4}B^{-1}\Big). \quad \text{(by Lemma 4)}
\end{aligned}$$

□

REMARK 4. Obviously, the upper bounds of inequalities (18) and (19) are stronger than that of [20, Theorem 2.1 (2.5, 2.6)], respectively.

Mohammad [1, Theorem 1.1] proved that

$$|T| \leqslant \frac{\sec(\alpha)}{2}[Re(T)+U^*(Re(T))U]. \tag{20}$$

for $T \in \mathbb{M}_n$, $W(T) \subseteq S_\alpha$ and U be the unitary part of T in the polar decomposition $T = U|T|$.

On the basis of (20), inequalities (5) and (6) are generalized to get two results which are different from that of Theorem 5.

THEOREM 7. *Let $A, B \in M_n$ be such that $W(A), W(B) \subset S_\alpha$. Then for $k = 1, \dots, n$*

$$\prod_{j=1}^{k} s_j(A+B) \leqslant \left(\frac{\sec(\alpha)}{2}\right)^k \prod_{j=1}^{k} s_j^2(I_n+Re(A)) \prod_{j=1}^{k} s_j^2(I_n+Re(B)), \tag{21}$$

and

$$\prod_{j=1}^{k} s_j(I_n+A+B)) \leqslant \sec^k(\alpha) \prod_{j=1}^{k} s_j^2\left(I_n+\frac{1}{2}Re(A)\right) \prod_{j=1}^{k} s_j^2\left(I_n+\frac{1}{2}Re(B)\right). \tag{22}$$

Proof.

$$\begin{aligned}
&\prod_{j=1}^{k} s_j(A+B)\\
&\leqslant \prod_{j=1}^{k} \frac{\sec(\alpha)}{2} s_j(Re(A+B)+U^*Re(A+B)U) \qquad \text{(by (20))}\\
&\leqslant \left(\frac{\sec(\alpha)}{2}\right)^k \prod_{j=1}^{k} s_j(I_n+Re(A+B)) \prod_{j=1}^{k} s_j(I_n+U^*Re(A+B)U) \qquad \text{(by (5))}\\
&= \left(\frac{\sec(\alpha)}{2}\right)^k \prod_{j=1}^{k} s_j^2(I_n+Re(A+B))\\
&= \left(\frac{\sec(\alpha)}{2}\right)^k \prod_{j=1}^{k} s_j^2(I_n+Re(A)+Re(B))\\
&\leqslant \left(\frac{\sec(\alpha)}{2}\right)^k \prod_{j=1}^{k} s_j^2(I_n+Re(A)) \prod_{j=1}^{k} s_j^2(I_n+Re(B)) \qquad \text{(by (6))}
\end{aligned}$$

$$\begin{aligned}
&\prod_{j=1}^{k} s_j(I_n+A+B)\\
&\leqslant \prod_{j=1}^{k} \frac{\sec(\alpha)}{2} s_j(Re(I_n+A+B)+U^*Re(I_n+A+B)U) \qquad \text{(by (20))}\\
&= \left(\frac{\sec(\alpha)}{2}\right)^k \prod_{j=1}^{k} s_j(I_n+Re(A+B)+I_n+U^*Re(A+B)U)
\end{aligned}$$

$$= \sec^k(\alpha)\prod_{j=1}^{k} s_j\Big(I_n + Re\Big(\frac{A+B}{2}\Big) + U^*Re\Big(\frac{A+B}{2}\Big)U\Big)$$

$$\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_j\Big(I_n + Re\Big(\frac{A+B}{2}\Big)\Big)\prod_{j=1}^{k} s_j\Big(I_n + U^*Re\Big(\frac{A+B}{2}\Big)U\Big) \qquad \text{(by (6))}$$

$$= \sec^k(\alpha)\prod_{j=1}^{k} s_j^2\Big(I_n + \frac{1}{2}Re(A) + \frac{1}{2}Re(B)\Big)$$

$$\leqslant \sec^k(\alpha)\prod_{j=1}^{k} s_j^2\Big(I_n + \frac{1}{2}Re(A)\Big)\prod_{j=1}^{k} s_j^2\Big(I_n + \frac{1}{2}Re(B)\Big). \qquad \text{(by (6))} \quad \square$$

COROLLARY 2. *Let $A,B \in M_n$ be such that $W(A),W(B) \subset S_\alpha$.*

$$||A+B|| \leqslant \frac{\sec(\alpha)}{2}||I_n + Re(A))||^2\ ||I_n + Re(B)||^2,$$

and

$$||I_n + A + B|| \leqslant \sec(\alpha)||I_n + \frac{1}{2}Re(A)||^2\ ||I_n + \frac{1}{2}Re(B)||^2. \tag{23}$$

Proof. From (21), we can get

$$\prod_{j=1}^{k} s_j^{\frac{1}{4}}(A+B) \leqslant \prod_{j=1}^{k}\Big(\frac{\sec(\alpha)}{2}\Big)^{\frac{1}{4}} s_j^{\frac{1}{2}}(I_n + Re(A))s_j^{\frac{1}{2}}(I_n + Re(B)),$$

for $k = 1,\cdots,n$.

By the property that weak log-majorization implies weak majorization and Cauchy-Schwarz inequality, we get

$$\sum_{j=1}^{k} s_j^{\frac{1}{4}}(A+B) \leqslant \Big(\frac{\sec(\alpha)}{2}\Big)^{\frac{1}{4}}\Big(\sum_{j=1}^{k} s_j(I_n + Re(A))\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{k} s_j(I_n + Re(B))\Big)^{\frac{1}{2}}, \tag{24}$$

for $k = 1,\cdots,n$.

Inequality (24) is equivalent to the following inequality

$$|||A+B|^{\frac{1}{4}}||_k^2 \leqslant \Big(\frac{\sec(\alpha)}{2}\Big)^{\frac{1}{2}}||I_n + Re(A))||_k\ ||I_n + Re(B)||_k, \tag{25}$$

for $k = 1,\cdots,n$.

According to the generalizations of Ky Fan's dominance theorem [12, Theorem 1.4], (25) implies

$$|||A+B|^{\frac{1}{4}}||^2 \leqslant \Big(\frac{\sec(\alpha)}{2}\Big)^{\frac{1}{2}}||I_n + Re(A))||\ ||I_n + Re(B)||.$$

Since $||A+B|| = ||\ |A+B|\ || = ||\ (|A+B|^{\frac{1}{4}})^4\ || \leqslant ||\ |A+B|^{\frac{1}{4}}\ ||^4$,

$$||A+B|| \leqslant \frac{\sec(\alpha)}{2}||I_n + Re(A))||^2\ ||I_n + Re(B)||^2.$$

Similarly, (23) can be proved in the same way. □

COROLLARY 3. *Let $A, B \in M_n$ be such that $W(A), W(B) \subset S_\alpha$. Then*

$$|\det(A+B)| \leqslant \left(\frac{\sec(\alpha)}{2}\right)^n \det^2(I_n + Re(A))\det^2(I_n + Re(B)),$$

and

$$\det(I_n + A + B)) \leqslant \sec^n(\alpha)\det^2\left(I_n + \frac{1}{2}Re(A)\right)\det^2\left(I_n + \frac{1}{2}Re(B)\right).$$

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REFERENCES

[1] M. ALAKHRASS, *A note on sectorial matrices*, Linear Multilinear Algebra, available from, https://doi.org/10.1080/03081087.2019.1575332.
[2] R. BHATIA, *Matrix Analysis*, GTM 169, Springer-Verlag, New York (NY), 1997.
[3] R. BHATIA, J. A. R. HOLBROOK, *On the Clarkson-McCarthy inequalities*, Math Ann. **281**, (1988), 7–12.
[4] R. BHATIA, F. KITTANEH, *Norm inequalities for positive operators*, Lett Math Phys. **43**, (1998), 225–231.
[5] J. C. BOURIN, M. UCHIYAM, *A matrix subadditivity inequality for $f(A+B)$ and $f(A)+f(B)$*, Linear Algebra Appl. **423**, (2007), 512–518.
[6] I. GARG, J. AUJLA, *Some singular value inequalities*, Linear Multilinear Algebra **66**, (2018), 776–784.
[7] I. H. GUMUS, O. HIRZALLAH, F. KITTANEH, *Norm inequalities involving accretive-dissipative 2×2 block matrices*, Linear Algebra Appl. **528**, (2017), 76–93.
[8] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.
[9] L. HOU, D. P. ZHANG, *Concave functions of partitioned matrices with numerical ranges in a sector*, Math Inequal Appl. **20**, (2017), 83–589.
[10] F. KITTANEH, M. SAKKIJHA, *Inequalities for accretive-dissipative matrices*, Linear Multilinear Algebra **67**, (2019), 1037–1042.
[11] E. Y. LEE, *Extension of Rotfel'd theorem*, Linear Algebra Appl. **435**, (2011), 735–741.
[12] C. K. LI, R. MATHIAS, *Generalizations of Ky Fans dominance theorem*, SIAM J. Matrix Anal. Appl. **19**, (1998), 99–106.
[13] C. K. LI, S. N. SZE, *Determinantal and eigenvalue inequalities for matrices with numerical ranges in a sector*, J. Math. Anal. Appl. **410**, (2014), 487–491.
[14] M. LIN, *Some inequalities for sector matrices*, Oper. Matrices **10**, (2016), 915–921.
[15] M. LIN, *Fischer type determinantal inequalities for accretive-dissipative matrices*, Linear Algebra Appl. **438**, (2013), 2808–2812.

[16] M. LIN, F. SUN, *A property of the geometric mean of accretive operator*, Linear Multilinear Algebra **65**, (2017), 433–437.

[17] M. LIN, D. ZHOU, *Norm inequalities for accretive-dissipative operator matrices*, J. Math. Anal. Appl. **407**, (2013), 436–442.

[18] S. LIN, X. FU, *On some inequalities for sector matrices*, Linear Multilinear Algebra, available from, https://doi.org/10.1080/03081087.2019.1600466.

[19] Y. MAO, X. LIU, *On some inequalities for accretive-dissipative matrices*, Linear Multilinear Algebra, available from, https://doi.org/10.1080/03081087.2019.1635566.

[20] L. NASIRI, S. FURUICHI, *New inequalities for sector matrices applying Garg-Aujla inequalities*, available from, https://arxiv.org/abs/2001.00687.

[21] C. YANG, F. LU, *Some generalizations of inequalities for sector matrices*, J. Inequal. Appl. **2018**, (2018), 183.

[22] D. ZHANG, L. HOU, L. MA, *Properties of matrices with numerical ranges in a sector*, Bull. Iranian Math. Soc. **43**, (2017), 1699–1707.

[23] F. ZHANG, *A matrix decomposition and its application*, Linear Multilinear Algebra **63**, (2015), 2033–2042.

[24] F. ZHANG, *Matrix Theory: Basic Results and Techniques*, Universitext, Springer, New York, 1999.

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