CORRIGENDUM TO: ON SUPERCYCLICITY
FOR ABELIAN SEMIGROUPS OF MATRICES ON
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SALAH HERZI AND HABIB MARZOUGUI

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Abstract. In this note we present corrected versions of some results in [1], due to an error in Lemma 3.6 of [1]. We also provide changes in the proof of Proposition 4.6.

This paper provides a corrigendum to some results caused by an error in Lemma 3.6 of [1] which can fairly be repaired. This error affected Theorems 3.1 and 4.1, Corollary 4.3 and Theorem 5.1. We also correct the proof of Proposition 4.6 in the case 2: $r = 0$. Below we go through the listed results and prove their corrected versions.

For Lemma 3.6, the correction is straightforward: we replace the false formula $\operatorname{ind}(G') = \operatorname{ind}(G)$ by the right one $\operatorname{ind}(G) \leqslant \operatorname{ind}(G')$ and $\operatorname{ind}(\mathbb{R}_+G) = \operatorname{ind}(G)$. Herewith a corrected version of Lemma 3.6:

LEMMA 3.6. *We have* $\operatorname{ind}(\mathbb{R}_+G) = \operatorname{ind}(G)$ *and* $\operatorname{ind}(G) \leqslant \operatorname{ind}(G')$.

(1) In Theorem 3.1, a corrected statement of it is as follows.

THEOREM 3.1. *Let* $n \in \mathbb{N}$, $n \geqslant 1$ *and let* G *be an abelian sub-semigroup of* $\mathscr{K}_\eta(\mathbb{R})$, *where* η *has length* $r + 2s$. *Then the following are equivalent:*

(i) *G is supercyclic,*

(ii) *u_η is a supercyclic vector for G,*

(iii) *$\mathrm{g}_\eta^2(u_\eta) + \mathbb{R}u_\eta$ is dense in $\mathbb{R}^n$ and $\operatorname{ind}(\mathbb{R}G) = r$.*

The proof of Theorem 3.1 doesn't use Lemmas 3.3, 3.7 and 3.9. However, we need the following lemmas.

We denote by

- $\mathbb{R}_+G := \{\lambda A : A \in G, \lambda \in \mathbb{R}_+\}$
- $\mathrm{g}_\eta^+ = \exp^{-1}(\mathbb{R}_+G) \cap \mathscr{K}_\eta(\mathbb{R})$.

LEMMA 0.1. *We have* $\mathrm{g}_\eta^+ = \mathrm{g}_\eta + \mathbb{R}I_n$.

Proof. The proof is similar to that of Lemma 3.2. □

The proof of Lemma 3.4 should be changed and may be proved simply as follows:

Proof of Lemma 3.4. This follows from the fact that

$$g'_\eta \subset \frac{1}{2}(g'_\eta)^2 \text{ and } (g'_\eta)^2 \subset g'_\eta. \quad \Box$$

LEMMA 0.2. ([3], Lemma 4.2) *Let G be an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$ and $g^*_\eta = \exp^{-1}(G^*) \cap \mathscr{K}_\eta(\mathbb{R})$. Then $g_\eta = g^*_\eta$.*

Proof of Theorem 3.1. The proof follows from Theorem 2.3, Lemmas 3.4 and 0.1. □

(2) A corrected version of Theorem 4.1 is:

THEOREM 4.1. *Let $n \in \mathbb{N}$, $n \geqslant 1$ and let G be an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$, where η has length $r + 2s$. Assume that G is generated by p matrices $A_1, \ldots, A_p$ ($p \geqslant 1$) and let $B_1, \ldots, B_p \in g_\eta$ such that $A_1^2 = e^{B_1}, \ldots, A_p^2 = e^{B_p}$. Then G is supercyclic if and only if*

$$\sum_{k=1}^{p} \mathbb{N}B_k u_\eta + \sum_{l=1}^{s} 2\pi\mathbb{Z}f_\eta^{(l)} + \mathbb{R}u_\eta$$

is dense in $\mathbb{R}^n$ and $\mathrm{ind}(\mathbb{R}G) = r$.

In Lemma 4.2, the reference "([3], Proposition 4.6)" should read "([3], Proposition 5.2)".

Proof of Theorem 4.1. The proof follows from Theorem 3.1, Lemmas 0.2 and 4.2. □

(3) Now we give a correct version of Corollary 4.3.

COROLLARY 4.3. *Let $n \in \mathbb{N}$, $n \geqslant 1$ and let G be an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$, where η has length $r + 2s$.*

(1) If G is a group and $\mathbb{R}_+$-supercyclic, then it is hypercyclic if and only if it has a somewhere dense orbit.

(2) If G is $\mathbb{R}_+$-supercyclic and generated by p matrices $A_1, \ldots, A_p$ ($p \geqslant 1$) such that $A_1^2 = e^{B_1}, \ldots, A_p^2 = e^{B_p}$, where $B_1, \ldots, B_p \in g_\eta$, then it is hypercyclic if and only if $\sum_{k=1}^{p} \mathbb{N}B_k u_\eta + \sum_{l=1}^{s} 2\pi\mathbb{Z}f_\eta^{(l)}$ is dense in $\mathbb{R}^n$.

Before the proof, we notice the following changes.
Lemma 3.8 should be changed as follows.

LEMMA 3.8. ([2], Proposition 4.3) *Let G be an abelian subgroup of $\mathscr{K}_\eta^*(\mathbb{R})$, where η has length $r+2s$. Then the following properties are equivalent:*

(i) $\overline{G(u_\eta)} = \mathbb{R}^n$,

(ii) $\overline{G(u_\eta)}$ *has non-empty interior and* $\mathrm{ind}(G) = r$.

Theorem 2.2, 1., (iii) should read "somewhere dense in $\mathbb{R}^n$" instead of "dense in $\mathbb{R}^n$" Similarly in Lemma 3.9, (iii).

Proof of Corollary 4.3. If G is $\mathbb{R}_+$-supercyclic, then by Theorem 3.1 and Lemma 3.6, $\mathrm{ind}(G) = r$. So Assertion (1) follows from Lemma 3.8 and Assertion (2) follows from Theorems 2.2 and 2.3.   □

(4) Theorem 5.1 and its proof need to be modified as follows.

THEOREM 5.1. *Let G be be an abelian sub-semigroup of $M_n(\mathbb{R})$ and $P \in GL(n, \mathbb{R})$ such that $P^{-1}GP \subset \mathscr{K}_\eta(\mathbb{R})$, where η has length $r+2s$, $n \in \mathbb{N}$, $n \geqslant 1$. Then the following are equivalent:*

(i) G *is $\mathbb{R}_+$-supercyclic,*

(ii) G *is supercyclic and* $\mathrm{ind}(G) = r$.

Proof. We may assume that G is an abelian sub-semigroup of $\mathscr{K}_\eta(\mathbb{R})$, for some partition η of n of length $r+2s$, by taking $P^{-1}GP$. $(i) \Rightarrow (ii)$ follows from Theorem 2.3 and Lemma 3.6. Let us prove $(ii) \Rightarrow (i)$. Suppose that G is supercyclic and $\mathrm{ind}(G) = r$. So by Theorem 3.1, $\mathrm{g}_\eta^2(u_\eta) + \mathbb{R}u_\eta$ is dense in $\mathbb{R}^n$ and by Lemma 3.6, $\mathrm{ind}(\mathbb{R}_+G) = r$. As $\mathrm{g}_\eta^2(u_\eta) \subset \mathrm{g}_\eta(u_\eta)$ and by Lemma 0.1, $\mathrm{g}_\eta^+ = \mathrm{g}_\eta + \mathbb{R}I_n$, it follows that $\mathrm{g}_\eta^+(u_\eta)$ is dense in $\mathbb{R}^n$. Hence by Theorem 2.3, $\mathbb{R}_+G(u_\eta)$ is dense in $\mathbb{R}^n$ and so G is $\mathbb{R}_+$-supercyclic. The proof is complete.   □

According to these corrections,

− The sentence in the abstract: "Furthemore, we show that supercyclicity and $\mathbb{R}_+$- supercyclicity are equivalent" should be "Furthemore, we investigate the relation between supercyclicity and $\mathbb{R}_+$- supercyclicity".

− In page 856, at the end of the introduction:

"Third, we prove that supercyclicity and positive (or $\mathbb{R}_+$)-supercyclicity are equivalent" should be "Third, we investigate the relation between supercyclicity and positive (or $\mathbb{R}_+$)-supercyclicity)"

"In Section 5, we prove the equivalence between supercyclicity and positive supercyclicity" should be "In Section 5, we investigate the relation between supercyclicity and positive supercyclicity."

− In page 864, at the first paragraph, "Actually we prove the same conclusion holds for any abelian semigroup of $M_n(\mathbb{R})$" should be "Actually for any abelian semigroup of $M_n(\mathbb{R})$, we have the following."

(5) Now we give a correct version of the proof of Proposition 4.6 in the case 2: $r = 0$. There is an elementary mistake in the computation of $(n-1) - s + 1$ in the

previous proof; it is in fact $(n-1)-(s-1)+1 = n-s+1$ matrices which is not the required value $n-s$. Thus the proof given in the paper must be changed. For this, we distinguish two cases:

— If $m_j \geqslant 2$, for some $1 \leqslant j \leqslant s$, say for example $m_1 \geqslant 2$, then $\eta_0 = (1, m_1 - 1, m_2, \ldots, m_s)$ is a partition of $n-1$ of length $1+2s$. By Lemma 4.7, there exist $(n-1)-s+1 = n-s$ matrices $A'_1, \ldots, A'_{n-s}$ in $\mathscr{K}^*_{\eta_0}(\mathbb{R})$ that generate a hypercyclic abelian semigroup G'. Set $A_j = \begin{bmatrix} 1 & O \\ O & A'_j \end{bmatrix}$, $j = 1, \ldots, n-s$ and let G be the semigroup generated by $A_1, \ldots, A_{n-s}$. It is clear that G is an abelian semigroup of $\mathscr{K}^*_{\eta'}(\mathbb{R})$, where $\eta' = (1, 1, m_1 - 1, m_2, \ldots, m_s)$ is a partition of n of length $2+2s$.

Let $x' \in \mathbb{R}^{n-1}$ so that $G'x'$ is dense in $\mathbb{R}^{n-1}$ and set $x = [1, x']^T$. By the same way as in Case 1, x is a supercyclic vector for G.

— Assume now that $m_j = 1$, for all $1 \leqslant j \leqslant s$. In this case, $\eta = (1, 1, \ldots, 1)$ is a partition of $n = 2s$ of length $2s$, and so $f^{(l)}_{\eta} = e_{2l}$, $l = 1, \ldots, s$ and $u_{\eta} = e_1 + e_3 + \ldots + e_{n-1}$. We shall construct a supercyclic abelian semigroup G of $\mathscr{K}^*_{\eta}(\mathbb{R})$ by applying Theorem 4.1.

CLAIM. There exist $n-s$ vectors $u_1, \ldots, u_{n-s}$ of $\mathbb{R}^n$ such that

$$\sum_{k=1}^{n-s} \mathbb{N}u_k + \sum_{l=1}^{s} 2\pi\mathbb{Z}e_{2l} + \mathbb{R}(e_1 + e_3 + \ldots + e_{n-1})$$

is dense in $\mathbb{R}^n$.

Proof. Let $\alpha_1, \ldots, \alpha_n$ be negative real numbers such that $1, \alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbb{Q}$. Define the matrix S by

$$Se_k = \begin{cases} 2\pi e_{2k}, & \text{if } k = 1, \ldots, s, \\ e_{i_k}, & \text{if } k = s+1, \ldots, n. \end{cases}$$

where $e_{i_{s+1}} = e_1, e_{i_{s+2}} = e_3, \ldots, e_{i_n} = e_{n-1}$. We see that $S \in GL(n; \mathbb{R})$ and $S(e_{s+1} + e_{s+2} + \ldots + e_n) = e_1 + e_3 + \ldots e_{n-1}$. Set $u = [\alpha_1, \ldots, \alpha_n]^T$ and define

$$u_k := \begin{cases} Se_{s+k}, & \text{if } k = 1, \ldots, n-s-1, \\ Su, & \text{if } k = n-s. \end{cases}$$

We let

$$H := \sum_{k=1}^{n-s-1} \mathbb{N}e_{s+k} + \mathbb{N}u + \sum_{l=1}^{s} \mathbb{Z}e_l + \mathbb{R}(e_{s+1} + e_{s+2} + \ldots + e_n).$$

We then have that

$$S(H) = \sum_{k=1}^{n-s-1} \mathbb{N}u_k + \mathbb{N}u_{n-s} + \sum_{l=1}^{s} 2\pi\mathbb{Z}e_{2l} + \mathbb{R}(e_1 + e_3 + \ldots + e_{n-1})$$

$$= \sum_{k=1}^{n-s} \mathbb{N}u_k + \sum_{l=1}^{s} 2\pi\mathbb{Z}e_{2l} + \mathbb{R}(e_1 + e_3 + \ldots + e_{n-1})$$

Observe that $\mathbb{N}^n + \mathbb{N}u \subset H$. By Kronecker's theorem (cf. [3]), $\mathbb{N}^n + \mathbb{N}u$ is dense in $\mathbb{R}^n$ and thus so is $S(H)$. This proves the claim. $\square$

Now set $u_k = [y_{1,1}^{(k)}, -y_{1,1}'^{(k)}, \ldots, y_{s,1}^{(k)}, -y_{s,1}'^{(k)}]^T$, $k = 1, \ldots, n-s$ and let $B_1, \ldots, B_{n-s} \in \mathscr{K}_\eta(\mathbb{R})$ be defined by $B_k = \mathrm{diag}(C_{1,1}^{(k)}, \ldots, C_{s,1}^{(k)})$, with

$$C_{l,1}^{(k)} = \begin{bmatrix} y_{l,1}^{(k)} & y_{l,1}'^{(k)} \\ -y_{l,1}'^{(k)} & y_{l,1}^{(k)} \end{bmatrix}, \ 1 \leqslant k \leqslant n-s, \ 1 \leqslant l \leqslant s.$$

Let G be the sub-semigroup of $\mathscr{K}_\eta^*(\mathbb{R})$ generated by $A_1, \ldots, A_{n-s}$, where

$$A_k = \mathrm{diag}\left(e^{\frac{1}{2}C_{1,1}^{(k)}}, \ldots, e^{\frac{1}{2}C_{s,1}^{(k)}}\right), \ k = 1, \ldots, n-s.$$

Then G is abelian and we have $A_j^2 = e^{B_j}$, $j = 1, \ldots, n-s$. Moreover $B_k u_\eta = u_k$. Therefore

$$S(H) = \sum_{k=1}^{n-s} \mathbb{N}u_k + \sum_{l=1}^{s} 2\pi\mathbb{Z}f_\eta^{(l)} + \mathbb{R}u_\eta = \sum_{k=1}^{n-s} \mathbb{N}B_k u_\eta + \sum_{l=1}^{s} 2\pi\mathbb{Z}f_\eta^{(l)} + \mathbb{R}u_\eta$$

is dense in $\mathbb{R}^n$. As $\mathrm{ind}(\mathbb{R}G) = 0$, then from Theorem 4.1, G is supercyclic.

There are three further minor typographical corrections:
— The reference [15] in Lemma 4.7 should be replaced by [3].
— In the statement of Proposition 4.6, add after "$1 + r + 2s$", "$2 + 2s$, $2s$".
— In the proof of Theorem 4.9, replace "length $2 + 2(s-1)$" by "length $2s$".

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Salah Herzi
University of Carthage
Bizerte Preparatory Engineering Institute, Tunisia
e-mail: herzisalah@yahoo.fr

Habib Marzougui
University of Carthage, Faculty of Science of Bizerte
(UR17ES21), "Dynamical systems and their applications"
7021, Jarzouna, Tunisia
e-mail: habib.marzougui@fsb.rnu.tn

Operators and Matrices
www.ele-math.com
oam@ele-math.com