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NEST ALGEBRAS IN AN ARBITRARY VECTOR SPACE

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Abstract. We examine the properties of algebras of linear transformations that leave invariant all subspaces in a totally ordered lattice of subspaces of an arbitrary vector space. We compare our results with those that apply for the corresponding algebras of bounded operators that act on a Hilbert space.

1. Introduction

The study of triangular forms for operators has long been an important part of the theory of non-self-adjoint operators and operator algebras. See [1] for a detailed account. In [5] Ringrose introduced the terms 'nest' and 'nest algebra'. For Ringrose a nest $\mathfrak{N}$ is a complete, totally ordered sublattice of the lattice of all closed subspaces of a Hilbert space $\mathfrak{H}$ that contains the trivial subspaces $\{0\}$ and $\mathfrak{H}$. The corresponding nest algebra $\operatorname{Alg}\mathfrak{N}$ is the algebra of all operators on $\mathfrak{H}$ that leave invariant each of the subspaces in $\mathfrak{N}$. For closed subspaces of a Hilbert space the lattice operations are: $\wedge = \cap$ and $\vee =$ *closed* linear span of the union.

In this paper we examine totally ordered lattices of linear subspaces of an arbitrary vector space and the associated operator algebras. Here a nest $\mathfrak{N}$ in a vector space $\mathfrak{X}$ is a totally ordered sublattice of the lattice of all subspaces of $\mathfrak{X}$ that contains the trivial subspaces $\{0\}$ and $\mathfrak{X}$, and is complete as a lattice, that is, $\mathfrak{N}$ contains the meet (intersection) and join (span of the union) of any family of subspaces in $\mathfrak{N}$. The corresponding nest algebra $\operatorname{Alg}\mathfrak{N}$ is the algebra of all linear transformations on $\mathfrak{X}$ that leave invariant each of the subspaces in $\mathfrak{N}$. We obtain results concerning the finite rank operators in $\operatorname{Alg}\mathfrak{N}$ that mirror those that apply in the Hilbert space case and some that do not. We also examine the Jacobson radical of $\operatorname{Alg}\mathfrak{N}$ and obtain a simple characterization when the nest satisfies a descending chain condition. We also show that the same characterization of the Jacobson radical holds for other types of nest algebras.

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1.1. Completely join irreducible elements

The lattice operations $\wedge$ and $\vee$ in $\mathscr{S}(\mathfrak{X})$, the lattice of all subspaces of the vector space $\mathfrak{X}$, are intersection and linear span of the union. In particular, if $\mathscr{M}$ and $\mathscr{N}$ are subspaces of $\mathfrak{X}$, $\mathscr{M} \vee \mathscr{N} = \operatorname{span}\{\mathscr{M}, \mathscr{N}\} = \{x+y : x \in \mathscr{M}, y \in \mathscr{N}\}$. However in a totally ordered sublattice the lattice operations are simply the set operations $\cap$ and $\cup$. So any nest $\mathfrak{N}$ is completely distributive (see [1]).

Suppose that $\mathfrak{N}$ is a nest in $\mathfrak{X}$. For each $0 \neq x \in \mathfrak{X}$ we define

$$\mathfrak{N}(x) = \bigcap\{\mathscr{M} \in \mathfrak{N} : x \in \mathscr{M}\} \text{ and } \mathfrak{N}(x)_- = \bigcup\{\mathscr{M} \in \mathfrak{N} : x \notin \mathscr{M}\}. \tag{1}$$

It follows easily from (1) that

$$x \in \mathscr{N} \iff \mathfrak{N}(x) \subseteq \mathscr{N} \text{ and } x \notin \mathscr{N} \iff \mathscr{N} \subseteq \mathfrak{N}(x)_- \tag{2}$$

An nonzero element a of a lattice $\mathscr{L}$ is *completely join irreducible* if and only if, whenever $a = \vee_{i \in I} a_i$ implies $a = a_i$ for some $i \in I$.

LEMMA 1. *The completely join-irreducible elements of $\mathfrak{N}$ are the subspaces of the form $\mathfrak{N}(x)$ where x is any non-zero vector in $\mathfrak{X}$.*

Proof. Suppose that $x \neq 0$, and that $\mathfrak{N}(x) = \bigcup\{N : N \in \mathfrak{N}^{\#}\}$ where $\mathfrak{N}^{\#} \subseteq \mathfrak{N}$. Then $x \in \bigcup\{N : N \in \mathfrak{N}^{\#}\}$ by (2). So $x \in N$ for some $N \in \mathfrak{N}^{\#}$, and it follows from (2) that $\mathfrak{N}(x) = N$. So $\mathfrak{N}(x)$ is completely join-irreducible.

Suppose now that N is a completely join-irreducible subspace in $\mathfrak{N}$. Clearly $N = \bigcup\{\mathfrak{N}(x) : x \in N\}$, and so $N = \mathfrak{N}(x)$ for some $x \in N$. $\square$

The completely join-irreducible subspaces 'separate' the subspaces in $\mathfrak{N}$, in the following sense.

LEMMA 2. *If $\mathscr{M}_1$ and $\mathscr{M}_2$ are subspaces in $\mathfrak{N}$ and $\mathscr{M}_1 \subset \mathscr{M}_2$, then*

$$\mathscr{M}_1 \subseteq \mathfrak{N}(x)_- \subset \mathfrak{N}(x) \subseteq \mathscr{M}_2,$$

for some $x \in \mathfrak{X}$.

Proof. Choose any $x \in \mathscr{M}_2 \setminus \mathscr{M}_1$, and now apply (2). $\square$

The existence of completely join irreducible elements distinguishes the vector space case from the Hilbert space case. Some of the most interesting nests of closed subspaces of a Hilbert space are 'continuous', have no completely join-irreducible elements.

EXAMPLE 3. Let $\mathscr{L}^2(\mathbb{R})$ denote the set of all (equivalence classes of) square-integrable complex-valued functions defined on the real line $\mathbb{R}$. For each $\alpha \in \mathbb{R}$, let $\mathscr{N}_\alpha = \{f \in \mathscr{L}^2(\mathbb{R}) : f(x) = 0 \text{ a.e. on } (\alpha, \infty)\}$, and let $\mathfrak{N} = \{0\} \cup \bigcup_{\alpha \in \mathbb{R}} \mathscr{N}_\alpha \cup \mathscr{L}^2(\mathbb{R})$. Then $\mathfrak{N}$ is a (complete) nest of closed subspaces of the Hilbert space $\mathscr{L}^2(\mathbb{R})$. It is 'continuous', in the sense that $\mathscr{N}_\alpha = \bigvee_{\beta<\alpha} \mathscr{N}_\beta$ for each $\alpha \in \mathbb{R}$, and hence there are no join-irreducible elements.

However $\mathfrak{N}$ is not complete as a lattice of (not necessarily closed) linear subspaces, because it is not closed under arbitrary unions. Let $\mathscr{M}_\alpha = \bigcup_{\beta<\alpha} \mathscr{N}_\beta$, for each $\alpha \in \mathbb{R}$, and let $\mathscr{M}_\infty = \bigcup_{\beta \in \mathbb{R}} \mathscr{N}_\beta$. Thus $f \in \mathscr{M}_\alpha \iff f(x) = 0$ a.e. on (β, ∞) for some $\beta < \alpha$, and $f \in \mathscr{M}_\infty \iff f(x) = 0$ a.e. on (β, ∞) for some $\beta \in \mathbb{R}$. Let $\mathfrak{N}^\# = \mathfrak{N} \cup \bigcup_{\alpha \in \mathbb{R}} \mathscr{M}_\alpha \cup \mathscr{M}_\infty$. Then $\mathfrak{N}^\#$ is a complete nest of linear subspaces of $\mathscr{L}^2(\mathbb{R})$. The subspaces in $\mathfrak{N}^\#$ are ordered as follows:

$$\{0\} \subset \mathscr{M}_\alpha \subset \mathscr{N}_\alpha \subset \mathscr{M}_\beta \subset \mathscr{N}_\beta \subset \mathscr{M}_\infty \subset \mathscr{L}^2(\mathbb{R}) \text{ for all } \alpha < \beta$$

The completely join-irreducible elements of $\mathfrak{N}^\#$ are the subspaces $\mathscr{N}_\alpha, \alpha \in \mathbb{R}$, and $\mathscr{L}^2(\mathbb{R})$ itself, and each subspace in $\mathfrak{N}^\#$ is a join of join-irreducibles.

2. Finite rank operators

The *rank* of an operator in $\mathscr{L}(\mathfrak{X})$ is the linear dimension of its range. In this section we examine the properties of operators in a nest algebra $\mathscr{A} = \operatorname{Alg}\mathfrak{N}$ whose ranks are finite. Let $\mathscr{R}$ denote the set of finite-rank operators in $\mathscr{L}(\mathfrak{X})$. Various authors have investigated the properties of $\mathscr{R} \cap \mathscr{A}$ in the Hilbert space context. For example, Erdos proved [2] that if $\mathscr{N}$ is a nest of closed subspaces of a Hilbert space then the closure of $\mathscr{R} \cap \mathscr{A}$ in the strong operator topology is $\mathscr{A}$.

Rank-one operators also have an important role. Let $\mathscr{R}_1$ denote the set of all rank-one operators in $\mathscr{L}(\mathfrak{X})$. For each $T \in \mathscr{R}_1$, there exist $x \in \mathfrak{X}$ and $\varphi \in \mathfrak{X}'$, where $\mathfrak{X}'$ denotes the algebraic dual of $\mathfrak{X}$, such that $Ty = \varphi(y)x$ for each $y \in \mathfrak{X}$. We write $T = x \otimes \varphi$.

A simple calculation shows that $(x_1 \otimes \varphi_1)(x_2 \otimes \varphi_2) = \varphi_1(x_2)(x_1 \otimes \varphi_2)$. So $x \otimes \varphi$ is idempotent if and only if $\varphi(x) = 1$.

The following lemma characterizes the rank-one operators in $\mathscr{A}$.

LEMMA 4. *Suppose that $x \in \mathscr{X}$ and $\varphi \in \mathscr{X}'$. Then $x \otimes \varphi \in \mathscr{R}_1 \cap \mathscr{A}$ if and only if $\mathfrak{N}_-(x) \subseteq \ker\varphi$.*

Proof. First suppose that $x \otimes \varphi \in \mathscr{R}_1 \cap \mathscr{A}$, and that $y \in \mathfrak{N}_-(x)$. Since $\mathfrak{N}_-(x) \in \mathfrak{N}$, $(x \otimes \varphi)(y) = \varphi(y)x \in \mathfrak{N}_-(x)$. Since $x \notin \mathfrak{N}_-(x)$ it follows that $\varphi(y) = 0$. So $\mathfrak{N}_-(x) \subseteq \ker\varphi$.

Now suppose that $\mathfrak{N}_-(x) \subseteq \ker\varphi$ and that $N \in \mathfrak{N}$. If $N \subset \mathfrak{N}(x)$ then $N \subseteq \mathfrak{N}_-(x)$ and $(x \otimes \varphi)N = \{0\} \subseteq N$. If $\mathfrak{N}(x) \subseteq N$ then $(x \otimes \varphi)N \subseteq \operatorname{span} x \subseteq \mathfrak{N}(x) \subseteq N$. So $x \otimes \varphi \in \mathscr{R}_1 \cap \mathscr{A}$. $\square$

2.1. Reflexivity of $\mathfrak{N}$

For any subset of $\mathfrak{A}$ of $\mathscr{L}(\mathfrak{X})$ let $\operatorname{Lat}\mathfrak{A}$ denote the sublattice of $\mathscr{S}(\mathfrak{X})$ consisting of all subspaces of $\mathfrak{X}$ that are invariant under each of the operators in $\mathfrak{A}$. We shall show that

$$\mathfrak{N} = \operatorname{Lat}(\mathscr{R}_1 \cap \mathscr{A}), \tag{3}$$

from which it follows that $\mathfrak{N}$ is reflexive, i.e., $\mathfrak{N} = \operatorname{Lat}\operatorname{Alg}\mathfrak{N}$. Longstaff shows [4] that (3) holds in the Hilbert space context.

The following lemma will be used to establish the reflexivity of $\mathfrak{N}$.

LEMMA 5. *If x and y are non-zero vectors in $\mathfrak{X}$ and $y \in \mathfrak{N}(x)$, then there exists $R \in \mathscr{R}_1 \cap \mathscr{A}$ such that $Rx = y$.*

Proof. Since $y \in \mathfrak{N}(x)$, $\mathfrak{N}(y)_- \subset \mathfrak{N}(y) \subseteq \mathfrak{N}(x)$. So $x \notin \mathfrak{N}(y)_-$, and hence there exists $\varphi \in X'$ such that $\varphi(x) = 1$ and $\mathfrak{N}(y)_- \subseteq \ker\varphi$. Then $R = \varphi \otimes y \in \mathscr{R}_1 \cap \mathscr{A}$ and $Rx = \varphi(x)y = y$. $\square$

THEOREM 6. *$\mathfrak{N}$ is reflexive.*

Proof. We shall show that $\mathfrak{N} = \operatorname{Lat}(\mathscr{R}_1 \cap \mathscr{A})$. Clearly, $\mathfrak{N} \subseteq \operatorname{Lat}(\mathscr{R}_1 \cap \mathscr{A})$. For the reverse inclusion, suppose that $N \in \operatorname{Lat}(\mathscr{R}_1 \cap \mathscr{A})$.

Suppose that x and y are non-zero vectors in N and $\mathfrak{N}(x)$ respectively. So, by Lemma 5, there exists $R \in \mathscr{R}_1 \cap \mathscr{A}$ such that $Rx = y$. Since $N \in \operatorname{Lat}(\mathscr{R}_1 \cap \mathscr{A})$, it follows that $y \in N$, and hence $\mathfrak{N}(x) \subseteq N$.

Thus

$$N \subseteq \bigcup\{\mathfrak{N}(x) : x \in N\} \subseteq N.$$

Hence $N = \bigcup\{\mathfrak{N}(x) : x \in N\} \in \mathfrak{N}$. $\square$

2.2. Finite rank idempotents

The following lemma concerning rank-one idempotents in $\mathscr{A}$ will be useful.

LEMMA 7. *Suppose that M is a finite-dimensional subspace of $\mathfrak{X}$. Then $M = \operatorname{ran}P$ for some idempotent $P \in \mathscr{A}$. Furthermore, P is the sum of n rank-one idempotents in $\mathscr{A}$, where $n = \dim M$.*

Proof. The proof is by induction on $\dim M$. First suppose that $\dim M = 1$, and choose a non-zero vector $x \in M$. Now choose $\varphi \in \mathfrak{X}'$ such that $\mathfrak{N}(x)_- \subseteq \ker\varphi$ and $\varphi(x) = 1$. Such a φ exists because $x \notin \mathfrak{N}(x)_-$.Then $x \otimes \varphi$ is the required idempotent.

Now suppose that $n = \dim M > 1$ and that the result is true for all subspaces of $\mathfrak{X}$ with dimension less than n. Choose a non-zero vector $y \in M$ and a subspace $M^{\#}$ of M such that $M^{\#}$ and $\operatorname{span} y$ are complementary subspaces of M, i.e., $M^{\#} + \operatorname{span} y = M$ and $M^{\#} \cap \operatorname{span} y = \{0\}$. By the induction hypothesis there exists an idempotent $P^{\#} \in \mathscr{A}$ such that $\operatorname{ran} P^{\#} = M^{\#}$, and rank-one idempotents $P_1, P_2, \cdots, P_{n-1}$ in $\mathscr{A}$ such that $P^{\#} = P_1 + P_2 + \cdots + P_{n-1}$. Let $x = y - P^{\#}y$. Then $0 \neq x \in M$ and $P^{\#}x = 0$. Assume, via contradiction, that $x = u + v$, with $u \in \mathfrak{N}(x)_-$ and $v \in M^{\#}$. Then $P^{\#}x = P^{\#}u + P^{\#}v$, i.e., $0 = P^{\#}u + v$, since $M^{\#} = \operatorname{ran} P^{\#}$ and $P^{\#}$ is idempotent. So $x = u - P^{\#}u$. Since $P^{\#} \in \mathscr{A}$, and $1 - P^{\#} \in \mathscr{A}$ and $u \in \mathfrak{N}(x)_-$ it follows that

$$x = u - P^{\#}u \in \mathfrak{N}(x)_-,$$

which is a contradiction. So $x \notin \mathfrak{N}(x)_- + M^\# = \mathfrak{N}(x)_- + \operatorname{ran} P^\#$, and hence there exists $\varphi \in \mathfrak{X}'$ such that

$$\varphi(x) = 1, \text{ and } \mathfrak{N}(x)_- + \operatorname{ran} P^\# \subseteq \ker \varphi$$

Let $P_n = x \otimes \varphi$. Then P_n is idempotent since $\varphi(x) = 1$, and $P_n \in \mathscr{A}$ since $\mathfrak{N}(x)_- \subseteq \ker \varphi$. Furthermore, $P^\# P_n = P^\# x \otimes \varphi = 0$, and $P_n P^\# = x \otimes \varphi P^\# = 0$ since $\operatorname{ran} P^\# \subseteq \ker \varphi$. Now let $P = P^\# + P_n$. Then

$$P^2 = (P^\#)^2 + P^\# P_n + P_n P^\# + P_n^2 = P^\# + P_n = P,$$

and $\operatorname{ran} P = \operatorname{ran} P^\# + \operatorname{ran} P_n = M^\# + \operatorname{span} x = M$, as required. $\square$

2.3. Rank decomposition

Lemma 7 provides an easy proof of a rank-decomposition property of finite rank operators in the nest algebra $\mathscr{A}$.

THEOREM 8. *Suppose that T is a finite rank operator in $\mathscr{A}$. Then T is the sum of n rank-one operators in $\mathscr{A}$, where $n = \operatorname{rank} T$.*

Proof. By Lemma 7, $\operatorname{ran} T = \operatorname{ran} P$ for some idempotent P in $\mathscr{A}$. Furthermore $P = P_1 + P_2 + \cdots + P_n$ where each P_k is a rank-one idempotent in $\mathscr{A}$. Let $T_k = P_k T$ for $1 \leqslant k \leqslant n$. Then $T_k \in \mathscr{A}$ and $\operatorname{rank} T_k \leqslant 1$ for each k. Furthermore,

$$T = PT = \sum_{k=1}^{n} P_k T = \sum_{k=1}^{n} T_k.$$

This is the required decomposition. $\square$

2.4. Density

Lemma 7 also provides an easy proof of a density property of the linear span of rank-one operators in $\mathscr{A}$. First we introduce a special topology on $\mathscr{L}(\mathfrak{X})$.

DEFINITION 9. The set of all subsets of $\mathscr{L}(\mathfrak{X})$ of the form

$$\mathscr{U}(T,x) = \{S \in \mathscr{L}(\mathfrak{X}) : Sx = Tx\},$$

where $x \in \mathfrak{X}$ and $T \in \mathscr{L}(\mathfrak{X})$, is a set of subbasic neighborhoods of T for the strict topology on $\mathscr{L}(\mathfrak{X})$

THEOREM 10. *The span of the rank-one operators in $\mathscr{A}$ is strictly dense in $\mathscr{A}$.*

Proof. Suppose that $T \in \mathscr{A}$ and that $\mathscr{F}$ is a finite subset of $\mathfrak{X}$. Let $\mathscr{R}_1^{\#} \cap \mathscr{A}$ denote the span of $\mathscr{R}_1 \cap \mathscr{A}$. We need to show that there exists $S \in \mathscr{R}_1^{\#} \cap \mathscr{A}$ such that $Sx = Tx$ for all $x \in \mathscr{F}$.

By Lemma 7 $\operatorname{span}\mathscr{F} = \operatorname{ran} P$ for some idempotent $P \in \mathscr{A}$. Furthermore, P is the sum of n rank-one idempotents in $\mathscr{A}$, where $n = \dim\operatorname{span}\mathscr{F}$. Let $T_k = TP_k$ for $1 \leqslant k \leqslant n$. Then $T_k \in \mathscr{A}$ and $\operatorname{rank} T_k \leqslant 1$ for each k. So $S = \sum_{k=1}^{n} T_k \in \mathscr{R}_1^{\#} \cap \mathscr{A}$. Furthermore, for each $x \in \operatorname{span}\mathscr{F}$,

$$Tx = TPx = \sum_{k=1}^{n} TP_k x = Sx,$$

as required. □

3. Dual nests

For any subset M of $\mathfrak{X}$, let $M^{\perp}$ denote the annihilator of M, i.e.,

$$M^{\perp} = \{\varphi \ : \ \varphi \in \mathfrak{X}' \text{ and } M \subseteq \ker\varphi\}$$

Suppose that $\mathfrak{N}$ is a nest of subspaces of $\mathfrak{X}$, and that $\mathfrak{N}^{\perp} = \{M^{\perp} : M \in \mathfrak{N}\}$. We call $\mathfrak{N}^{\perp}$ the dual of the nest $\mathfrak{N}$. Since the map $M \mapsto M^{\perp}$ is order reversing, i.e., $M_1 \subseteq M_2 \iff M_1^{\perp} \supseteq M_2^{\perp}$, $\mathfrak{N}^{\perp}$ is a linearly ordered family of subspaces of $\mathfrak{X}'$ that is anti-order isomorphic to $\mathfrak{N}$.

We are interested in the issue of completeness of $\mathfrak{N}^{\perp}$.

LEMMA 11. *For any family $\{M_\alpha : \alpha \in \Psi\}$ of subspaces in $\mathfrak{N}$,*

$$\bigcap_{\alpha\in\Psi} M_\alpha^{\perp} = \left(\bigcup_{\alpha\in\Psi} M_\alpha\right)^{\perp} \text{ and } \bigcup_{\alpha\in\Psi} M_\alpha^{\perp} \subseteq \left(\bigcap_{\alpha\in\Psi} M_\alpha\right)^{\perp}$$

Proof. Suppose that $\varphi \in \mathfrak{X}'$. It is easy to see that

$$\varphi \in \bigcap_{\alpha\in\Psi} M_\alpha^{\perp} \iff M_\alpha \subseteq \ker\varphi \text{ for all } \alpha \in \Psi \iff \varphi \in \left(\bigcup_{\alpha\in\Psi} M_\alpha\right)^{\perp}.$$

Similarly, if $\varphi \in \bigcup_{\alpha\in\Psi} M_\alpha^{\perp}$ then $M_{\alpha^{\#}} \subseteq \ker\varphi$ for some $\alpha^{\#} \in \Psi$. It follows that $\bigcap_{\alpha\in\Psi} M_\alpha \subseteq \ker\varphi$, i.e., $\varphi \in \left(\bigcap_{\alpha\in\Psi} M_\alpha\right)^{\perp}$. □

COROLLARY 12. *$\mathfrak{N}^{\perp}$ is complete if and only if $\bigcup_{\alpha\in\Psi} M_\alpha^{\perp} = \left(\bigcap_{\alpha\in\Psi} M_\alpha\right)^{\perp}$ for each family $\{M_\alpha : \alpha \in \Psi\}$ of subspaces in $\mathfrak{N}$.*

Proof. By Lemma 11 it is sufficient to show that if $\{M_\alpha : \alpha \in \Psi\}$ is a family of subspaces in $\mathfrak{N}$ and $\mathfrak{N}^{\perp}$ is complete, then $\left(\bigcap_{\alpha\in\Psi} M_\alpha\right)^{\perp} \subseteq \bigcup_{\alpha\in\Psi} M_\alpha^{\perp}$.

If $\mathfrak{N}^{\perp}$ is complete, $\bigcup_{\alpha\in\Psi} M_\alpha^{\perp} = M_{\#}^{\perp}$ for some $M_{\#} \in \mathfrak{N}$. Suppose that $\alpha_0 \in \Psi$. Then $M_{\alpha_0}^{\perp} \subseteq \bigcup_{\alpha\in\Psi} M_\alpha^{\perp} = M_{\#}^{\perp}$, and so $M_{\#} \subseteq M_{\alpha_0}$. Therefore $M_{\#} \subseteq \bigcap_{\alpha\in\Psi} M_\alpha$, and so $\left(\bigcap_{\alpha\in\Psi} M_\alpha\right)^{\perp} \subseteq M_{\#}^{\perp} = \bigcup_{\alpha\in\Psi} M_\alpha^{\perp}$, as required. □

EXAMPLE 13. Suppose that $\mathfrak{X} = c_{00}(\mathbb{N})$, the vector space of all finitely non-zero $\mathbb{F}$-valued sequences. Then $\mathfrak{X}'$ can be regarded as the vector space of all $\mathbb{F}$-valued sequences. If $f = (f(k))_{k=1}^{\infty} \in \mathfrak{X}$ and $\varphi = (\varphi(k))_{k=1}^{\infty} \in \mathfrak{X}'$, then $\varphi(f) = \sum_{k=1}^{\infty} \varphi(k)f(k)$. This sum converges because only finitely many of the numbers $f(k) : k \in \mathbb{N}$ are non-zero.

For each $n \in \mathbb{N}$, let $M_n = \{f \in \mathfrak{X} : \operatorname{supp} f \subseteq \{1,2,3,\cdots,n\}\}$, where $\operatorname{supp}(f(k))_{k=1}^{\infty} = \{k : f(k) \neq 0\}$, and let

$$\mathfrak{N} = \{\{0\}, M_1, M_2, M_3, \cdots, \mathfrak{X}\}.$$

Then $\mathfrak{N}$ is a complete, totally ordered family of subspaces of $\mathfrak{X}$, i.e., $\mathfrak{N}$ is a nest.

Note that $M_n^{\perp} = \{\varphi \in \mathfrak{X}' : \operatorname{supp} \varphi \subseteq \{n+1, n+2, n+3, \cdots\}$. It is easy to see that $\mathfrak{N}^{\perp} = \{\mathfrak{X}', M_1^{\perp}, M_2^{\perp}, M_3^{\perp}, \cdots, \{0\}\}$ is a complete, totally ordered family of subspaces of $\mathfrak{X}'$, i.e., $\mathfrak{N}^{\perp}$ is a nest.

EXAMPLE 14. Suppose that $\mathfrak{X} = c_{00}(\mathbb{N})$ as in Example 13, and let

$$\mathfrak{N}^{\#} = \{\mathfrak{X}, M_1^{\#}, M_2^{\#}, M_3^{\#}, \cdots, \{0\}\},$$

where $M_n^{\#} = \{f \in \mathfrak{X} : \operatorname{supp} f \subseteq \{n+1, n+2, n+3, \cdots\}$ for each $n \in \mathbb{N}$. Then $\mathfrak{N}^{\#}$ is a complete, totally ordered family of subspaces of $\mathfrak{X}$, i.e., $\mathfrak{N}^{\#}$ is a nest.

Note that $(M_n^{\#})^{\perp} = M_n = \{\varphi \in \mathfrak{X}' : \operatorname{supp} \varphi \subseteq \{1,2,3,\cdots,n\}\}$ as in Example 13. So $(M_1^{\#})^{\perp}, (M_2^{\#})^{\perp}, (M_3^{\#})^{\perp}, \cdots$ is a strictly increasing sequence in $(\mathfrak{N}^{\#})^{\perp}$, and $\bigcup_{n=1}^{\infty} (M_n^{\#})^{\perp} = \mathfrak{X} \notin \left(\mathfrak{N}^{\#}\right)^{\perp}$. So $(\mathfrak{N}^{\#})^{\perp}$ is not complete.

The nest $\mathfrak{N}^{\#}$ in Example 14 has a strictly decreasing, infinite sequence of subspaces, i.e., it is not well-ordered. The following lemma shows that this is the key to the incompleteness of $(\mathfrak{N}^{\#})^{\perp}$.

LEMMA 15. *Suppose that $\mathfrak{N}$ is a complete nest of subspaces of a vector space $\mathfrak{X}$. Then $\mathfrak{N}^{\perp}$ is complete if and only if $\mathfrak{N}$ is well-ordered.*

Proof. First suppose that $\mathfrak{N}$ is well-ordered, and that $\{M_\alpha : \alpha \in \Psi\}$ is a family of subspaces in $\mathfrak{N}$. In the light of Corollary 12 it is sufficient to show that $\left(\bigcap_{\alpha\in\Psi} M_\alpha\right)^{\perp} \subseteq \bigcup_{\alpha\in\Psi} M_\alpha^{\perp}$.

Since $\mathfrak{N}$ is well-ordered, $\cap_{\alpha\in\Psi} M_\alpha = M_{\alpha^{\#}}$ for some $\alpha^{\#} \in \Psi$. So

$$\left(\bigcap_{\alpha\in\Psi} M_\alpha\right)^{\perp} = M_{\alpha^{\#}}^{\perp} \subseteq \bigcup_{\alpha\in\Psi} M_\alpha^{\perp}, \text{ as required.}$$

Now suppose that $\mathfrak{N}$ is not well-ordered, and that $M_1, M_2, M_3, \cdots$ is a strictly decreasing infinite sequence of subspaces in $\mathfrak{N}$. For each $n \in \mathbb{N}$ choose x_n such that $x_n = M_n \setminus M_{n+1}$. Then $\{x_1, x_2, x_3, \cdots\}$ is a linearly independent set and $\operatorname{span}\{x_1, x_2, x_3, \cdots\} \cap M_\infty = \{0\}$, where $M_\infty = \cap_{n=1}^{\infty} M_n$. So there exists $\varphi \in \mathfrak{X}'$ such that

$$\varphi(x_n) = 1 \text{ for each } n \in \mathbb{N} \text{ and } M_\infty \subseteq \ker \varphi \tag{4}$$

It follows easily from (4) that $\varphi \in M_\infty^\perp \setminus \left(\bigcup_{n=1}^\infty M_n^\perp\right)$. *So*

$$\bigcup_{n=1}^{\infty} M_n^\perp \subset M_\infty^\perp \tag{5}$$

Suppose that $\bigcup_{n=1}^\infty M_n^\perp \in \mathfrak{N}^\perp$, *i.e.,* $\bigcup_{n=1}^\infty M_n^\perp = M^\perp$ *for some* $M \in \mathfrak{N}$. *Then* $M_n^\perp \subseteq M^\perp$ *and* $M \subseteq M_n$ *for each* $n \in \mathbb{N}$. *So* $M \subseteq M_\infty$, *and hence* $M_\infty^\perp \subseteq M^\perp$. *But this contradicts (5), and so there is no such subspace* M *in* $\mathfrak{N}$. *So* $\mathfrak{N}^\perp$ *is not complete.* □

4. The Jacobson radical

Suppose that $\mathscr{R}$ is a ring with identity 1. The Jacobson radical $\operatorname{Rad}\mathscr{R}$ is the intersection of all maximal left ideals of $\mathscr{R}$. It is also the intersection of all maximal right ideals of $\mathscr{R}$. See ([3]). A more useful characterization of $\operatorname{Rad}\mathscr{R}$ is the following:

PROPOSITION 16. *Suppose that* $T \in \mathscr{R}$. *The following are equivalent:*
1. $T \in \operatorname{Rad}\mathscr{R}$
2. $1 - AT$ *is invertible in* $\mathscr{R}$ *for each* $A \in \mathscr{A}$
3. $1 - TA$ *is invertible in* $\mathscr{R}$ *for each* $A \in \mathscr{A}$

DEFINITION 17. Suppose that $\mathfrak{N}$ is a nest on $\mathfrak{X}$ and that $\mathscr{A} = \operatorname{Alg}\mathfrak{N}$. The strictly triangular ideal $\mathscr{A}_-$ is defined by

$$\mathscr{A}_- = \{T : T \in \mathscr{A} \text{ and } Tx \in \mathfrak{N}(x)_- \text{ for all } x \in \mathfrak{X}\}$$

LEMMA 18. *Suppose that* $\mathfrak{N}$ *is a nest on* $\mathfrak{X}$ *and that* $\mathscr{A} = \operatorname{Alg}\mathfrak{N}$. *Then*

$$\operatorname{Rad}\mathscr{A} \subseteq \mathscr{A}_-.$$

Proof. Suppose that $T \in \mathscr{A} \setminus \mathscr{A}_-$. Then $Tx \notin \mathfrak{N}(x)_-$ for some $x \in \mathfrak{X}$. Choose $\varphi \in X'$ such that $\varphi(Tx) = 1$ and $\mathfrak{N}(x)_- \subseteq \ker\varphi$. It follows from (4) that $x \otimes \varphi \in \mathscr{A}$.

Now $(1 - (x \otimes \varphi)T)x = x - \varphi(Tx)x = 0$. So $1 - (x \otimes \varphi)T$ is not invertible and so $T \notin \operatorname{Rad}\mathscr{A}$ by Proposition 16. □

EXAMPLE 19. Suppose that $\mathfrak{X} = c_{00}(\mathbb{N})$, and that $\mathfrak{N}$ is the nest of subspaces of $\mathfrak{X}$, as defined in Example 13. Then $\operatorname{Alg}\mathfrak{N}$ and $\operatorname{Alg}\mathfrak{N}_-$ can be identified with, respectively, the set of all upper triangular, and the set of all strictly upper triangular, $\mathbb{F}-$valued matrices. It is a simple exercise in matrix algebra to show that if T is any strictly upper triangular matrix, and A is upper triangular, then there a strictly upper triangular matrix B such that $I + B = (I - AT)^{-1}$. So for this nest $\operatorname{Rad}\mathscr{A} = \mathscr{A}_-$.

EXAMPLE 20. Suppose that $\mathfrak{X} = c_{00}(\mathbb{N})$, and that $\mathfrak{N}^\#$ is the nest of subspaces of $\mathfrak{X}$, as defined in Example 14. Then $\operatorname{Alg}\mathfrak{N}$ can be identified with the set of all lower triangular, column-finite, matrices. That is, $A = (a_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}} \in \operatorname{Alg}\mathfrak{N}^\#$ if and only if $a_{ij} = 0$ if $i < j$ and for each $j \in \mathbb{N}$, $a_{ij} = 0$ if j is sufficiently large. Similarly, $\operatorname{Alg}\mathfrak{N}_-$ can be identified with the set of all strictly lower triangular, column-finite, matrices. Let

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

that is, $T(f(1), f(2), f(3), \cdots) = (0, f(1), f(2), f(3), \cdots)$ for all $f = (f(k))_{k=1}^{\infty}$. It is easy to check that $I - T$ has no column-finite lower triangular inverse. So for this nest $\operatorname{Rad}\mathscr{A} \neq \mathscr{A}_-$.

We now seek conditions which are either necessary or sufficient for the equality of the radical $\operatorname{Rad}\mathscr{A}$ and the strictly triangular ideal $\mathscr{A}_-$ The notion of local nilpotence will be useful.

DEFINITION 21. We say that $T \in \mathscr{L}(\mathfrak{X})$ is nilpotent at $x \in \mathfrak{X}$ if $T^n x = 0$ for sufficiently large n. We say that T is locally nilpotent if it is nilpotent at each $x \in \mathfrak{X}$.

LEMMA 22. *If each $T \in \mathscr{A}_-$ is locally nilpotent, then* $\operatorname{Rad}\mathscr{A} = \mathscr{A}_-$.

Proof. Suppose that $T \in \mathscr{A}_-$ and that $A \in \mathscr{A}$. Then $AT \in \mathscr{A}_-$ and hence is locally nilpotent by assumption.

Let $S = 1 + \sum_{n=1}^{\infty}(AT)^n$. The sum S is well-defined as an operator in $\mathscr{L}(\mathfrak{X})$, because the local nilpotence of AT ensures that for each $x \in \mathfrak{X}$ the series $\sum_{n=1}^{\infty}(AT)^n x$ has only finitely many non-zero terms. If $x \in M$ for some $M \in \mathfrak{N}$, it is clear that $Sx \in M$. So $S \in \mathscr{A}$. Furthermore, it is easy to see that $S(1 - AT) = (1 - AT)S = 1$. So S is the inverse of $1 - AT$ in $\mathscr{A}$, and hence $T \in \operatorname{Rad}\mathscr{A}$. $\square$

LEMMA 23. *If $\mathfrak{N}$ is well-ordered then each $T \in \mathscr{A}_-$ is locally nilpotent.*

Proof. Suppose that $T \in \mathscr{A}_-$ is not locally nilpotent. Then there exists $x \in \mathfrak{X}$ such that $T^n x \neq 0$ for all $n \in \mathbb{N}$. Since $T \in \mathscr{A}_-$, we see that $T^{n+1}x \in \mathfrak{N}(T^n x)_-$; hence for each $n \in \mathbb{N}$,

$$\mathfrak{N}(T^{n+1}x) \subseteq \mathfrak{N}(T^n x)_- \subset \mathfrak{N}(T^n x).$$

So $\mathfrak{N}(T^n x) : n = 1, 2, 3, \cdots$ is a strictly decreasing, infinite sequence of subspaces in $\mathfrak{N}$, and hence $\mathfrak{N}$ is not well-ordered. $\square$

COROLLARY 24. *If $\mathfrak{N}$ is well-ordered then* $\operatorname{Rad}\mathscr{A} = \mathscr{A}_-$.

The following result shows that for dual nests, well-ordering is not essential for the equality of the radical and the strictly triangular ideal.

THEOREM 25. *Suppose that $\mathfrak{N}$ is a nest of subspaces of a vector space $\mathfrak{X}$ whose order type is ω, the first infinite ordinal. Then $\mathfrak{N}^{\perp}$ is a nest of subspaces of $\mathfrak{X}'$, whose order type is anti-isomorphic to ω, and* $(\operatorname{Alg}\mathfrak{N}^{\perp})_- = \operatorname{Rad}(\operatorname{Alg}\mathfrak{N}^{\perp})$.

Proof. In view of Lemma 15 it is sufficient to show that $\mathscr{A}_- = \operatorname{Rad}\mathscr{A}$, where $\mathscr{A} = \operatorname{Alg}\mathfrak{N}^\perp$.

Let $\mathscr{M}_0 = \{0\}$, and for each $n > 0$ let $\mathscr{M}_n$ denote the immediate successor of $\mathscr{M}_{n-1}$ in $\mathfrak{N}$. Since the order type of $\mathfrak{N}$ is ω, $\bigcup_{n=1}^\infty \mathscr{M}_n = \mathfrak{X}$.

Suppose that $T \in \mathscr{A}_-$ and that $\varphi \in \mathscr{M}_n^\perp$. Then $T\varphi \in \mathfrak{N}^\perp(\varphi)_- \subset \mathfrak{N}^\perp(\varphi) \subseteq \mathscr{M}_n^\perp$. Since $\mathscr{M}_{n+1}^\perp$ is the immediate predecessor of $\mathscr{M}_n^\perp$ in $\mathfrak{N}^\perp$, it follows that $T\varphi \in \mathscr{M}_{n+1}$, and so $T(\mathscr{M}_n^\perp) \subseteq \mathscr{M}_{n+1}^\perp$.

Suppose that $A \in \mathscr{A}$. Then $AT \in \mathscr{A}_-$ and so $AT(\mathscr{M}_n^\perp) \subseteq \mathscr{M}_{n+1}^\perp$ for each $n \geqslant 0$ and so $(AT)^n(\mathfrak{X}') = (AT)^n(\mathscr{M}_0^\perp) \subseteq \mathscr{M}_n^\perp$ for each $n \geqslant 0$.

Let $S = \ 1 + \sum_{n=1}^\infty (AT)^n$. The sum S is well-defined as an operator in $\mathscr{L}(\mathfrak{X}')$ because, for each $x \in \mathfrak{X}$ and each $\varphi \in \mathfrak{X}'$, the series $\sum_{n=1}^\infty ((AT)^n)(\varphi)(x)$ has only finitely many non-zero terms. (To see this note that $x \in \mathscr{M}_{n^\#}$ for some $n^\# \geqslant 0$, and $((AT)^n\varphi)(x) = 0$ if $n \geqslant n^\#$.) Furthermore, $S(1-AT)\varphi(x) = (1-AT)S\varphi(x) = \varphi(x)$, and so $S = (1-AT)^{-1}$. Finally, it is easy to check that $S(\mathscr{M}_n^\perp) \subseteq \mathscr{M}_n^\perp$ for each $n \geqslant 0$ and so $S \in \mathscr{A}$. Thus $T \in \operatorname{Rad}\mathscr{A}$, and hence $\mathscr{A}_- \subseteq \operatorname{Rad}\mathscr{A}$. It follows from Lemma 18 that $\mathscr{A}_- = \operatorname{Rad}\mathscr{A}$. $\square$

4.1. Examples

The nest $\mathfrak{N}$ defined in Example 13 satisfies the conditions of Theorem 25, and so $(\operatorname{Alg}\mathfrak{N}^\perp)_- = \operatorname{Rad}(\operatorname{Alg}\mathfrak{N}^\perp)$.

Note that $\mathfrak{N}^\perp$ is not well-ordered. It does, however, satisfy the ascending chain condition, i.e., each subset of $\mathfrak{N}^\perp$ contains a maximal element.

DEFINITION 26. Suppose that $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are vector spaces over the same field $\mathbb{F}$, and that $\mathfrak{N}_k$ is a nest of subspaces of $\mathfrak{X}_k$ for $k \in \{1,2\}$. The ordinal sum $\mathfrak{N}_1 \dotplus \mathfrak{N}_2$ is a nest of subspaces of $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2$ defined by

$$\mathfrak{N}_1 \dotplus \mathfrak{N}_2 = \{\mathscr{N}_1 \oplus \{0\} : \mathscr{N}_1 \in \mathfrak{N}_1\} \cup \{\mathfrak{X}_1 \oplus \mathscr{N}_2 : \mathscr{N}_2 \in \mathfrak{N}_2\}$$

Let $\mathscr{A} = \operatorname{Alg}(\mathfrak{N}_1 \dotplus \mathfrak{N}_2)$ and let $\mathscr{A}_k = \operatorname{Alg}\mathfrak{N}_k$ for $k \in \{1,2\}$. Every T in $\mathscr{L}(\mathfrak{X})$ has an operator matrix,

$$T = \begin{pmatrix} A_1 & B \\ C & A_2 \end{pmatrix}$$

relative to the decomposition $\mathfrak{X} = \mathfrak{X}_1 \oplus \mathfrak{X}_2$. It is easy to check that

$$T \in \mathscr{A} \text{ if and only if } A_k \in \mathscr{A}_k \text{ for } k \in \{1,2\} \text{ and } C = 0, \text{ and} \tag{6}$$

$$T \in \mathscr{A}_- \text{ if and only if } A_k \in (\mathscr{A}_k)_- \text{ for } k \in \{1,2\} \text{ and } C = 0. \tag{7}$$

LEMMA 27. *With the above notation and* $C = 0$,

$$T \in \operatorname{Rad}\mathscr{A} \textit{ if and only if } A_k \in \operatorname{Rad}\mathscr{A}_k \textit{ for } k \in 1,2\}, \textit{ and} \tag{8}$$

$$\operatorname{Rad}\mathscr{A} = \mathscr{A}_- \textit{ if and only if } \operatorname{Rad}\ \mathscr{A}_k = (\mathscr{A}_k)_- \textit{ for } k \in \{1,2\} \tag{9}$$

Proof. A simple matrix computation shows that if $\begin{pmatrix} D & E \\ 0 & F \end{pmatrix} = \begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}^{-1}$ if and only if $D = A_1^{-1}$, $F = A_2^{-1}$ and $E = -A_1^{-1}BA_2^{-1}$. So $\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}^{-1} \in \mathscr{A}$ if and only if $A_1^{-1} \in \mathscr{A}_1$ and $A_2^{-1} \in \mathscr{A}_2$. Statement (8) is now obvious. Statement (9) follows from (7)and (8). $\square$

EXAMPLE 28. Let $\mathfrak{X} = \mathfrak{Y} \oplus \mathfrak{X}$, where $\mathfrak{Y}$ is the vector space of all $\mathbb{F}$-valued sequences and $\mathfrak{X} = c_{00}(\mathbb{N})$. So $\mathfrak{Y} = \mathfrak{X}'$. Let

$$\mathfrak{N}_1 = \{\mathfrak{Y}, M_1^{\perp}, M_2^{\perp}, M_3^{\perp}, \cdots, \{0\}\}$$

where $M_n^{\perp} = \{\varphi \in \mathfrak{Y} : \operatorname{supp}\varphi \subseteq \{n+1, n+2, n+3, \cdots\}$, as in Example 13, and let

$$\mathfrak{N}_2 = \{\{0\}, (M_1^{\#})^{\perp}, (M_2^{\#})^{\perp}, (M_3^{\#})^{\perp}, \cdots, \mathfrak{X}\}$$

where $(M_n^{\#})^{\perp} = \{\varphi \in \mathfrak{X} : \operatorname{supp}\varphi \subseteq \{1, 2, \cdots, n\}\}$, as in Example 14.

Note that $\mathfrak{N}_1 = \mathfrak{N}^{\perp}$, where $\mathfrak{N}$ is as defined in Example 13, and sine $\mathfrak{N}$ is well-ordered with order type ω, it follows from Theorem 25 that $\operatorname{Rad}\mathscr{A}_1 = (\mathscr{A}_1)_-$. Note also that $\mathfrak{N}_2$ satisfies the descending chain condition, so we know from Corollary 24 $\operatorname{Rad}\mathscr{A}_2 = (\mathscr{A}_2)_-$. So by Lemma 27 $\operatorname{Rad}(\mathfrak{N}_1 \dot{+} \mathfrak{N}_2) = (\mathfrak{N}_1 \dot{+} \mathfrak{N}_2)_-$.

But $\mathfrak{N}_1 \dot{+} \mathfrak{N}_2$ satisfies neither the ascending chain condition nor the descending chain condition. Its order type is $1 + \omega^* + \omega + 1$, i.e., the order type of $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, where $\mathbb{Z}$ denote the set of integers, and it contains both strictly decreasing and strictly increasing infinite sequences of subspaces.

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