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UNCERTAINTY PRINCIPLES IN TERM OF SUPPORTS IN HANKEL WAVELET SETTING

S. HKIMI AND S. OMRI*

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Abstract. Uncertainty principles in term of supports, namely Amrein-Berthier and Logvinenko-Sereda theorems are proved for the continuous Hankel wavelet transform.

1. Introduction

During the last decades, many developments in harmonic analysis and signal theory showed that despite the power of the Fourier transform as a main tool in studying and analyzing signals. This transform revealed some inabilities to localize the frequency spectrum of some non-stationary signals. To get over this problem, Gabor [6] introduced the short time Fourier transform (STFT). The author considered a nonzero function $g \in L^2(\mathbb{R}^d)$ called a window and defined the short time Fourier transform of a function $f \in L^2(\mathbb{R}^d)$ on the so-called time-frequency plan, by

$$\forall (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}^d},\ \mathscr{V}_g(f)(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-i\langle\omega|t\rangle}\frac{dt}{(2\pi)^{\frac{d}{2}}}.$$

Even though the short time Fourier transform solved the localization problem, a given window couldn't be well adapted to study every multi-frequency signals though. To solve this issue, Grossman and Morlet [10] introduced the wavelet transform, that is, given a nonzero function $g \in L^2(\mathbb{R}^d)$ satisfying the following relation for every $\xi \in S^{d-1}$, known as the admissibility condition,

$$C_g = \int_0^{+\infty} |\hat{g}(a\xi)|^2 \frac{da}{a} < +\infty.$$

The classical wavelet transform of a function $f \in L^2(\mathbb{R}^d)$ is defined on the so-called time-scale plan $\mathbb{R}_+^* \times \mathbb{R}^d$, by

$$\mathscr{T}_g(f)(a,x) = \frac{1}{a^{\frac{d}{2}}}\int_{\mathbb{R}^d} f(t)\overline{g\left(\frac{t-x}{a}\right)}\frac{dt}{(2\pi)^{\frac{d}{2}}}.$$

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* Corresponding author.

The classical wavelet transform is closely related to signal theory, for more details we refer the reader to [4, 24].

Let μ_α be the measure defined on $\mathbb{R}_+$ by

$$d\mu_\alpha(r) = \frac{r^{2\alpha+1}dr}{2^\alpha\Gamma(\alpha+1)}.$$

For every $p \in [1,+\infty]$, let $L^p_\alpha(\mathbb{R}_+)$ the Banach space of measurable functions f on $\mathbb{R}_+$ satisfying

$$\|f\|_{p,\mu_\alpha} = \left(\int_0^{+\infty} |f(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}} < +\infty \quad \text{if } p \in [1,+\infty[,$$
$$\|f\|_{\infty,\mu_\alpha} = \text{esssup}_{x\in\mathbb{R}_+} |f(x)| < +\infty \quad \text{if } p = +\infty.$$

The Hankel transform of order $\alpha \geqslant -\frac{1}{2}$, is defined on $L^1_\alpha(\mathbb{R}_+)$ by [13]

$$\forall\lambda \in \mathbb{R},\ \mathscr{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(r) j_\alpha(\lambda r)\mathrm{d}\mu_\alpha(r), \tag{1.1}$$

where j_α is the Bessel function of the first kind and index α given by [16]

$$j_\alpha(z) = \frac{2^\alpha\Gamma(\alpha+1)}{z^\alpha}J_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\Gamma(\alpha+n+1)}\left(\frac{z}{2}\right)^{2n}, \quad z \in \mathbb{C}.$$

The Hankel transform given by Relation (1.1) is closely related to the Bessel operator ℓ_α, $\alpha \geqslant -\dfrac{1}{2}$, defined on $]0,+\infty[$ by

$$\ell_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r},$$

knowing that $j_\alpha(\lambda.)$ is the unique differentiable function satisfying the following Cauchy problem

$$\begin{cases} \ell_\alpha(\varphi) = -\lambda^2\varphi, \\ \varphi'(0) = 0, \\ \varphi(0) = 1. \end{cases}$$

The Hankel transform of index $\dfrac{d}{2} - 1$ is the Fourier transform of radial functions on $\mathbb{R}^d$ and involves in many physical problems [12, 19].

Let $g \in L^2_\alpha(\mathbb{R}_+)$ be a nonzero function satisfying the following admissibility relation

$$0 < C_g = \int_0^{+\infty} |\mathscr{H}_\alpha(g)(a)|^2\frac{da}{a} < +\infty.$$

Then the Hankel wavelet transform of a function $f \in L^2_\alpha(\mathbb{R}_+)$ is defined on $\mathbb{R}^*_+ \times \mathbb{R}_+$, by

$$\mathscr{W}_g(f)(a,x) = \int_0^{+\infty} f(t)\overline{g_{x,a}(t)}d\mu_\alpha(t),$$

where $g_{x,a}(t) = \tau_x^\alpha \delta_a g(t)$, τ_x^α is the generalized shift operator defined for every $x \in \mathbb{R}_+$ on $L^p_\alpha(\mathbb{R}_+)$, $1 \leqslant p \leqslant +\infty$, by

$$\tau_x^\alpha(h)(y) = \begin{cases} \dfrac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})} \displaystyle\int_0^\pi h(\sqrt{x^2+y^2+2xy\cos(\theta)})\sin(\theta)^{2\alpha}\mathrm{d}\theta, & \text{if } \alpha > -\frac{1}{2}, \\ \dfrac{h(x+y)+h(|x-y|)}{2}, & \text{if } \alpha = -\frac{1}{2}, \end{cases} \tag{1.2}$$

and δ_a is the dilation operator defined for every $a > 0$ on $\mathscr{M}(\mathbb{R}_+)$, the space of all measurable functions on $\mathbb{R}_+$, by

$$\delta_a(h)(z) = a^{\alpha+1}h(az).$$

The Hankel wavelet transform $\mathscr{W}_g$ has many interesting applications in signal analysis, namely in Laser and ultrasound area [23]. In the last years many works have been interested in studying and developing the harmonic analysis related to the transform $\mathscr{W}_g$ namely Baccar, Pathak, Dixit, Mahato, Prasad, Ben Hamadi, Omri and Ünalmis [3, 11, 18, 20, 21, 25].

The uncertainty principles in harmonic analysis state that a nonzero function can't be arbitrarily localized simultaneously with its Fourier transform. The idea of uncertainty was first introduced in 1927 by Heisenberg [15] and constituted one of the most important foundation of quantum mechanics. Roughly speaking, Heisenberg showed that we can't localize simultaneously with an arbitrary precision the position and the momentum of a high speed particle by providing a lower bound of the product of their variances. As mentioned above, in harmonic analysis, the uncertainty principles claim the impossibility for a nonzero function and its Fourier transform to be arbitrary small at the same time. In the last decades, the general idea of the uncertainty has been interpreted differently by many authors who have given many formulations of the localization and the smallness, we cite for instance Amrein-Berthier and Logvinenko-Sereda theorems [1, 17], who studied the localization in term of supports. For more details, we refer the reader to [5, 14, 22].

Uncertainty principles remain closely related to Gabor and wavelet analysis on which they play an important role by improving the knowledge and the localization of the frequency spectrum of a given signal [27], in this context many results have been already established notably by Wilckzok [26]. In the Hankel sitting, many uncertainty principles have been also proved namely by Baccar, Ben Hamadi and Omri [3, 11]. In Ben Hamadi and Omri [11] the authors proved several uncertainty principles associated to the transform $\mathscr{W}_g$ directly related to the dispersions, namely the authors established Shapiro's dispersion theorem as well as Price's local theorem. In Baccar [3] the author showed in particular the well-known Amrein-Berthier theorem related to the transform $\mathscr{W}_g$, unfortunately the theorem was proved with a constraining additional hypothesis which covers only a particular case of admissible Hankel wavelets. The aim of this work is to prove the Amrein-Berthier theorem for the Hankel wavelet transform $\mathscr{W}_g$ in the general case, in addition we also generalize Logvinenko-Sereda uncertainty principle for the same transform. Even if this paper turns out as part of a series of papers about uncertainty principles related to the transform $\mathscr{W}_g$, the results proved here are interested for the first time in the studying of the support of this transform.

The paper is organized as follows, in the second section we recall some preliminary harmonic analysis results related to the Hankel transform and the Hankel wavelet transform. The last section will be devoted to the main results of this paper that are Amrein-Berthier and Logvinenko-Sereda theorems associated to the Hankel wavelet transform $\mathscr{W}_g$.

2. Preliminaries

In this section we introduce the Hankel transform as well as the continuous Hankel wavelet transform for which we recall some basic harmonic analysis results.

2.1. The Hankel transform

For every $\alpha \geqslant -\frac{1}{2}$, the modified Bessel function j_α has the following integral representation [2, 16],

$$\forall z \in \mathbb{C},\; j_\alpha(z) = \begin{cases} \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+(1/2))}\displaystyle\int_0^1 (1-t^2)^{\alpha-1/2}\cos(zt)\mathrm{d}t, & \text{if } \alpha > -\frac{1}{2}, \\ \cos(z), & \text{if } \alpha = -\frac{1}{2}. \end{cases}$$

In particular, for every $n \in \mathbb{N}$ and $x \in \mathbb{R}$

$$|j_\alpha^{(n)}(x)| \leqslant 1.$$

For every $x, y, \lambda \geqslant 0$, we have

$$\tau_x^\alpha(j_\alpha(\lambda.))(y) = j_\alpha(\lambda x) j_\alpha(\lambda y),$$

where τ_x^α is the generalized shift operator defined by Relation (1.2).

For $\alpha > -\frac{1}{2}$ and for every measurable function f and $x, y > 0$ the generalized shift operator defined by Relation (1.2) may be expressed as an integral operator with kernel by

$$\tau_x^\alpha(f)(y) = \int_0^{+\infty} f(t) W_\alpha(t,x,y) d\mu_\alpha(t), \tag{2.1}$$

where W_α is the kernel given by

$$W_\alpha(t,x,y) = \begin{cases} \frac{\Gamma(\alpha+1)^2}{\sqrt{\pi}2^{\alpha-1}\Gamma(\alpha+\frac{1}{2})} \frac{\left([(x+y)^2-t^2][t^2-(x-y)^2]\right)^{\alpha-1/2}}{(xyt)^{2\alpha}}, & \text{if } |x-y| < t < x+y, \\ 0, & \text{otherwise.} \end{cases}$$

The kernel W_α is symmetric with respect to the variables $t, x, y,$ and satisfies

$$\int_0^{+\infty} W_\alpha(t,x,y) d\mu_\alpha(t) = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})} \int_0^\pi \sin(\theta)^{2\alpha} \mathrm{d}\theta = 1. \tag{2.2}$$

Relations (2.1) and (2.2) show that for every $p \in [1,+\infty]$ and for every $x \in \mathbb{R}_+$ the generalized shift operator τ_x^α is a bounded linear operator from $L^p_\alpha(\mathbb{R}_+)$ into itself satisfying for every $f \in L^p_\alpha(\mathbb{R}_+)$

$$\|\tau_x^\alpha(f)\|_{p,\mu_\alpha} \leqslant \|f\|_{p,\mu_\alpha}.$$

Moreover, for every $f \in L^1_\alpha(\mathbb{R}_+)$, and for every $x \in \mathbb{R}_+$, we have

$$\int_0^{+\infty} \tau_x^\alpha(f)(y)d\mu_\alpha(y) = \int_0^{+\infty} f(y)d\mu_\alpha(y).$$

The generalized convolution product associated to the Bessel operator is defined by

$$f *_\alpha g(x) = \int_0^{+\infty} \tau_x^\alpha(f)(y)g(y)d\mu_\alpha(y),$$

whenever the integral on the right hand side is well defined.

The Hankel transform defined on $L^1_\alpha(\mathbb{R}_+)$, by Relation (1.1), by

$$\forall\lambda \in \mathbb{R}_+,\ \mathscr{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(t)j_\alpha(t\lambda)d\mu_\alpha(t),$$

is a bounded linear operator from $L^1_\alpha(\mathbb{R}_+)$ into $L^\infty_\alpha(\mathbb{R}_+)$ and that for every $f \in L^1_\alpha(\mathbb{R}_+)$, we have

$$\|\mathscr{H}_\alpha(f)\|_{\infty,\mu_\alpha} \leqslant \|f\|_{1,\mu_\alpha}.$$

It's well-known that the Hankel transform $\mathscr{H}_\alpha$ satisfies an inversion formula on $L^1_\alpha(\mathbb{R}_+)$, that is for every $f \in L^1_\alpha(\mathbb{R}_+)$ such that $\mathscr{H}_\alpha(f) \in L^1_\alpha(\mathbb{R}_+)$, we have

$$f(r) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(x)j_\alpha(rx)\mathrm{d}\mu_\alpha(x) \quad \text{a.e.}$$

Moreover, $\mathscr{H}_\alpha$ can be extended by the standard density argument from the Schwartz class to $L^2_\alpha(\mathbb{R}_+)$, and satisfies for every $f,g \in L^2_\alpha(\mathbb{R}_+)$, the following Parseval's formula

$$\int_0^{+\infty} f(x)\overline{g(x)}\mathrm{d}\mu_\alpha(x) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}\mathrm{d}\mu_\alpha(\lambda).$$

In particular, for every $f \in L^2_\alpha(\mathbb{R}_+)$, we have

$$\|\mathscr{H}_\alpha(f)\|_{2,\mu_\alpha} = \|f\|_{2,\mu_\alpha}.$$

PROPOSITION 2.1.

i) For every $f \in L^p_\alpha(\mathbb{R}_+)$, $p = 1,2$, *and* $x \in \mathbb{R}_+$,

$$\forall \lambda \in \mathbb{R}_+, \quad \mathscr{H}_\alpha(\tau^\alpha_x f)(\lambda) = j_\alpha(x\lambda)\mathscr{H}_\alpha(f)(\lambda).$$

2i) For every $f \in L^1_\alpha(\mathbb{R}_+)$ *and* $g \in L^p_\alpha(\mathbb{R}_+)$, $p = 1,2$, *the function* $f *_\alpha g$ *is in* $L^p_\alpha(\mathbb{R}_+)$, $p = 1,2$ *and*

$$\mathscr{H}_\alpha(f *_\alpha g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g).$$

PROPOSITION 2.2. *Let* f *and* $g \in L^2_\alpha(\mathbb{R}_+)$. *Then,*

$$f *_\alpha g = \mathscr{H}_\alpha(\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)).$$

Moreover, $f *_\alpha g \in L^2_\alpha(\mathbb{R}_+)$ *if and only if* $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g) \in L^2_\alpha(\mathbb{R}_+)$ *and in this case*

$$\mathscr{H}_\alpha(f *_\alpha g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g).$$

Proof. Consider the space $S_e(\mathbb{R})$ of Schwartz class functions, even it's well-known that for every $1 \leqslant p < +\infty$, $S_e(\mathbb{R})$ is dense in $L^p_\alpha(\mathbb{R}_+)$. It's also well-known that the Hankel transform is an isomorphism from $S_e(\mathbb{R})$ into itself. Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be two sequences of $S_e(\mathbb{R})$ which converge respectively to f and g in $L^2_\alpha(\mathbb{R}_+)$. Then, we have

$$f_n *_\alpha g_n - f *_\alpha g = (f_n - f) *_\alpha g_n + f *_\alpha (g_n - g).$$

Hence by using the Cauchy-Schwarz inequality, we deduce that for every $n \in \mathbb{N}$ and $y \in \mathbb{R}_+$,

$$\begin{aligned} |f_n *_\alpha g_n(y) - f *_\alpha g(y)| &\leqslant \int_0^{+\infty} |(f_n - f)(x)\tau^\alpha_y(g_n - g)(x)| d\mu_\alpha(x) \\ &\quad + \int_0^{+\infty} |f(x)\tau^\alpha_y(g_n - g)(x)| d\mu_\alpha(x) \\ &\leqslant \|g_n\|_{2,\mu_\alpha}\|f_n - f\|_{2,\mu_\alpha} + \|f\|_{2,\mu_\alpha}\|g_n - g\|_{2,\mu_\alpha}. \end{aligned}$$

Then

$$\|f_n *_\alpha g_n - f *_\alpha g\|_{\infty,\mu_\alpha} \leqslant \|g_n\|_{2,\mu_\alpha}\|f_n - f\|_{2,\mu_\alpha} + \|f\|_{2,\mu_\alpha}\|g_n - g\|_{2,\mu_\alpha},$$

in particular

$$\lim_{n\to+\infty} \|f_n *_\alpha g_n - f *_\alpha g\|_{\infty,\mu_\alpha} \leqslant \lim_{n\to+\infty} \|g_n\|_{2,\mu_\alpha}\|f_n - f\|_{2,\mu_\alpha} + \|f\|_{2,\mu_\alpha}\|g_n - g\|_{2,\mu_\alpha} = 0,$$

which means that the sequence $(f_n *_\alpha g_n)_{n\in\mathbb{N}}$ converges uniformly to $f *_\alpha g$. On the other hand, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} &\mathscr{H}_\alpha(f_n)\mathscr{H}_\alpha(g_n) - \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g) \\ &= (\mathscr{H}_\alpha(f_n) - \mathscr{H}_\alpha(f))\mathscr{H}_\alpha(g_n) + (\mathscr{H}_\alpha(g_n) - \mathscr{H}_\alpha(g))\mathscr{H}_\alpha(f), \end{aligned}$$

and therefore by the same way using the Cauchy-Schwarz inequality, we deduce that for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\|\mathscr{H}_\alpha(f_n)\mathscr{H}_\alpha(g_n) - \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)\|_{1,\mu_\alpha} \\
&= \|(\mathscr{H}_\alpha(f_n) - \mathscr{H}_\alpha(f))\mathscr{H}_\alpha(g_n) + (\mathscr{H}_\alpha(g_n) - \mathscr{H}_\alpha(g))\mathscr{H}_\alpha(f)\|_{1,\mu_\alpha} \\
&\leqslant \|(\mathscr{H}_\alpha(f_n) - \mathscr{H}_\alpha(f))\mathscr{H}_\alpha(g_n)\|_{1,\mu_\alpha} + \|(\mathscr{H}_\alpha(g_n) - \mathscr{H}_\alpha(g))\mathscr{H}_\alpha(f)\|_{1,\mu_\alpha} \\
&\leqslant \|\mathscr{H}_\alpha(g_n)\|_{2,\mu_\alpha}\|\mathscr{H}_\alpha(f_n) - \mathscr{H}_\alpha(f)\|_{2,\mu_\alpha} + \|\mathscr{H}_\alpha(f)\|_{2,\mu_\alpha}\|\mathscr{H}_\alpha(g_n) - \mathscr{H}_\alpha(g)\|_{2,\mu_\alpha} \\
&= \|g_n\|_{2,\mu_\alpha}\|f_n - f\|_{2,\mu_\alpha} + \|f\|_{2,\mu_\alpha}\|g_n - g\|_{2,\mu_\alpha}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\lim_{n\to+\infty} \|\mathscr{H}_\alpha(f_n)\mathscr{H}_\alpha(g_n) - \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)\|_{1,\mu_\alpha} \\
&\leqslant \lim_{n\to+\infty} \|g_n\|_{2,\mu_\alpha}\|f_n - f\|_{2,\mu_\alpha} + \|f\|_{2,\mu_\alpha}\|g_n - g\|_{2,\mu_\alpha} = 0,
\end{aligned}$$

which implies that the sequence $(\mathscr{H}_\alpha(f_n)(\mathscr{H}_\alpha(g_n))_{n\in\mathbb{N}}$ converges to $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)$ in $L^1_\alpha(\mathbb{R}_+)$. However, for every $n \in \mathbb{N}$, we have

$$\mathscr{H}_\alpha(f_n *_\alpha g_n) = \mathscr{H}_\alpha(f_n)\mathscr{H}_\alpha(g_n),$$

then $(\mathscr{H}_\alpha(f_n *_\alpha g_n))_{n\in\mathbb{N}}$ converges to $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)$ in $L^1_\alpha(\mathbb{R}_+)$. Since the Hankel transform is continuous from $L^1_\alpha(\mathbb{R}_+)$ into $\mathscr{C}_{*,0}(\mathbb{R}_+)$ (Space of even continuous functions f on $\mathbb{R}$ such that $\lim_{x\to+\infty} |f(x)| = 0$), we get that the sequence $(f_n *_\alpha g_n)_{n\in\mathbb{N}}$ converges uniformly to $\mathscr{H}_\alpha(\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g))$, in particular $f *_\alpha g = \mathscr{H}_\alpha(\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g))$. Thus, $f *_\alpha g \in L^2_\alpha(\mathbb{R}_+)$ is equivalent to the fact that $\mathscr{H}_\alpha(\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)) \in L^2_\alpha(\mathbb{R}_+)$ which implies that $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g) \in L^2_\alpha(\mathbb{R}_+)$. Conversely, if $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g) \in L^2_\alpha(\mathbb{R}_+)$ then $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g) \in L^1_\alpha(\mathbb{R}_+) \cap L^2_\alpha(\mathbb{R}_+)$ and consequently

$$f *_\alpha g = \mathscr{H}_\alpha(\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g)) \in L^2_\alpha(\mathbb{R}_+). \quad \square$$

2.2. The continuous Hankel Wavelet transform

In the following we denote by $d\nu_\alpha$ the measure defined on $\mathbb{R}^*_+ \times \mathbb{R}_+$ by

$$d\nu_\alpha(a,x) = \frac{a^{2\alpha+1}x^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}dadx.$$

We denote also by $L^p_\alpha(\mathbb{R}^*_+ \times \mathbb{R}_+)$, $p \in [1,+\infty]$ its related Lebesgue spaces of measurable functions F on $\mathbb{R}^*_+ \times \mathbb{R}_+$ such that

$$\|F\|_{p,\nu_\alpha} = \left(\int_0^{+\infty} |F(a,x)|^p \mathrm{d}\nu_\alpha(a,x)\right)^{1/p} < +\infty \quad \text{if } p \in [1,+\infty[,$$

$$\|F\|_{\infty,\nu_\alpha} = \operatorname{esssup}_{(a,x)\in\mathbb{R}^*_+\times\mathbb{R}_+} |F(a,x)| < +\infty \quad \text{if } p = +\infty.$$

For every measurable subset $M \subset \mathbb{R}_+^* \times \mathbb{R}_+$ and for all $\lambda \in \mathbb{R}_+^*$, we denote by

$$\lambda M = \left\{ \left(\lambda a, \frac{x}{\lambda} \right) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid (a,x) \in M \right\}$$
$$= \left\{ (a,x) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid \left(\frac{a}{\lambda}, \lambda x \right) \in M \right\}.$$

For every $\lambda \in \mathbb{R}_+^*$, the dilation operator $\mathscr{D}_\lambda$ is defined on $\mathscr{M}(\mathbb{R}_+^* \times \mathbb{R}_+)$ the set a measurable function on $\mathbb{R}_+^* \times \mathbb{R}_+$, by

$$\mathscr{D}_\lambda(F)(a,x) = F\left(\frac{a}{\lambda}, \lambda x \right).$$

In the following we denote by $\langle\ |\ \rangle_{\mu_\alpha}$ the inner product defined on $L^2_\alpha(\mathbb{R}_+)$ by

$$\langle f|g\rangle_{\mu_\alpha} = \int_0^{+\infty} f(x)\overline{g(x)}d\mu_\alpha(x).$$

We denote also by $\langle\ |\ \rangle_{\nu_\alpha}$ the inner product defined on $L^2_\alpha(\mathbb{R}_+^* \times \mathbb{R}_+)$, by

$$\langle \varphi|\psi\rangle_{\nu_\alpha} = \int_0^{+\infty}\int_0^{+\infty} \varphi(a,x)\overline{\psi(a,x)}d\nu_\alpha(a,x).$$

Then we have the following properties.

PROPERTIES 2.3.

i) For every $f \in L^2_\alpha(\mathbb{R}_+)$ and $a > 0$, we have

$$\|\delta_a(f)\|_{2,\mu_\alpha} = \|f\|_{2,\mu_\alpha}.$$

2i) For every $f,g \in L^2_\alpha(\mathbb{R}_+)$ and $a > 0$, we have

$$\langle \delta_a(f)|g\rangle_{\mu_\alpha} = \langle f|\delta_{\frac{1}{a}} g\rangle_{\mu_\alpha}.$$

3i) For every $x \geqslant 0$ and $a > 0$, we have

$$\delta_a \tau_x^\alpha = \tau_{\frac{x}{a}} \delta_a.$$

4i) For every $a > 0$,

$$\mathscr{H}_\alpha \delta_a = \delta_{\frac{1}{a}} \mathscr{H}_\alpha.$$

5i) For every measurable function $F \in \mathscr{M}(\mathbb{R}_+^* \times \mathbb{R}_+)$, we have

$$Supp(\mathscr{D}_\lambda(F)) = \lambda Supp(F),$$

where the support of a given measurable function h on $\mathbb{R}_+^* \times \mathbb{R}_+$ is defined by

$$Supp(h) = \left\{ (a,x) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid h(a,x) \neq 0 \right\}.$$

DEFINITION 2.4. A nonzero function $g \in L^2_\alpha(\mathbb{R}_+)$ is said to be an admissible Hankel wavelet if

$$0 < C_g = \int_0^{+\infty} |\mathscr{H}_\alpha(g)(a)|^2 \frac{da}{a} < +\infty.$$

DEFINITION 2.5. Let g be an admissible Hankel wavelet. The continuous Hankel wavelet transform $\mathscr{W}_g$ is defined on $L^2_\alpha(\mathbb{R}_+)$ by

$$\forall (a,x) \in \mathbb{R}_+^* \times \mathbb{R}_+,\ \mathscr{W}_g(f)(a,x) = \int_0^{+\infty} f(t)\overline{g_{x,a}(t)}d\mu_\alpha(t), \tag{2.3}$$

where $g_{x,a} = \tau_x^\alpha \delta_a g$.

Relation (2.3) can also be written as

$$\begin{aligned} \mathscr{W}_g(f)(a,x) &= f *_\alpha \delta_a(\overline{g})(x) \\ &= \langle f|g_{x,a}\rangle_{\mu_\alpha}. \end{aligned}$$

PROPOSITION 2.6. *Let g be an admissible Hankel wavelet. Then, the continuous Hankel wavelet transform $\mathscr{W}_g$ is a bounded linear operator from $L^2_\alpha(\mathbb{R}_+)$ onto $L^\infty_\alpha(\mathbb{R}_+^* \times \mathbb{R}_+)$ and we have*

$$\|\mathscr{W}_g(f)\|_{\infty,\nu_\alpha} \leqslant \|f\|_{2,\mu_\alpha}\|g\|_{2,\mu_\alpha}.$$

THEOREM 2.7. (Plancherel) *Let g be an admissible Hankel wavelet. Then,*

i) For every $f,h \in L^2_\alpha(\mathbb{R}_+)$, we have

$$\langle \mathscr{W}_g(f)|\mathscr{W}_g(h)\rangle_{\nu_\alpha} = C_g\langle f|h\rangle_{\mu_\alpha}.$$

2i) For every $f \in L^2_\alpha(\mathbb{R}_+)$, we have

$$\|\mathscr{W}_g(f)\|_{2,\nu_\alpha} = \sqrt{C_g}\|f\|_{2,\mu_\alpha}.$$

Moreover, the Hankel wavelet transform satisfies the following properties (see [3]).

PROPOSITION 2.8. *Let g be an admissible Hankel wavelet and $f \in L^2_\alpha(\mathbb{R}_+)$. Then, we have*

i) $\mathscr{W}_g(f) \in \mathscr{C}_b(\mathbb{R}_+^* \times \mathbb{R}_+)$ (the space of continuous functions bounded on $\mathbb{R}_+^* \times \mathbb{R}_+$).

2i) For every $\lambda > 0$, we have

$$\forall (a,x) \in \mathbb{R}_+^* \times \mathbb{R}_+,\ \mathscr{W}_g(\delta_\lambda f)(a,x) = \mathscr{D}_\lambda(\mathscr{W}_g(f))(a,x). \tag{2.4}$$

3i) For $x_0 \geqslant 0$,

$$\tau_{x_0}(\mathscr{W}_g(f)(a,.))(x) = \mathscr{W}_g(\tau_{x_0}(f))(a,x), \quad (a,x) \in \mathbb{R}_+^* \times \mathbb{R}_+.$$

4i) For every $(a,x) \in \mathbb{R}_+^* \times \mathbb{R}_+$, we have

$$\mathscr{W}_g(f)(a,x) = \frac{1}{a^{\alpha+1}}\mathscr{H}_\alpha\left(\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(\overline{g})\left(\frac{\cdot}{a}\right)\right)(x).$$

3. Amrein-Berthier uncertainty principles associated to the Hankel wavelet transform

In [1] Amrein and Berthier showed that for every measurable subsets $S,\Sigma \subset \mathbb{R}^d$ with finite Lebesgue measure, there exists a nonnegative constant $C(S,\Sigma)$ which depends on S and Σ, such that for every $f \in L^2(\mathbb{R}^d)$, we have

$$\|f\|_{2,d}^2 \leqslant C(S,\Sigma)(\|\chi_{S^c} f\|_{2,d}^2 + \|\chi_{\Sigma^c} \hat{f}\|_{2,d}^2),$$

where χ denotes the characteristic function and S^c,Σ^c denote respectively the complements of S and Σ. Many generalizations of this theorem have been proved in the last decades, namely by Ghobber and Jaming [7, 9] for the Hankel transform and by Wilczok [26] for the classical short time Fourier transform and classical wavelet transform.

In [3], Baccar proved the Amrein-Berthier for the continuous Hankel wavelet transform, however the author imposed a constraint on the support of the admissible Hankel wavelet g so that the theorem doesn't cover the general case. The aim of this section is to prove the main results of this work, more precisely we will prove the Amrein-Berthier theorem for the continuous Hankel wavelet transform in the general case, which implies in particular that the continuous Hankel wavelet transform can not has its support in a set of finite measure unless f is zero. In the second part of this section we will also prove an analogue of Logvinenko-Sereda uncertainty for the transform $\mathscr{W}_g$ with which we will characterize annihilating set for the $\mathscr{W}_g$.

The following proposition has been given by Wilczok [26].

PROPOSITION 3.1. *Let (X,τ,μ) be a σ-finite measurable space. Let $M \subset X$ be a measurable subspace with finite measure and let $\mathscr{P}_M$ be the orthogonal projection on $L^2(d\mu)$ defined by $\mathscr{P}_M(F) = \chi_M F$. Let H be a reproducing kernel Hilbert subspace of $L^2(X)$ with reproducing kernel K. Assuming that*

$$\sup_{(x,y)\in M\times M} |K(x,y)| < +\infty.$$

Then, we have

$$dim\,(H \cap Im\,\mathscr{P}_M) \leqslant \mu(M) \sup_{(x,y)\in M\times M} |K(x,y)| < +\infty.$$

Proof. The proof can be found in [26]. In the following for the sake of completeness, we summarize Wilczok's arguments. Let $\alpha = \sup_{(x,y)\in M\times M} |K(x,y)|$. Then,

$$\iint_{M\times M} |K(x,y)|^2 d\mu(x) \otimes d\mu(y) \leqslant \alpha^2 \mu(M)^2 < +\infty,$$

and therefore $K \in L^2(M\times M)$. Let $m \in \mathbb{N}$ and $(e_n)_{1\leqslant n\leqslant m}$ be an orthonormal sequence of $H \cap Im\,\mathscr{P}_M$ and for every $1 \leqslant n \leqslant m$ let v_n be the the tensor product defined on $M\times M$ by

$$v_n(x,y) = e_n(x)\overline{e_n(y)}.$$

Then for every $1 \leqslant p,q \leqslant m$, we have

$$\iint_{M\times M} v_p(x,y)\overline{v_q(x,y)}d\mu(x)\otimes d\mu(y) = \left(\int_M e_p(x)\overline{e_q(x)}d\mu(x)\right)\left(\int_M e_p(y)\overline{e_q(y)}d\mu(y)\right) = \delta_{pq},$$

so that $(v_n)_{1\leqslant n\leqslant m}$ is an orthonormal sequence in $L^2(M\times M)$. Thus, using Bessel's inequality, we deduce that

$$\begin{aligned}\|K\|_{L^2(M\times M)} &\geqslant \sum_{n=1}^{m}\langle K|v_n\rangle_{M\times M}\\ &= \sum_{n=1}^{m}\iint_{M\times M} K(x,y)\overline{v_n(x,y)}d\mu(x)\otimes d\mu(y)\\ &= \int_M\left(\sum_{n=1}^{m}\int_M K(x,y)\overline{e_n(x)}d\mu(x)\right)e_n(y)d\mu(y)\\ &= \int_M\sum_{n=1}^{m}\overline{\langle e_n|K(.,y)\rangle}_M e_n(y)d\mu(y)\\ &= \int_M\sum_{n=1}^{m}e_n(y)\overline{e_n(y)}d\mu(y)\\ &= m.\end{aligned}$$

This shows that the cardinal of each orthonormal family of $H\cap Im\,\mathscr{P}_M$ is less than $\|K\|_{L^2(M\times M)}$, in particular

$$dim\,(H\cap Im\,\mathscr{P}_M) \leqslant \|K\|_{L^2(M\times M)} \leqslant \alpha\mu(M) < +\infty. \quad \square$$

LEMMA 3.2. *Let $p\in[1,+\infty[$. Then for every $F\in L^p_\alpha(\mathbb{R}_+^*\times\mathbb{R}_+)$ such that $\nu_\alpha(M) < +\infty$, we have*

$$\lim_{\lambda\to 1}\|\mathscr{D}_\lambda F - F\|_{p,\nu_\alpha} = 0.$$

Proof. The idea is basic, the result will be shown first for continuous functions with compact support and then extended by the usual density argument. Actually, let $G\in\mathscr{C}_c(\mathbb{R}_+^*\times\mathbb{R}_+)$ (space of continuous function on $\mathbb{R}_+^*\times\mathbb{R}_+$ with compact support). Then for every $\lambda>0$, we have $\mathscr{D}_\lambda G\in\mathscr{C}_c(\mathbb{R}_+^*\times\mathbb{R}_+)$ and thus for all $\varepsilon>0$ sufficiently small there are $a,b>0$ such that for every $\lambda\in]1-\varepsilon,1+\varepsilon[$, $Supp\,(\mathscr{D}_\lambda G)\subset[a,b]\times[a,b]$ and therefore by using the dominate convergence theorem, we get

$$\lim_{\lambda\to 1}\|\mathscr{D}_\lambda G - G\|^p_{p,\nu_\alpha} = 0. \quad \square$$

LEMMA 3.3. *Let $M,M_0\subset\mathbb{R}_+^*\times\mathbb{R}_+$ be two measurable subsets such that $M_0\subset M$, $\nu_\alpha(M_0)>0$ and $\nu_\alpha(M)<+\infty$. Then, the function φ defined on $\mathbb{R}_+^*$ by*

$$\varphi(\lambda) = \nu_\alpha(M\cup\lambda M_0),$$

is a non constant continuous function on $\mathbb{R}_+^*$.

Proof. For every $\lambda \in \mathbb{R}_+^*$, we have

$$\begin{aligned}\varphi(\lambda) &= \nu_\alpha(M \cup \lambda M_0)\\ &= \nu_\alpha(M) + \nu_\alpha(\lambda M_0) - \nu_\alpha(M \cap \lambda M_0)\\ &= \nu_\alpha(M) + \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \chi_{\lambda M_0}(a,x) d\nu_\alpha(a,x)\\ &\quad - \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \chi_M(a,x)\chi_{\lambda M_0}(a,x) d\nu_\alpha(a,x)\\ &= \nu_\alpha(M) + \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \chi_{M^c}(a,x)\chi_{M_0}(\frac{a}{\lambda}, \lambda x) d\nu_\alpha(a,x).\end{aligned}$$

Consequently, for every $\lambda, \lambda_0 \in \mathbb{R}_+^*$, we have

$$\begin{aligned}|\varphi(\lambda) - \varphi(\lambda_0)| &= \left| \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \chi_{M^c}(a,x) \left(\chi_{M_0}\left(\frac{a}{\lambda}, \lambda x\right) - \chi_{M_0}\left(\frac{a}{\lambda_0}, \lambda_0 x\right)\right) d\nu_\alpha(a,x) \right|\\ &\leqslant \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} |\chi_{M^c}(a,x)| \left| \chi_{M_0}\left(\frac{a}{\lambda}, \lambda x\right) - \chi_{M_0}\left(\frac{a}{\lambda_0}, \lambda_0 x\right)\right| d\nu_\alpha(a,x)\\ &\leqslant \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \left| \chi_{M_0}\left(\frac{a}{\lambda}, \lambda x\right) - \chi_{M_0}\left(\frac{a}{\lambda_0}, \lambda_0 x\right)\right| d\nu_\alpha(a,x)\\ &= \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \left|\chi_{M_0}\left(\frac{\lambda_0}{\lambda}a, \frac{\lambda}{\lambda_0}x\right) - \chi_{M_0}(a,x)\right| d\nu_\alpha(a,x).\end{aligned}$$

So by Lemma 3.2, we deduce that

$$\begin{aligned}\lim_{\lambda \to \lambda_0} |\varphi(\lambda) - \varphi(\lambda_0)| &\leqslant \lim_{\frac{\lambda}{\lambda_0} \to 1} \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \left|\chi_{M_0}\left(\frac{\lambda_0}{\lambda}a, \frac{\lambda}{\lambda_0}x\right) - \chi_{M_0}(a,x)\right| d\nu_\alpha(a,x)\\ &= \lim_{h \to 1} \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \left|\chi_{M_0}\left(\frac{a}{h}, hx\right) - \chi_{M_0}(a,x)\right| d\nu_\alpha(a,x)\\ &= \lim_{h \to 1} \|\mathscr{D}_h(\chi_{M_0}) - \chi_{M_0}\|_{1,\nu_\alpha}\\ &= 0.\end{aligned}$$

Hence, φ is continuous on $\mathbb{R}_+^*$. Now we shall prove that φ is non constant, so one can see that

$$\varphi(1) = \nu_\alpha(M \cup M_0) = \nu_\alpha(M).$$

For every $p \in \mathbb{N}^*$, let $B_p = \left\{(a,x) \in \mathbb{R}_+^* \times \mathbb{R}_+ \mid \frac{1}{p} \leqslant x \leqslant p\right\}$ and $C_p = M \cap B_p^c$. Then the sequence $(C_p)_{p \in \mathbb{N}^*}$ is decreasing in the sense of inclusion and we have

$$\nu_\alpha(C_1) \leqslant \nu_\alpha(M) < +\infty.$$

Consequently, by the right continuous of the measure ν_α, we have

$$\begin{aligned}\lim_{p\to+\infty}\nu_\alpha(C_p) &= \nu_\alpha\left(\bigcap_{p=1}^{+\infty} C_p\right)\\ &= \nu_\alpha\left(\bigcap_{p=1}^{+\infty} M\cap B_p^c\right) = \nu_\alpha\left(M\bigcap\left(\bigcap_{p=1}^{+\infty} B_p^c\right)\right)\\ &= \nu_\alpha(M\cap\mathbb{R}_+^*\times\{0\}) = 0.\end{aligned}$$

Let $0<\delta<\dfrac{\nu_\alpha(M_0)}{2}$. Then there exists $p_0\in\mathbb{N}^*$ such that $\nu_\alpha(C_{p_0})<\delta$. So, we have

$$\nu_\alpha(M\cap B_{p_0}) = \nu_\alpha(M)-\nu_\alpha(C_{p_0}) > \nu_\alpha(M)-\delta. \tag{3.1}$$

On the other hand, for every $\lambda\in\mathbb{R}_+^*$, and for all subsets $F,G\subset\mathbb{R}_+^*\times\mathbb{R}_+$, we have

$$\lambda F\cap G = \lambda\Big(F\cap\frac{1}{\lambda}G\Big).$$

Let $\lambda\in\mathbb{R}_+^*$ such that $\lambda>p_0^2$. Since $\nu_\alpha(\lambda M)=\nu_\alpha(M)$, thus, we have

$$\nu_\alpha(\lambda M\cap B_{p_0}) = \nu_\alpha\left(\lambda\Big(M\cap\frac{1}{\lambda}B_{p_0}\Big)\right) = \nu_\alpha\left(M\cap\frac{1}{\lambda}B_{p_0}\right).$$

Since $M_0\subset M$, thus for every $\lambda>p_0^2$, we have $\frac{1}{\lambda}B_{p_0}\subset B_{p_0}^c$ and

$$\begin{aligned}\nu_\alpha(\lambda M_0\cap B_{p_0}) &= \nu_\alpha\Big(M_0\cap\frac{1}{\lambda}B_{p_0}\Big)\\ &\leqslant \nu_\alpha\left(M\cap\frac{1}{\lambda}B_{p_0}\right)\\ &\leqslant \nu_\alpha\left(M\cap B_{p_0}^c\right) = \nu_\alpha(C_{p_0})\\ &< \delta,\end{aligned}$$

and then

$$\begin{aligned}\nu_\alpha(\lambda M_0\cap B_{p_0}^c) &= \nu_\alpha(\lambda M_0)-\nu_\alpha(\lambda M_0\cap B_{p_0})\\ &\geqslant \nu_\alpha(\lambda M_0)-\delta\\ &= \nu_\alpha(M_0)-\delta.\end{aligned} \tag{3.2}$$

Using now Relations (3.1) and (3.2), we get

$$\begin{aligned}\varphi(\lambda) &= \nu_\alpha(M\cup\lambda M_0)\\ &= \nu_\alpha\left((M\cup\lambda M_0)\cap B_{p_0}\right)+\nu_\alpha\left((M\cup\lambda M_0)\cap B_{p_0}^c\right)\\ &\geqslant \nu_\alpha\left(M\cap B_{p_0}\right)+\nu_\alpha\left(\lambda M_0\cap B_{p_0}^c\right)\end{aligned}$$

$$\geqslant \nu_\alpha(M) + \nu_\alpha(\lambda M_0) - 2\delta$$
$$= \nu_\alpha(M) + \nu_\alpha(M_0) - 2\delta$$
$$> \nu_\alpha(M) = \varphi(1),$$

and then φ is not constant. □

COROLLARY 3.4. *Let $M, M_0 \subset \mathbb{R}_+^* \times \mathbb{R}_+$ be two measurable subsets such that $M_0 \subset M$, $\nu_\alpha(M_0) > 0$ and $\nu_\alpha(M) < +\infty$. Then, for every $\varepsilon \in]0, \nu_\alpha(M_0)[$ there exists $\lambda_\varepsilon \in \mathbb{R}_+^*$ such that*

$$\nu_\alpha(M) < \nu_\alpha(M \cup \lambda_\varepsilon M_0) < \nu_\alpha(M) + \varepsilon.$$

PROPOSITION 3.5. *Let g be an admissible Hankel wavelet. The space $\mathscr{W}_g(L^2_\alpha(\mathbb{R}_+))$ is a reproducing kernel Hilbert space with kernel*

$$\mathscr{K}_g\left((x',a');(x,a)\right) = \frac{1}{C_g} g_{x',a'} *_\alpha \overline{\delta_a(g)}(x) = \frac{1}{C_g}\mathscr{W}_g(g_{x',a'})(x,a).$$

Proof. The proof is identical to that given by [11]. In fact, it is obvious that

$$\|\mathscr{K}_g((.,.);(x,a))\|^2_{2,\nu_\alpha} = \frac{1}{C_g^2}\|\mathscr{W}_g(g_{x,a})\|^2_{2,\nu_\alpha} = \frac{1}{C_g}\|g_{x,a}\|^2_{2,\mu_\alpha} \leqslant \frac{1}{C_g}\|g\|^2_{2,\mu_\alpha} < +\infty. \quad \square$$

The Proposition 3.5 allows us to define for every admissible Hankel wavelet g, the orthogonal projection operator $\mathscr{P}_g$ from $L^2_\alpha(\mathbb{R}_+^* \times \mathbb{R}_+)$ over $\mathscr{W}_g\left(L^2_\alpha(\mathbb{R}_+)\right)$. We also define for every given measurable subset $M \subset \mathbb{R}_+^* \times \mathbb{R}_+$, the orthogonal projection $\mathscr{P}_M$ on $L^2(\mathbb{R}_+^* \times \mathbb{R}_+)$ by

$$\mathscr{P}_M(F) = \chi_M F.$$

PROPOSITION 3.6. *Let g be an admissible Hankel wavelet and $M \subset \mathbb{R}_+^* \times \mathbb{R}_+$ be a measurable subset such that $\nu_\alpha(M) < +\infty$. Then, $\mathscr{P}_M\mathscr{P}_g$ is a Hilbert-Schmidt operator, with*

$$\|\mathscr{P}_M\mathscr{P}_g\|_{HS} \leqslant \sqrt{\nu_\alpha(M)}\|g\|_{2,\mu_\alpha},$$

where $\| \ \|_{HS}$ denotes the Hilbert-Schmidt norm. In particular, $\mathscr{P}_M\mathscr{P}_g$ is compact.

Proof. For every $F \in L^2_\alpha(\mathbb{R}_+^* \times \mathbb{R}_+)$, we have

$$\begin{aligned}\mathscr{P}_M\mathscr{P}_g(F)(a,x) &= \chi_M(a,x)\mathscr{P}_g(F)(a,x)\\ &= \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \chi_M(a,x)F(b,y)\overline{\mathscr{K}_g((b,y),(a,x))}d\nu_\alpha(a,x).\end{aligned}$$

Then, $\mathscr{P}_M\mathscr{P}_g$ is an integral operator with kernel $N_{g,M}$ defined on $(\mathbb{R}_+^* \times \mathbb{R}_+) \times (\mathbb{R}_+^* \times \mathbb{R}_+)$ by

$$N_{g,M}((b,y),(a,x)) = \chi_M(a,x)\overline{\mathscr{K}_g((b,y),(a,x))}.$$

Furthermore,

$$\begin{aligned}
&\iint_{\mathbb{R}_+^*\times\mathbb{R}_+}\iint_{\mathbb{R}_+^*\times\mathbb{R}_+}|N_{g,M}((b,y),(a,x))|^2 d\nu_\alpha(b,y)d\nu_\alpha(a,x)\\
&=\iint_{\mathbb{R}_+^*\times\mathbb{R}_+}\chi_M(a,x)\left(\iint_{\mathbb{R}_+^*\times\mathbb{R}_+}|\mathscr{K}_g((b,y),(a,x))|^2 d\nu_\alpha(b,y)\right)d\nu_\alpha(a,x)\\
&=\iint_{\mathbb{R}_+^*\times\mathbb{R}_+}\chi_M(a,x)\left(\iint_{\mathbb{R}_+^*\times\mathbb{R}_+}\frac{1}{C_g^2}|\mathscr{W}_g(g_{a,x})(b,y)|^2 d\nu_\alpha(b,y)\right)d\nu_\alpha(a,x)\\
&=\iint_{\mathbb{R}_+^*\times\mathbb{R}_+}\chi_M(a,x)\|g_{a,x}\|_{2,\mu_\alpha}^2 d\nu_\alpha(a,x)\\
&\leqslant\iint_{\mathbb{R}_+^*\times\mathbb{R}_+}\chi_M(a,x)\|g\|_{2,\mu_\alpha}^2 d\nu_\alpha(a,x)\\
&=\nu_\alpha(M)\|g\|_{2,\mu_\alpha}^2. \quad \square
\end{aligned}$$

PROPOSITION 3.7. *Let $g\in L^2_\alpha(\mathbb{R}_+)$ be an admissible Hankel wavelet and $M\subset\mathbb{R}_+^*\times\mathbb{R}_+$ be a measurable subset such that $\nu_\alpha(M)<+\infty$. Then,*

$$Im\,\mathscr{P}_g\cap Im\,\mathscr{P}_M=\{0\}.$$

Proof. Suppose that $Im\,\mathscr{P}_g\cap Im\,\mathscr{P}_M\neq\{0\}$ and let $F_0\in Im\,\mathscr{P}_g\cap Im\,\mathscr{P}_M$ such that $F_0\neq 0$. Let $M_0=Supp(F_0)$. Since $F_0\neq 0$ and $F_0\in\mathscr{P}_M$, then

$$0<\nu_\alpha(M_0)\leqslant\nu_\alpha(M)<+\infty.$$

Let $(M_k)_{k\in\mathbb{N}}$ be the sequence of subsets of $\mathbb{R}_+^*\times\mathbb{R}_+$ defined by $M_0=M_0$ and

$$\forall k\geqslant 1,\ M_k=M_{k-1}\cup\left(\prod_{j=1}^{k}\lambda_j M_0\right),$$

where $(\lambda_j)_{1\leqslant j\leqslant k}\subset\mathbb{R}_+^*$ are chosen according to Corollary 3.4, such that for every $k\geqslant 1$

$$\nu_\alpha(M_{k-1})<\nu_\alpha(M_k)=\nu_\alpha\left(M_{k-1}\cup\left(\prod_{j=1}^{k}\lambda_j M_0\right)\right)<\nu_\alpha(M_{k-1})+\frac{1}{2^{k-1}}. \tag{3.3}$$

Therefore, we obtain

$$\forall k\in\mathbb{N}^*,\ \nu_\alpha(M_0)<\nu_\alpha(M_k)<\nu_\alpha(M_0)+\sum_{j=1}^{k}\frac{1}{2^j}. \tag{3.4}$$

Note that by Relation (3.3), we have in particular

$$\forall k\in\mathbb{N},\ M_k\subsetneq M_{k+1}. \tag{3.5}$$

Let $\widetilde{M} = \bigcup_{k=0}^{\infty} M_k$. Then by Relations (3.4) and (3.5), we deduce that

$$\begin{aligned} \nu_\alpha(\widetilde{M}) &= \nu_\alpha(\bigcup_{k=0}^{\infty} M_k) \\ &= \lim_{k\to\infty} \nu_\alpha(M_k) \\ &\leqslant \nu_\alpha(M_0) + \sum_{i=0}^{\infty} \frac{1}{2^i} \\ &= \nu_\alpha(M_0) + 2 \\ &< \infty. \end{aligned}$$

Let $(F_k)_{k\in\mathbb{N}^*}$ be the sequence of functions defined on $\mathbb{R}_+^* \times \mathbb{R}_+$ by

$$\forall k \geqslant 1, \;\; F_k = \mathscr{D}_{\lambda_k}(F_{k-1}).$$

Since $F_0 \in \mathscr{W}_g(L^2_\alpha(\mathbb{R}_+))$. Then by Relation (2.4), we deduce that

$$\forall k \in \mathbb{N}, \;\; F_k \in \mathscr{W}_g(L^2_\alpha(\mathbb{R}_+)). \tag{3.6}$$

However, using Relation (3.5), we have

$$\forall k \in \mathbb{N}, \;\; Supp(F_k) \subset M_k \subset \widetilde{M}. \tag{3.7}$$

Thus, by Relations (3.6) and (3.7), we deduce that

$$\forall k \in \mathbb{N}, \;\; F_k \in (Im\,\mathscr{P}_{\widetilde{M}} \cap Im\,\mathscr{P}_g).$$

Now, we suppose that there exist $k \geqslant 2$ and $\gamma_0, \ldots, \gamma_{k-1} \in \mathbb{R}_+$ such that

$$F_k = \sum_{j=1}^{k-1} \gamma_j F_j.$$

Then, we get

$$Supp(F_k) = Supp(\sum_{j=0}^{k-1} \gamma_j F_j) \subset \bigcup_{j=0}^{k-1} \gamma_j Supp(F_j) = M_{k-1}.$$

which leads to

$$\nu_\alpha(M_k) \leqslant \nu_\alpha(M_{k-1}),$$

and contradicts then Relation (3.5). Thus, the sequence $(F_k)_{k\in\mathbb{N}^*}$ is linearly independent which contradicts the fact that

$$dim\,(Im\mathscr{P}_{\widetilde{M}} \cap Im\,\mathscr{P}_g) < +\infty. \quad \square$$

DEFINITION 3.8. (Annihilating set) Let $g \in L^2_\alpha(\mathbb{R}_+)$ be an admissible Hankel wavelet and $M \subset \mathbb{R}^*_+ \times \mathbb{R}_+$ be a measurable subset. We say that M is a strong annihilating set for the continuous Hankel wavelet transform $\mathscr{W}_g$, if there exists a non negative constant $C_{g,M}$ such that for every $f \in L^2_\alpha(\mathbb{R}_+)$, we have

$$\|\chi_{M^c}\mathscr{W}_g(f)\|_{2,\nu_\alpha} \geqslant C_{g,M}\|f\|_{2,\mu_\alpha}.$$

THEOREM 3.9. (Amrein-Berthier) *Let $M \subset \mathbb{R}^*_+ \times \mathbb{R}_+$ be a measurable subset. If $\nu_\alpha(M) < +\infty$, then for every admissible Hankel wavelet $g \in L^2_\alpha(\mathbb{R}_+)$, M is a strong annihilating set for the continuous Hankel wavelet transform $\mathscr{W}_g$.*

Proof. According to [14, Theorem A, p 88] and [14, Theorem A, p 90], and using Propositions 3.6 and 3.7, we deduce that there exists $C > 0$ such that for every $f \in L^2_\alpha(\mathbb{R}_+)$, we have

$$\|\chi_{M^c}\mathscr{W}_g(f)\|_{2,\nu_\alpha} \geqslant C\sqrt{C_g}\|f\|_{2,\mu_\alpha}. \qquad \square$$

4. Logvinenko-Sereda uncertainy principle associated to the Hankel Wavelet transfom

In [17] Logvinenko and Sereda showed that if Σ is a closed interval in $\mathbb{R}$ then for every measurable set $S \subset \mathbb{R}$, (S,Σ) is a strong annihilating pair if and only if S^c is relatively dense. This result has been generalized by Ghobber and Jaming [8] in the Hankel setting. In this part we are interesting in proving an analogue of this theorem for the continuous Hankel wavelet transform. More precisely we will prove that if a measurable subset $M \subset \mathbb{R}^*_+ \times \mathbb{R}_+$ is a strong annihilating set, then M^c satisfy some topological density property.

DEFINITION 4.1. Let $M \subset \mathbb{R}^*_+ \times \mathbb{R}_+$ be a measurable subset and let $N, \gamma > 0$. The subset M is said to be (N,γ)-dense if for every $\lambda \in \mathbb{R}^*_+$, we have

$$\nu_\alpha(M \cap \lambda K_N) \geqslant \gamma,$$

where $K_N = [0,N]^2$.

THEOREM 4.2. *Let $g \in L^2_\alpha(\mathbb{R}_+)$ be an admissible Hankel wavelet, and let M be a measurable subset of $\mathbb{R}^*_+ \times \mathbb{R}_+$. If M is an annihilating set for $\mathscr{W}_g$, then there exist $N, \gamma > 0$ such that M^c is (N,γ)-dense.*

Proof. Let $f \in L^2_\alpha(\mathbb{R}_+)$ such that $\|f\|_{2,\mu_\alpha} = 1$. For every $\sigma \geqslant 0$ we denote by

$$T_g(\sigma) = \sup_{\nu_\alpha(E)\leqslant\sigma}\{\|\chi_E\mathscr{W}_g(f)\|^2_{2,\nu_\alpha}\}.$$

Then, for every $\lambda > 0$ and for every measurable subset $E \subset \mathbb{R}_+^* \times \mathbb{R}_+$, we have

$$\begin{aligned}\iint_E |\mathscr{D}_\lambda(\mathscr{W}_g(f))(a,x)|^2 d\nu_\alpha(a,x) &= \iint_E \left|\mathscr{W}_g(f)\left(\frac{a}{\lambda},\lambda x\right)\right|^2 d\nu_\alpha(a,x) \\ &= \iint_{\frac{1}{\lambda}E} |\mathscr{W}_g(f)(a,x)|^2 d\nu_\alpha(a,x) \\ &\leqslant T_g\left(\nu_\alpha\left(\frac{1}{\lambda}E\right)\right).\end{aligned}$$

Since

$$\nu_\alpha(\lambda E) = \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \chi_{(\lambda E)}(a,x) d\nu_\alpha(a,x) = \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} \chi_E\left(\frac{a}{\lambda},\lambda x\right) d\nu_\alpha(a,x) = \nu_\alpha(E),$$

then we get

$$\iint_E |\mathscr{D}_\lambda(\mathscr{W}_g(f))(a,x)|^2 d\nu_\alpha(a,x) \leqslant T_g(\nu_\alpha(E)). \tag{4.1}$$

On the other hand, for every $n \in \mathbb{N}^*$, let $K_n = [0,n]^2$, then we have

$$\lim_{n\to+\infty} \iint_{K_n^c} |\mathscr{W}_g(f)(a,x)|^2 d\nu_\alpha(a,x) = 0,$$

so that there exists $N \in \mathbb{N}^*$ such that

$$\iint_{K_N^c} |\mathscr{W}_g(f)(a,x)|^2 d\nu_\alpha(a,x) \leqslant \frac{C_g^2}{2C_{g,M}^2}.$$

Consequently, for every $\lambda > 0$, we get

$$\begin{aligned}\iint_{\lambda K_N^c} |\mathscr{D}_\lambda \mathscr{W}_g(f)(a,x)|^2 d\nu_\alpha(a,x) &= \iint_{\lambda K_N^c} \left|\mathscr{W}_g(f)\left(\frac{a}{\lambda},\lambda x\right)\right|^2 d\nu_\alpha(a,x) \\ &= \iint_{K_N^c} |\mathscr{W}_g(f)(a,x)|^2 d\nu_\alpha(a,x),\end{aligned}$$

and therefore by using Definition 3.8, we have

$$\iint_{\lambda K_N^c} |\mathscr{D}_\lambda \mathscr{W}_g(f)(a,x)|^2 d\nu_\alpha(a,x) \leqslant \frac{C_g^2}{2C_{g,M}^2}. \tag{4.2}$$

Hence, by using Relations (4.1) and (4.2), we deduce that

$$\begin{aligned}C_g^2 &= \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} |\mathscr{D}_\lambda \mathscr{W}_g(f)(a,x)|^2 d\nu_\alpha(a,x) \\ &= \iint_{\mathbb{R}_+^* \times \mathbb{R}_+} |\mathscr{W}_g(\delta_\lambda f)(a,x)|^2 d\nu_\alpha(a,x) \\ &\leqslant C_{g,M}^2 \iint_{M^c} |\mathscr{W}_g(\delta_\lambda f)(a,x)|^2 d\nu_\alpha(a,x)\end{aligned}$$

$$\begin{aligned}&= C_{g,M}^2 \iint_{M^c\cap\lambda K_N} |\mathscr{D}_\lambda \mathscr{W}_g(f)(a,x)|^2 d\nu_\alpha(a,x) \\ &\quad + C_{g,M}^2 \iint_{M^c\cap(\lambda K_N)^c} |\mathscr{D}_\lambda \mathscr{W}_g(f)(a,x)|^2 d\nu_\alpha(a,x) \\ &\leqslant \frac{C_g^2}{2} + C_{g,M}^2 T_g(\nu_\alpha(M^c\cap\lambda K_N)),\end{aligned}$$

and then

$$T_g(\nu_\alpha(M^c\cap\lambda K_N)) \geqslant \frac{C_g^2}{2C_{g,M}^2}.$$

However, $\lim_{\sigma\to 0} T_g(\sigma) = 0$, then there exists $\gamma > 0$ such that for every $\lambda > 0$

$$\nu_\alpha(M^c\cap\lambda K_N) \geqslant \gamma,$$

thus M^c is (N,γ)-dense. □

In the following let

$$\forall r\in\mathbb{R}_+,\quad \psi_\alpha(r) = \frac{1}{(1+r^2)^{\alpha+\frac{3}{2}}},$$

and let Ψ_α the function defined on $\mathbb{R}_+^2$ by

$$\Psi_\alpha(a,x) = \psi_\alpha(a)\psi_\alpha(x).$$

For every $\lambda > 0$ we denote by

$$d\eta_{\alpha,\lambda} = \mathscr{D}_\lambda \Psi_\alpha d\nu_\alpha.$$

In the following we give an equivalent characterization of the $(N-\gamma)$ density property.

THEOREM 4.3. *Let M be a measurable subset of $\mathbb{R}_+^*\times\mathbb{R}_+$, then the following assertions are equivalents*

i) There exist $N,\gamma > 0$ such that M is (N,γ)-dense.

ii) $\inf\{\eta_{\alpha,\lambda}(M) \mid \lambda > 0\} > 0$.

Proof. Suppose that there exist $N,\gamma > 0$ such that M is (N,γ)-dense, then

$$\begin{aligned}\eta_{\alpha,\lambda}(M) &\geqslant \eta_{\alpha,\lambda}(M\cap\lambda K_N) \\ &= \iint_{M\cap\lambda K_N} \mathscr{D}_\lambda \Psi_\alpha(a,x) d\nu_\alpha(a,x) \\ &= \iint_{\lambda K_N} \chi_M(a,x)\mathscr{D}_\lambda \Psi_\alpha(a,x) d\nu_\alpha(a,x) \\ &= \int_0^{\lambda N}\int_0^{\frac{N}{\lambda}} \chi_M(a,x)\mathscr{D}_\lambda \Psi_\alpha(a,x) d\nu_\alpha(a,x)\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\lambda N} \int_0^{\frac{N}{\lambda}} \chi_M(a,x) \psi_\alpha\left(\frac{a}{\lambda}\right) \psi_\alpha(\lambda x) d\nu_\alpha(a,x) \\
&\geqslant \psi_\alpha(N)^2 \int_0^{\lambda N} \int_0^{\frac{N}{\lambda}} \chi_M(a,x) d\nu_\alpha(a,x) \\
&= \psi_\alpha(N)^2 \iint_{\lambda K_N} \chi_M(a,x) d\nu_\alpha(a,x) \\
&= \psi_\alpha(N)^2 \nu_\alpha(M \cap \lambda K_n) \\
&\geqslant \psi_\alpha(N)^2 \gamma.
\end{aligned}
$$

In particular $\inf\{\eta_{\alpha,\lambda}(M) \mid \lambda > 0\} > 0$. Suppose now that

$$\inf\{\eta_{\alpha,\lambda}(M) \mid \lambda > 0\} = \sigma > 0,$$

and let $N \geqslant \sqrt{\dfrac{2^{3\alpha+5}}{\Gamma(\alpha+1)\sigma}}$. For every $p,q \in \mathbb{N}^*$, let

$$I_{p,q} = \left\{(a,x) \in [0,+\infty[\times[0,+\infty[\mid N2^p \leqslant a < N2^{p+1},\ N2^q \leqslant x < N2^{q+1}\right\},$$

then for every $\lambda > 0$, we have

$$\mathbb{R}_+^2 = \lambda K_N \bigcup \left(\bigcup_{p,q=1}^{+\infty} \lambda I_{p,q} \right).$$

Therefore, we have

$$
\begin{aligned}
\sigma &\leqslant \eta_{\alpha,\lambda}(M) \\
&= \int_0^{+\infty} \int_0^{+\infty} \chi_M(a,x) \mathscr{D}_\lambda(\Psi_\alpha)(a,x) d\nu_\alpha(a,x) \\
&= \iint_{\lambda K_N} \chi_M(a,x) \mathscr{D}_\lambda(\Psi_\alpha)(a,x) d\nu_\alpha(a,x) \\
&\quad + \sum_{p,q=1}^{+\infty} \iint_{\lambda I_{p,q}} \chi_M(a,x) \mathscr{D}_\lambda(\Psi_\alpha)(a,x) d\nu_\alpha(a,x) \\
&= \iint_{\lambda K_N} \chi_M(a,x) \psi_\alpha\left(\frac{a}{\lambda}\right) \psi_\alpha(\lambda x) d\nu_\alpha(a,x) \\
&\quad + \sum_{p,q=1}^{+\infty} \iint_{\lambda I_{p,q}} \chi_M(a,x) \psi_\alpha\left(\frac{a}{\lambda}\right) \psi_\alpha(\lambda x) d\nu_\alpha(a,x) \\
&\leqslant \iint_{\lambda K_N} \chi_M(a,x) d\nu_\alpha(a,x) \\
&\quad + \sum_{p,q=1}^{+\infty} \iint_{\lambda I_{p,q}} \chi_M(a,x) \psi_\alpha\left(\frac{a}{\lambda}\right) \psi_\alpha(\lambda x) d\nu_\alpha(a,x) \\
&\leqslant \nu_\alpha(M \cap \lambda K_N) + \sum_{p,q=1}^{+\infty} \psi_\alpha(N2^p) \psi_\alpha(N2^q) \iint_{\lambda I_{p,q}} \chi_M(a,x) d\nu_\alpha(a,x)
\end{aligned}
$$

$$
\begin{aligned}
&= \nu_\alpha(M\cap\lambda K_N) + \sum_{p,q=1}^{\infty} \frac{1}{(1+N^2 2^{2p})^{\alpha+\frac{3}{2}}} \frac{1}{(1+N^2 2^{2q})^{\alpha+\frac{3}{2}}} \nu_\alpha(M\cap\lambda I_{p,q}) \\
&\leqslant \nu_\alpha(M\cap\lambda K_N) + \frac{1}{N^{4\alpha+6}} \sum_{p,q=1}^{\infty} \frac{1}{(2^{2p})^{\alpha+\frac{3}{2}}} \frac{1}{(2^{2q})^{\alpha+\frac{3}{2}}} \nu_\alpha(I_{p,q}) \\
&= \nu_\alpha(M\cap\lambda K_N) + \frac{(2^{2\alpha+2}-1)^2}{N^2\Gamma(\alpha+1)(\alpha+1)^2 2^{2\alpha+2}} \sum_{p,q=1}^{\infty} \frac{1}{2^p}\frac{1}{2^q} \\
&\leqslant \nu_\alpha(M\cap\lambda K_N) + \frac{2^{3\alpha+4}}{N^2\Gamma(\alpha+1)} \\
&\leqslant \nu_\alpha(M\cap K_N) + \frac{\sigma}{2}.
\end{aligned}
$$

Hence,

$$\nu_\alpha\left(M\cap\lambda K_N\right) \geqslant \frac{\sigma}{2},$$

in particular M is $\left(\frac{\sigma}{2},N\right)$-dense. $\square$

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S. Hkimi
Département de mathématiques
Faculté des sciences de Tunis
Campus universitaire El Manar, Tunis-Tunisie
e-mail: siwar.hkimi@fst.utm.tn

S. Omri
Département de mathématiques
Faculté des sciences de Tunis
Campus universitaire El Manar, Tunis-Tunisie
e-mail: slim.omri@ipein.rnu.tn