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THE SPECTRUM OF q–CESÀRO MATRICES ON c AND ITS VARIOUS SPECTRAL DECOMPOSITION FOR $0 < q < 1$

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Abstract. One of q-analogs of the Cesáro matrix of order one is the triangular matrix with nonzero entries $c_{nk} = \frac{q^{n-k}}{1+q+\cdots+q^n}$, $0 \leqslant k \leqslant n$, where $q \in [0,1]$. In this article, we will determine the spectrum of this matrix on the space of convergent sequences c. We will also obtain the fine spectral decomposition in the sense of Goldberg and a non-discrete spectral decomposition of the obtained spectrum.

1. Introduction

This section is devoted firstly to make a brief introduction to symbols of q-mathematics and q-Cesàro matrices. The subject of q-mathematics has many applications in mathematics, and the beginnings of q-mathematics date back to time of Euler. The q-analogue of the integer n, is defined by

$$[n]_q = \frac{1-q^n}{1-q} \; (q \neq 1). \tag{1.1}$$

Then, the q-analog of the factorial, i.e q-factorial, is defined as

$$[n]_q! = \begin{cases} \frac{q-1}{q-1}\frac{q^2-1}{q-1}\cdots\frac{q^n-1}{q-1} & , n = 1,2, \ldots.. \\ 1 & , n = 0. \end{cases} \tag{1.2}$$

and then q-binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}. \tag{1.3}$$

Note that, as $q \to 1$, the q-binomial coefficients approach the usual binomial coefficients.

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An ordinary Hausdorff matrix H has the representation $H = \delta\mu\delta$, where μ is the diagonal matrix with diagonal entries

$$\delta_{nk} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}. \tag{1.4}$$

The matrix δ is its own inverse.

A q-Hausdorff matrix H_q can be written in the form

$$H_q = \delta_q \mu \left(\delta_q\right)^{-1} \tag{1.5}$$

where μ is again the diagonal matrix with entries μ_k, and

$$\left(\delta_q\right)_{nk} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \tag{1.6}$$

For H_q matrices, δ_q is not its own inverse. However, as is the case for ordinary Hausdorff matrices, the row sum of every q-Hasudorff matrix is μ_0.

Cesàro matrix, C_1 is defined by

$$c_{nk} = \begin{cases} \frac{1}{n+1} &, 0 \leqslant k \leqslant n \\ 0 &, k > n \end{cases}. \tag{1.7}$$

For $0 < q < 1$, $C_1(q)$, the q-Hausdorff analog of C_1 which is the lower triangular matrix, has nonzero entries

$$C_1(q) = \frac{q^{n-k}(1-q)}{1-q^{n+1}}, 0 \leqslant k \leqslant n. \tag{1.8}$$

For $q > 1$, entries of C_q which is the lower triangular matrix, have nonzero entries

$$C_1(q) = \frac{q^k(q-1)}{q^{n+1}-1}, 0 \leqslant k \leqslant n. \tag{1.9}$$

Also, if we take the limit for $q \to 1^-$ in (1.6), then the Cesàro matrix C_1 is obtained.

A remarkable production of research involving q-series and q-differences in [28] has seen in the last thirty years.

Akgün and Rhoades show that the $C_1(q)$ matrices, the q-Hausdorff analogs of the Cèsaro matrix of order 1, are all equivalent to convergence, and that the same is true of the q-Hölder matrices in [5].

The Fourier theory containing specific q-analogs of trigonometric functions was first discussed in [12] and [14]. A complete development of q-Fourier theory should include a suitable summability theory. An introduction was made to q-Fourier theory using the q-analog of the Cesàro summability in [13].

Recently, there are many studies on q-analogs of matrix methods such as Cesàro, Hölder, Euler and Hausdorff methods [5], [13], [42], [10]. There are also new studies such as q-density and q-statistical convergence [3], [16].

1.1. The spectrum

Let X and Y be the Banach spaces, and $L : X \to Y$ also be a bounded linear operator. By $R(L)$, we denote the range of L, i.e.,

$$R(L) = \{y \in Y : y = Lx,\ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $L \in B(X)$ then the adjoint L^* of L is a bounded linear operator on the dual X^* of X defined by $(L^* f)(x) = f(Lx)$ for all $f \in X^*$ and $x \in X$.

Let $L : D(L) \to X$ be a linear operator, defined on $D(L) \subset X$, where $D(L)$ denote the domain of L and X is a complex normed linear space. For $L \in B(X)$ we associate a complex number α with the operator $(\alpha I - L)$ denoted by L_α defined on the same domain $D(L)$, where I is the identity operator. The inverse $(\alpha I - L)^{-1}$, denoted by L_α^{-1} is known as the resolvent operator of L.

A regular value of L is a complex number α of L such that L_α^{-1} exists, is bounded and, is defined on a set which is dense in X.

The resolvent set of L is the set of all such regular values of L, denoted by $\rho(L,X)$. Its complement is given by $\mathbb{C} \setminus \rho(L;X)$ in the complex plane $\mathbb{C}$ is called the spectrum of L, denoted by $\sigma(L,X)$. Thus the spectrum $\sigma(L,X)$ consist of those values of $\alpha \in \mathbb{C}$, for which L_α is not invertible.

The spectrum $\sigma(L,X)$ is union of three disjoint sets as follows: The point (discrete) spectrum $\sigma_p(L,X)$ is the set such that L_α^{-1} does not exist. Further $\alpha \in \sigma_p(L,X)$ is called the eigenvalue of L. We say that $\alpha \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(L,X)$ of L if the resolvent operator L_α^{-1} is defined on a dense subspace of X and is unbounded. Furthermore, we say that $\alpha \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(L,X)$ of L if the resolvent operator L_α^{-1} exists, but its domain of definition (i.e. the range $R(\alpha I - L)$ of $\alpha I - L$) is not dense in X; in this case L_α^{-1} may be bounded or unbounded. Together with the point spectrum, these two subspectra form a disjoint subdivision

$$\sigma(L,X) = \sigma_p(L,X) \cup \sigma_c(L,X) \cup \sigma_r(L,X) \tag{1.10}$$

of the spectrum of L.

1.2. Subdivision of the spectrum

The spectrum $\sigma(L,X)$ is partitioned into three sets which are not necessarily disjoint as follows:

If there exists a sequence (x_n) in X such that $\|x_n\| = 1$ and $\|Lx_n\| \to 0$ as $n \to \infty$ then (x_n) is called Weyl sequence for L.

We call the set

$$\sigma_{ap}(L,X) := \{\alpha \in \mathbb{C} : \text{there exists a Weyl sequence for } \alpha I - L\} \tag{1.11}$$

as the approximate point spectrum of L. Moreover, the set

$$\sigma_\delta(L,X) := \{\alpha \in \sigma(L,X) : \alpha I - L \text{ is not surjective}\} \tag{1.12}$$

is called defect spectrum of L. Finally, the set

$$\sigma_{co}(L,X) = \{\alpha \in \mathbb{C} : \overline{R(\alpha I - L)} \neq X\} \tag{1.13}$$

is called compression spectrum in the literature.

The following Proposition is quietly useful for calculating the separation of the spectrum of linear operator in Banach spaces.

PROPOSITION 1.1. ([4], Proposition 1.3) *The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:*
(a) $\sigma(L^*,X^*) = \sigma(L,X)$, (b) $\sigma_c(L^*,X^*) \subseteq \sigma_{ap}(L,X)$,
(c) $\sigma_{ap}(L^*,X^*) = \sigma_\delta(L,X)$, (d) $\sigma_\delta(L^*,X^*) = \sigma_{ap}(L,X)$,
(e) $\sigma_p(L^*,X^*) = \sigma_{co}(L,X)$, (f) $\sigma_{co}(L^*,X^*) \supseteq \sigma_p(L,X)$,
(g) $\sigma(L,X) = \sigma_{ap}(L,X) \cup \sigma_p(L^*,X^*) = \sigma_p(L,X) \cup \sigma_{ap}(L^*,X^*)$.

1.3. Goldberg's classification of spectrum

If $T \in B(X)$, then there are three cases for $R(T)$:

$(\mathcal{I})$ $R(T) = X$, $(\mathcal{II})$ $\overline{R(T)} = X$, but $R(T) \neq X$, $(\mathcal{III})$ $\overline{R(T)} \neq X$ and three cases for T^{-1}:

(1) T^{-1} exists and continuous, (2) T^{-1} exists but discontinuous, (3) T^{-1} does not exist.

If these cases are combined in all possible ways, nine different states are created. These are labelled by: $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{II}_1$, $\mathcal{II}_2$, $\mathcal{II}_3$, $\mathcal{III}_1$, $\mathcal{III}_2$, $\mathcal{III}_3$ (see [29]).

$\sigma(L,X)$ can be divided into subdivisions $\mathcal{I}_2\sigma(L,X) = \emptyset$, $\mathcal{I}_3\sigma(L,X)$, $\mathcal{II}_2\sigma(L,X)$, $\mathcal{II}_3\sigma(L,X)$, $\mathcal{III}_1\sigma(L,X)$, $\mathcal{III}_2\sigma(L,X)$, $\mathcal{III}_3\sigma(L,X)$. For example, if $T = \alpha I - L$ is in a given state, $\mathcal{III}_2$ (say), then we write $\alpha \in \mathcal{III}_2\sigma(L,X)$.

By the definitions given above, we can write following table

		1	2	3
		L_α^{-1} exists and is bounded	L_α^{-1} exists and is unbounded	L_α^{-1} does not exists
$\mathcal{I}$	$R(\alpha I - L) = X$	$\alpha \in \rho(L,X)$	–	$\alpha \in \sigma_p(L,X)$ $\alpha \in \sigma_{ap}(L,X)$
$\mathcal{II}$	$\overline{R(\alpha I - L)} = X$	$\alpha \in \rho(L,X)$	$\alpha \in \sigma_c(L,X)$ $\alpha \in \sigma_{ap}(L,X)$ $\alpha \in \sigma_\delta(L,X)$	$\alpha \in \sigma_p(L,X)$ $\alpha \in \sigma_{ap}(L,X)$ $\alpha \in \sigma_\delta(L,X)$
$\mathcal{III}$	$\overline{R(\alpha I - L)} \neq X$	$\alpha \in \sigma_r(L,X)$ $\alpha \in \sigma_\delta(L,X)$ $\alpha \in \sigma_{co}(L,X)$	$\alpha \in \sigma_r(L,X)$ $\alpha \in \sigma_{ap}(L,X)$ $\alpha \in \sigma_\delta(L,X)$ $\alpha \in \sigma_{co}(L,X)$	$\alpha \in \sigma_p(L,X)$ $\alpha \in \sigma_{ap}(L,X)$ $\alpha \in \sigma_\delta(L,X)$ $\alpha \in \sigma_{co}(L,X)$

Table 1: *Subdivision of the spectrum*

The spectrum of matrix transformations on sequence spaces has also been found by various authors in [11], [15], [37].

The discrete spectral decomposition of the spectrum of bounded linear operators (in the sense of Goldberg classification or residual, point and continuous spectra) has been made by many authors to this day in [1], [24], [26], [27], [30], [31], [32], [34], [35], [38], [39], [40], [41], [43], [46], [49]. Durna and Yildirim gave the table above in [19] and [20]. After this date, many authors have used this table to perform spectral decomposition of the spectrum of bounded linear operators (in terms of approximate point spectrum, defect spectrum and compression spectrum) in [2], [6], [7], [8], [9], [17], [18], [21], [22], [23], [25], [33], [44], [45], [47] and [48].

2. Generalized

In this section, boundedness, spectra, fine spectra and various spectral separations of spectrum of $C_1(q) = (a_{nk}(q))$ on the sequence space c have been determined.

The following theorem gives us necessary and sufficient condition to have a bounded linear transformation of an infinite matrix $A = (a_{nk})$ on c.

THEOREM 2.1. (Kojima-Schur (Maddox [36, p. 166])) *$A = (a_{nk}) \in B(c)$ if and only if i) $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$ and ii) for each $p \in \mathbb{N}$ there exists $\lim_{n\to\infty} \sum_{k=p}^{\infty} a_{nk} = a_p$.*

Let us now show that the q-analog of the Cesàro operator has a bounded linear transformation on c:

LEMMA 2.2. *$C_1(q) \in B(c)$ and $\|C_1(q)\| = 1$.*

Proof. For each k

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=p}^{\infty} c_{nk} &= \lim_{n\to\infty} \sum_{k=p}^{n} \frac{q^{n-k}}{1+q+\cdots+q^n} \\
&= \lim_{n\to\infty} \frac{1}{1+q+\cdots+q^n} \sum_{k=p}^{n} q^{n-k} \\
&= \lim_{n\to\infty} \frac{\frac{1-q^{n-p+1}}{1-q}}{\frac{1-q^{n+1}}{1-q}} \\
&= \begin{cases} \left(\frac{1}{q}\right)^p & ,q > 1 \\ 1 & ,0 < q < 1 \end{cases}
\end{aligned}$$

and

$$\|C_1(q)\| = \sup_n \sum_{k=0}^{\infty} |c_{nk}| = \sup_n \left\{ \frac{1}{1+q+\cdots+q^n} \sum_{k=0}^{n} q^{n-k} \right\} = 1.$$

From the above statements and Theorem 2.1, the desired result is obtained. $\square$

It can be easily seen that there are two different matrices for states $0 < q < 1$ and $q > 1$.

Throughout this article we will cover the matrix obtained with case $0<q<1$, the matrix obtained with case $q>1$ can be further studied in another study.

We need the following Lemma:

LEMMA 2.3. [50, p. 267] *If $T : c \to c$ is a linear transformation and $T^* : \ell_1 \to \ell_1$, $T^*g = g \circ T$, $g \in c^* \cong \ell_1$, then T and T^* have matrix representations, also $T^* : \ell_1 \to \ell_1$ is given by*

$$
\begin{aligned}
T^* = A^* &= \begin{pmatrix} \chi(\lim A) & (\vartheta_n)_{n=0}^{\infty} \\ (a_k)_{k=0}^{\infty} & A^t \end{pmatrix} \\
&= \begin{pmatrix} \chi(\lim A) & \vartheta_0 & \vartheta_1 & \vartheta_2 & \cdots \\ a_0 & a_{00} & a_{10} & a_{20} & \cdots \\ a_1 & a_{01} & a_{11} & a_{21} & \cdots \\ a_2 & a_{02} & a_{12} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
a_k &= \lim_n a_{nk} \\
\chi(A) &= \lim Ae - \sum_{k=0}^{\infty} \lim Ae_k = \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk} \\
\vartheta_n &= \chi(P_n \circ T) = (P_n \circ T)e - \sum_k a_{nk}, \\
a_{nk} &= P_n(T(e_k)) = (T(e_k))_n .
\end{aligned}
$$

If $0<q<1$, then one can get the following result from the above lemma for q-Cesàro matrix.

Making use of the relations given above and Lemma 2.3, we easily arrive at the following Lemma:

LEMMA 2.4. *The adjoint of $C_1(q)$ on c is given by*

$$
C_1^*(q) = \begin{pmatrix} 1 & 0\ 0 \cdots \\ 0 & \\ 0 & C_1^t(q) \\ \vdots & \end{pmatrix}. \tag{2.1}
$$

Proof. Since $0<q<1$, we have

$$
c_k = \lim_n c_{nk} = \lim_{n\to\infty} \frac{q^{n-k}(1-q)}{1-q^{n+1}} = q^{-k}(1-q) \lim_{n\to\infty} \frac{q^n}{1-q^{n+1}} = 0
$$

and

$$
\sum_{k=0}^{n} c_{nk} = \frac{(1-q)}{1-q^{n+1}} \sum_{k=0}^{n} q^{n-k} = \frac{(1-q)}{1-q^{n+1}} \sum_{k=0}^{n} q^k = \frac{(1-q)}{1-q^{n+1}} \frac{1-q^{n+1}}{(1-q)} = 1.
$$

Hence

$$\chi\left(C_1(q)\right)=\lim_n\sum_k c_{nk}-\sum_k\lim_n c_{nk}=1.$$

Also, since

$$\left(P_n\circ C_1(q)\right)e=\left\{\sum_{k=0}^n c_{nk}x_k\right\}_{x=e}=\sum_{k=0}^n c_{nk}=1,$$

we have

$$\vartheta_n=\left(P_n\circ C_1(q)\right)e-\sum_{k=0}^n c_{nk}=1-1=0$$

Hence, by Lemma 2.3, we have

$$C_1^*(q)=\begin{pmatrix}1 & 0\,0\,\cdots \\ 0 & \\ 0 & C_1^t(q) \\ \vdots & \end{pmatrix}.$$

This proves Lemma. $\square$

In the following theorems, let's calculate $\sigma_p\left(C_1(q),c\right)$, $\sigma_p\left(C_1^*(q),\ell_1\right)$, $\sigma\left(C_1(q),c\right)$ and $\sigma_r\left(C_1(q),c\right)$, respectively.

THEOREM 2.5. $\sigma_p\left(C_1(q),c\right)=\{1\}$.

Proof. Let $\sigma_p\left(C_1(q),c\right)\neq\emptyset$. Then, in this case there is $0\neq x\in c$, such that $C_1(q)x=\alpha x$. Hence

$$\begin{aligned}
x_0 &= \alpha x_0\\
\frac{1}{1+q}\left(qx_0+x_1\right) &= \alpha x_1\\
\frac{1}{1+q+q^2}\left(q^2x_0+qx_1+x_2\right) &= \alpha x_2\\
&\vdots\\
\frac{1}{1+q+q^2+\cdots+q^n}\left(q^nx_0+q^{n-1}x_1+\cdots+qx_{n-1}+x_n\right) &= \alpha x_n\\
&\vdots
\end{aligned}$$

the above equalities are obtained. If $x_m,\ m\geqslant 1$ is the first component that is different from zero of sequence $(x_n)=x$, then

$$\alpha=\frac{1}{1+q+\cdots+q^m}$$

is obtained. Furthermore, the statement

$$x_{m+1}=\frac{1+q+\cdots+q^m}{q^m}x_m$$

is provided. Similarly we obtain

$$
\begin{aligned}
x_{m+2} &= \frac{(1+q+\cdots+q^m)\left(1+q+\cdots+q^{m+1}\right)}{q^{2m}(1+q)}x_m,\\
x_{m+3} &= \frac{(1+q+\cdots+q^m)\left(1+q+\cdots+q^{m+1}\right)\left(1+q+\cdots+q^{m+2}\right)}{q^{3m}(1+q)\left(1+q+q^2\right)}x_m,\\
x_{m+4} &= \frac{(1+q+\cdots+q^m)\left(1+q+\cdots+q^{m+1}\right)\left(1+q+\cdots+q^{m+2}\right)\left(1+q+\cdots+q^{m+3}\right)}{q^{4m}(1+q)\left(1+q+q^2\right)\left(1+q+q^2+q^3\right)}x_m.
\end{aligned}
$$

If we continue in this way then we have

$$
x_n = \frac{(1+q+\cdots+q^m)\left(1+q+\cdots+q^{m+1}\right)\cdots\left(1+q+\cdots+q^{n-1}\right)}{q^{(n-m)m}(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-m-1})} \tag{2.2}
$$

is obtained for $n>m$ with $0<q<1$, so $q^m>q^n$. Thereby for $n>m$, $1-q^m<1-q^n$ and so $\frac{1-q^n}{1-q^m}>1$. In that case, from (2.2)

$$
\begin{aligned}
x_n &= \frac{\frac{1-q^{m+1}}{1-q}\frac{1-q^{m+2}}{1-q}\cdots\frac{1-q^n}{1-q}}{q^{(n-m)m}\frac{1-q^2}{1-q}\frac{1-q^3}{1-q}\cdots\frac{1-q^{n-m}}{1-q}}\\
&= \frac{\left(1-q^{m+1}\right)\left(1-q^{m+2}\right)\cdots(1-q^n)}{(1-q^2)(1-q^3)\cdots(1-q^{n-m})}\frac{1}{q^{(n-m)m}}\\
&= \frac{1-q^{m+1}}{1-q}\frac{1-q^{m+2}}{1-q^2}\cdots\frac{1-q^n}{1-q^{n-m}}\frac{1-q}{q^{(n-m)m}}\\
&\geqslant \frac{1-q}{q^{(n-m)m}} \to \infty,\ n\to\infty.
\end{aligned}
$$

That is, $x=(x_n)\notin c$. Also, for $\alpha=1$, $C_1(q)x=\alpha x$ with $x=(1,1,1,\ldots)\in c$. This indicates that $\sigma_p(C_1(q),c)=\{1\}$. □

THEOREM 2.6. $\sigma_p(C_1^*(q),\ell_1)=\left\{\alpha\in\mathbb{C}:\left|\alpha-\frac{1}{1+q}\right|<\frac{q}{1+q}\right\}\cup\{1\}$.

Proof. Let $x\neq 0$ and $C_1^*(q)x=\alpha x$.
Then, the below equalities are obtained

$$
\begin{aligned}
x_0 &= \alpha x_0\\
x_1+\frac{q}{1+q}x_2+\frac{q^2}{1+q+q^2}x_3+\frac{q^3}{1+q+q^2+q^3}x_4+\cdots &= \alpha x_1\\
\frac{1}{1+q}x_2+\frac{q}{1+q+q^2}x_3+\frac{q^2}{1+q+q^2+q^3}x_4+\cdots &= \alpha x_2\\
&\ \vdots
\end{aligned}
$$

Thus

$$x_n = \frac{x_1}{q^n}\prod_{k=1}^{n}\left(1-\frac{1-q}{1-q^k}\frac{1}{\alpha}\right),\ n=2,3,\ldots.$$

Suppose $\alpha \in \left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\}$.

If $\alpha = 1$, then $x = (x_0, x_1, 0, \ldots) \neq 0$ and $C_1^*(q)x = x$ are obtained so $1 \in \sigma_p(C_1^*(q), \ell_1)$.

If $\alpha = \frac{1}{1+q}$, then $x = (x_0, x_1, -x_1, 0, 0, \ldots) \neq 0$ and $C_1^*(q)x = \frac{1}{1+q}x$ so $\frac{1}{1+q} \in \sigma_p(C_1^*(q), \ell_1)$.

If $\alpha = \frac{1}{1+q+q^2}$, then $x = (x_0, x_1, -(1+q)x_1, qx_1, 0, 0, \ldots) \neq 0$ and $C_1^*(q)x = \frac{1}{1+q+q^2}x$ hence $\frac{1}{1+q+q^2} \in \sigma_p(C_1^*(q), \ell_1)$.

In a similar way, $\left\{\frac{1}{1+q+q^2+q^3}, \frac{1}{1+q+q^2+q^3+q^4}, \ldots\right\} \subset \sigma_p(C_1^*(q), \ell_1)$ is obtained. Is there any eigenvalues other than $\left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\}$?

Now, let us accept $\alpha \notin \left\{1, \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\}$. By the ratio test

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\frac{1}{q}\left|1-\frac{1-q}{1-q^{n+1}}\frac{1}{\alpha}\right| = \frac{1}{q}\left|1-\frac{1-q}{\alpha}\right|$$

is obtained. Move from here

$$\frac{1}{q}\left|1-\frac{1-q}{\alpha}\right| < 1 \Leftrightarrow \left|1-\frac{1-q}{\alpha}\right| < q$$

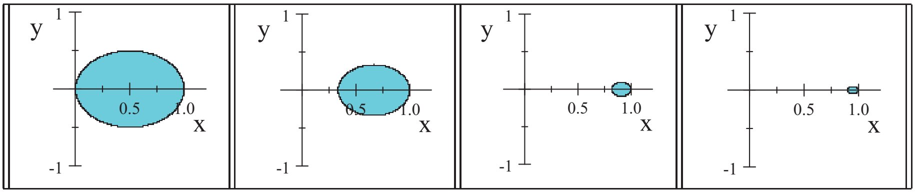
Take $\alpha = \lambda + i\mu$ in the above equation we get

$$\begin{aligned}
&\Leftrightarrow \left|1-\frac{1-q}{\lambda^2+\mu^2}\lambda + \frac{1-q}{\lambda^2+\mu^2}\mu i\right| < q \\
&\Leftrightarrow 1-\frac{2(1-q)}{\lambda^2+\mu^2}\lambda + \frac{(1-q)^2}{\lambda^2+\mu^2} < q^2 \\
&\Leftrightarrow 1-q^2 < (1-q)\left(\frac{2\lambda}{\lambda^2+\mu^2} - \frac{1-q}{\lambda^2+\mu^2}\right) \\
&\Leftrightarrow 1+q < \frac{2\lambda}{\lambda^2+\mu^2} - \frac{1-q}{\lambda^2+\mu^2} \\
&\Leftrightarrow \lambda^2+\mu^2 < \frac{2\lambda}{1+q} - \frac{1-q}{1+q} \\
&\Leftrightarrow \left(\lambda-\frac{1}{1+q}\right)^2 + \mu^2 < \frac{1}{(1+q)^2} - \frac{1-q}{1+q} = \frac{q^2}{(1+q)^2}
\end{aligned}$$

$$\Leftrightarrow \left|\alpha - \frac{1}{1+q}\right| < \frac{q}{1+q}$$

has the same meaning. So from ratio test, if $\left|\alpha - \frac{1}{1+q}\right| < \frac{q}{1+q}$ then $\sigma_p\left(C_1^*(q), \ell_1\right) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{1+q}\right| < \frac{q}{1+q}\right\} \cup \{1\}$ with $x_n \in \ell_1$ is obtained. $\square$

$$\sigma\left(C_1(q), c\right) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{1+q}\right| \leqslant \frac{q}{1+q}\right\}$$



$q = 0.99$	$q = 0.5$	$q = 0.1$	$q = 0.05$

Table 2: *Spectrum of q-Cesàro matrices*

THEOREM 2.7. $\sigma\left(C_1(q), c\right) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{1+q}\right| \leqslant \frac{q}{1+q}\right\}$.

Proof. We will show the proof of Theorem by showing that complex numbers α with $\left|\alpha - \frac{1}{1+q}\right| > \frac{q}{1+q}$ are present in $\rho\left(C_1(q), c\right)$. From Theorem 2.5, $\alpha I - C_1(q)$ is one to one. Let us now show that $\left(\alpha I - C_1(q)\right)^{-1} \in B(c)$ for complex numbers α with $\left|\alpha - \frac{1}{1+q}\right| > \frac{q}{1+q}$.

From the equality $\left(\alpha I - C_1(q)\right)x = y$,

$$x_n = \frac{1}{\alpha - \frac{1}{1+q+\ldots+q^n}} y_n + \frac{1}{1+q+\ldots+q^n} \sum_{k=0}^{n-1} \frac{y_k}{q} \frac{1}{\alpha^2} \prod_{i=k}^{n} \frac{\alpha q}{\alpha - \frac{1}{1+q+\ldots+q^n}}$$

is obtained. Now we have to find the value of

$$\begin{aligned}
&\lim_{n\to\infty} x_n \\
&= \lim_{n\to\infty} \left[\frac{1}{\alpha - \frac{1}{1+q+\ldots+q^n}} y_n + \frac{1}{1+q+\ldots+q^n} \sum_{k=0}^{n-1} \frac{y_k}{q} \frac{1}{\alpha^2} \prod_{i=k}^{n} \frac{\alpha q}{\alpha - \frac{1}{1+q+\ldots+q^i}}\right] \\
&= \lim_{n\to\infty} \frac{1}{\alpha - \frac{1}{1+q+\ldots+q^n}} y_n + \lim_{n\to\infty} \frac{1}{1+q+\ldots+q^n} \sum_{k=0}^{n-1} \frac{y_k}{q} \frac{1}{\alpha^2} \prod_{i=k}^{n} \frac{\alpha q}{\alpha - \frac{1-q}{1-q^{i+1}}}
\end{aligned}$$

$$= \frac{\ell}{\alpha+q-1} + \lim_{n\to\infty} \frac{1-q}{1-q^{n+1}} \sum_{k=0}^{n-1} \frac{y_k}{q} \frac{1}{\alpha^2} \prod_{i=k}^{n} \frac{\alpha q}{\alpha - \frac{1-q}{1-q^{i+1}}}$$

$$= \frac{\ell}{\alpha+q-1} + \lim_{n\to\infty} \alpha^{n-1} \frac{1-q}{1-q^{n+1}} \frac{\sum_{k=0}^{n-1} \frac{y_k \prod_{i=k}^{n} \frac{\left(1-q^{i+1}\right)}{\alpha\left(1-q^{i+1}\right)-(1-q)}}{(\alpha q)^k}}{\left(\frac{1}{q}\right)^n}$$

where $\ell = \lim_{n\to\infty} y_n$ for $y_n \in c$.

Let $a_n = \sum_{k=0}^{n} \frac{y_k \prod_{i=k}^{n} \frac{1-q}{1-q-\alpha\left(1-q^{i+1}\right)}}{q^k}$ and $b_n = \left(\frac{1}{q}\right)^n$. That is; let

$$x_n = \frac{a_n}{b_n}. \tag{2.3}$$

So, since

$$\begin{aligned}
a_{n+1} - a_n &= \sum_{k=0}^{n} \frac{y_k \prod_{i=k}^{n+1} \frac{\left(1-q^{i+1}\right)}{\alpha\left(1-q^{i+1}\right)-(1-q)}}{(\alpha q)^k} - \sum_{k=0}^{n-1} \frac{y_k \prod_{i=k}^{n} \frac{\left(1-q^{i+1}\right)}{\alpha\left(1-q^{i+1}\right)-(1-q)}}{(\alpha q)^k} \\
&= \frac{y_n \frac{\left(1-q^{n+1}\right)}{\alpha\left(1-q^{n+1}\right)-(1-q)} \frac{\left(1-q^{n+2}\right)}{\alpha\left(1-q^{n+2}\right)-(1-q)}}{(\alpha q)^n} \\
&\quad + \sum_{k=0}^{n-1} \frac{\frac{\left(1-q^{n+2}\right)}{\alpha\left(1-q^{n+2}\right)-(1-q)} y_k \prod_{i=k}^{n} \frac{\left(1-q^{i+1}\right)}{\alpha\left(1-q^{i+1}\right)-(1-q)}}{q^k} \\
&\quad - \sum_{k=0}^{n-1} \frac{y_k \prod_{i=k}^{n} \frac{\left(1-q^{i+1}\right)}{\alpha\left(1-q^{i+1}\right)-(1-q)}}{(\alpha q)^k} \\
&= \frac{y_n \frac{\left(1-q^{n+1}\right)}{\alpha\left(1-q^{n+1}\right)-(1-q)} \frac{\left(1-q^{n+2}\right)}{\alpha\left(1-q^{n+2}\right)-(1-q)}}{(\alpha q)^n} \\
&\quad + \sum_{k=0}^{n-1} \frac{\left[\frac{\left(1-q^{n+2}\right)}{\alpha\left(1-q^{n+2}\right)-(1-q)} - 1\right] y_k \prod_{i=k}^{n} \frac{\left(1-q^{i+1}\right)}{\alpha\left(1-q^{i+1}\right)-(1-q)}}{q^k} \\
&= \frac{y_n \frac{\left(1-q^{n+1}\right)}{\alpha\left(1-q^{n+1}\right)-(1-q)} \frac{\left(1-q^{n+2}\right)}{\alpha\left(1-q^{n+2}\right)-(1-q)}}{(\alpha q)^n} \\
&\quad + \frac{(1-\alpha)\left(1-q^{n+2}\right)+(1-q)}{\alpha\left(1-q^{n+2}\right)-(1-q)} a_n
\end{aligned}$$

we get

$$\begin{aligned}
&\frac{a_{n+1}-a_n}{b_{n+1}-b_n} \\
&= \frac{\dfrac{y_n \dfrac{(1-q^{n+1})}{\alpha(1-q^{n+1})-(1-q)} \dfrac{(1-q^{n+2})}{\alpha(1-q^{n+2})-(1-q)}}{(\alpha q)^n} + \dfrac{(1-\alpha)(1-q^{n+2})+(1-q)}{\alpha(1-q^{n+2})-(1-q)} a_n}{\frac{1}{q^{n+1}}-\frac{1}{q^n}} \\
&= \frac{y_{n+1}}{1-q-\alpha(1-q^{n+2})} + \frac{q}{1-q}\left[\frac{\alpha(1-q^{n+2})}{1-q-\alpha(1-q^{n+2})}\right]\frac{a_n}{b_n}. \qquad (2.4)
\end{aligned}$$

From (b_n) increasing and Stolz theorem, $\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{a_n}{b_n}$ is valid. Let's approve $\lim_{n\to\infty}\frac{a_n}{b_n} = L$. Also, $\lim_{n\to\infty} y_n = \ell$ with $y \in c$. In (2.4), if we switch to the limit for $n \longrightarrow \infty$

$$L = \frac{\ell}{1-q-\alpha} + \frac{q}{1-q}\left[\frac{\alpha}{1-q-\alpha}\right]L$$

is obtained. Because of the fact that $\alpha \neq 1-q$ and $\alpha \neq (1-q)^2$ for complex numbers α with $\left|\alpha - \frac{1}{1+q}\right| > \frac{q}{1+q}$

$$L = \frac{(1-q)\ell}{(1-q)^2-\alpha}$$

is given. In (2.4), if we switch to the limit for $n \to \infty$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{a_n}{b_n} = \frac{(1-q)\ell}{(1-q)^2-\alpha}$$

is valid. So, for complex numbers α with $\left|\alpha - \frac{1}{1+q}\right| > \frac{q}{1+q}$, $x = (x_n) \in c$ is obtained. Now let's find $\|x_n\|_c$. Since $y \in c$, $\|x_n\|_c$ is bounded. Hence, for each $k \in \mathbb{N}$, $|y_k| \leqslant M$ with $M > 0$. Also,

$$\begin{aligned}
\left|\alpha - \frac{1}{1+q}\right| > \frac{q}{1+q} &\Leftrightarrow \frac{1}{|\alpha(1+q)-1|} < \frac{1}{q} \\
&\Leftrightarrow \left|\frac{(1-q)q}{\alpha(1-q^{i+1})-(1-q)}\right| < 1
\end{aligned}$$

the above is the equivalent $\left|\frac{(1-q)q}{\alpha(1-q^{i+1})-(1-q)}\right| = p_i$ with $0 < p_i < 1$. If, we choose

$p := \max_{k=0}^{n} p_k$, then

$$
\begin{aligned}
\|x_n\|_{c_0} = \sup_n |x_n| = \sup_n & \left| \sum_{k=0}^{n} \frac{y_k}{q} \prod_{i=k}^{n} \frac{(1-q)\,q}{\alpha\,(1-q^{i+1})-(1-q)} \right| \\
& \leqslant \frac{M}{q} \sup_n \sum_{k=0}^{n} \prod_{i=k}^{n} \left| \frac{(1-q)\,q}{\alpha\,(1-q^{i+1})-(1-q)} \right| \\
& \leqslant \frac{M}{q} \sup_n \sum_{k=0}^{n} \prod_{i=k}^{n} \left| \frac{(1-q)\,q}{\alpha\,(1-q^{i+1})-(1-q)} \right| \\
& \leqslant \frac{M}{q} \sup_n \sum_{k=0}^{n} \prod_{i=k}^{n} p = \frac{M}{q} \sup_n \sum_{k=0}^{n} p^{n-k+1} \\
& \leqslant \tfrac{Mp}{(1-p)q} \sup_n \left(1-p^{n+1}\right) = \frac{Mp}{q}
\end{aligned}
$$

is obtained so $(\alpha I - C_1(q))^{-1} \in B(c)$. Consequently from equality $\sigma(C_1(q),c) = \mathbb{C} \setminus \rho(C_1(q),c)$, the proof is completed. $\square$

REMARK 2.8. As shown in Table 1, $\sigma(C_1(q_1),c) \subset \sigma(C_1(q_2),c)$ is valid for $0 < q_1 < q_2 < 1$. Also

$$
\lim_{q\to 1^-} \sigma(C_1(q),c) = \left\{ \alpha \in \mathbb{C} : \left| \alpha - \frac{1}{2} \right| \leqslant \frac{1}{2} \right\}
$$

is valid and this is equal to the set of $\sigma(C_1,c)$ from [37]. Moreover, let's note that $\lim_{q\to 0^+} \sigma(C_1(q),c) = \{1\}$. So the spectrum collapses for $q \to 0^+$.

THEOREM 2.9. $\sigma_r(C_1(q),c) = \left\{ \alpha \in \mathbb{C} : \left| \alpha - \frac{1}{1+q} \right| < \frac{q}{1+q} \right\} \cup \{1\}.$

Proof. Since $\sigma_r(C_1(q),c) = \sigma_p(C_1^*(q),\ell_1) \setminus \sigma_p(C_1(q),c)$, Theorems 2.5 and 2.6 give us required result. $\square$

Next, we appoint the two parts $\mathcal{III}_3\sigma(C_1(q),c)$ and $\mathcal{III}_2\sigma(C_1(q),c)$; it gives a finer subdivision of the spectrum. For this, we need the following lemmas.

LEMMA 2.10. (Golberg [29, p. 60]) *T has a bounded inverse if and only if T^* is onto.*

LEMMA 2.11. ([29], Theorem II 3.7) *A linear operator T has a dense range if and only if the adjoint operator T^* is one to one.*

THEOREM 2.12. $\mathcal{III}_2\sigma(C_1(q),c) = \left\{ \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots \right\}.$

Proof. From Theorem 2.9, $\left\{\frac{1}{1+q},\frac{1}{1+q+q^2},\dots\right\}\subseteq\sigma_r(C_1(q),c)$ is valid and it is clearly known that $\left\{\frac{1}{1+q},\frac{1}{1+q+q^2},\dots\right\}\subseteq\mathcal{III}$, where $\mathcal{III}$ is in Table1.

Now we have to show whether this cluster belongs to (1) or (2). For this we will use the Lemma 2.10.

For every $y\in\ell_1$, is there $x\in\ell_1$ such that $(C_1(q)-\alpha I)^*x=y$?

Let $(C_1(q)-\alpha I)^*x=y$. In this case

$$\begin{aligned}(1-\alpha)x_0&=y_0\\ (1-\alpha)x_1+\frac{q}{1+q}x_2+\frac{q^2}{1+q+q^2}x_3+\frac{q^3}{1+q+q^2+q^3}x_4+\cdots&=y_1\\ \left(\frac{1}{1+q}-\alpha\right)x_2+\frac{q}{1+q+q^2}x_3+\frac{q^2}{1+q+q^2+q^3}x_4+\cdots&=y_2\\ &\vdots\end{aligned}$$

is obtained. If we take x_n from this equation system,

$$\begin{aligned}x_0&=\tfrac{1}{1-\alpha}y_0\\ x_2&=\frac{1}{q\alpha}\left[(\alpha-1)x_1+y_1-qy_2\right]\\ x_3&=\frac{1}{q\alpha}\left[\frac{1}{\alpha q}(\alpha-1)\left(\alpha-\frac{1}{1+q}\right)x_1+\frac{1}{\alpha q}\left(\alpha-\frac{1}{1+q}\right)y_1+\frac{1}{\alpha(1+q)}y_1-qy_2\right]\\ x_4&=\frac{1}{q\alpha}\left[\frac{1}{(\alpha q)^2}(\alpha-1)\left(\alpha-\frac{1}{1+q}\right)\left(\alpha-\frac{1}{1+q+q^2}\right)x_1\right.\\ &\quad+\frac{1}{(\alpha q)^2}\left(\alpha-\frac{1}{1+q}\right)\left(\alpha-\frac{1}{1+q+q^2}\right)y_1\\ &\quad+\frac{1}{\alpha q}\left(\alpha-\frac{1}{1+q+q^2}\right)\frac{1}{\alpha(1+q)}y_2\\ &\quad\left.+\left(1-\frac{1}{\alpha}\left(\alpha-\frac{1}{1+q+q^2}\right)\right)y_2-qy_3\right]\\ &\vdots\end{aligned}$$

are obtained. So, for $x_0 = \frac{1}{1-\alpha} y_0$ and $n = 1, 2, \ldots$

$$x_{n+1} = \frac{x_1}{(q\alpha)^n} \prod_{v=0}^{n} \left(\alpha - \frac{1}{\sum\limits_{k=0}^{v} q^k} \right) + \frac{y_1}{(q\alpha)^n} \prod_{v=1}^{n} \left(\alpha - \frac{1}{\sum\limits_{k=0}^{v} q^k} \right) + \sum_{i=1+1}^{n} \frac{1}{\alpha \sum\limits_{k=0}^{i} q^k} \frac{y_i}{(q\alpha)^{n-1}} \prod_{v=i+1}^{n} \left(\alpha - \frac{1}{\sum\limits_{k=0}^{v} q^k} \right) - \frac{1}{\alpha} y_{n+1} \tag{2.5}$$

is obtained. If $\alpha \in \left\{ \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots \right\}$, then $x_{n+1} = -\frac{1}{\alpha} y_{n+1}$ with $x \in \ell_1$. This means that the operator $(C_1(q) - \alpha I)^*$ is onto. So, from Lemma 2.10, when

$$\alpha \in \left\{ \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots \right\},$$

$C_1(q) - \alpha I$ has bounded inverse and so

$$\left\{ \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots \right\} \subseteq (2)\, \sigma(C_1(q), c)$$

is obtained, where (2) is n, Table 1.

Let us accept

$$\alpha \notin \left\{ \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots \right\}.$$

In this case, considering that $y \in \ell_1$ from (2.5), sequence of (x_n) is convergent if and only if infinite product

$$\prod_{v=0}^{\infty} \left(\alpha - \frac{1}{\sum\limits_{k=0}^{v} q^k} \right)$$

is convergent. The limit of the general term of this product is as follows

$$\lim_{v \to \infty} \left(\alpha - \frac{1}{\sum\limits_{k=0}^{v} q^k} \right) = \alpha - \frac{1}{\frac{1}{1-q}} = \alpha - 1 + q$$

with $\alpha - 1 + q \neq 1$, that is, if $\alpha \neq 2 - q$ then the infinite product becomes divergent. Hence, if $\alpha \neq 2 - q$, then $x \notin c$ and so also $x \notin \ell_1$. This is, when

$$\alpha \notin \left\{ \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots \right\} \cup \{2 - q\},$$

it means that $(C_1(q) - \alpha I)^*$ is not onto operator. So from Lemma 2.10, when

$$\alpha \notin \left\{ \frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots \right\} \cup \{2 - q\},$$

$C_1(q) - \alpha I$ has not bounded inverse. Thus,

$$\left\{\frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\} \subseteq (2)\sigma(C_1(q), c) \subseteq \left\{\frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\} \cup \{2-q\}$$

is obtained. Finally, let's accept that $\alpha = 2-q$. From (2.5), the first component of sequence (x_n) which is

$$\frac{x_1}{(q(2-q))^n} \prod_{v=0}^{n} \left(\alpha - \frac{1}{\sum_{k=0}^{v} q^k}\right),$$

infinite product, even if bounded, must be $\dfrac{1}{(q(2-q))^n} \to \infty$. However, we know that $0 < q < 1$, so $0 < q(2-q) < 1$ is valid. Hence, $(q(2-q))^n \to 0$ that happens if $\alpha = 2-q$, $x \notin c$ and so also $x \notin \ell_1$. Hence, when $\alpha = 2-q$, it means that the operator of $(C_1(q) - \alpha I)^*$ is not onto. Thus, from Lemma 2.10, when $\alpha = 2-q$, $C_1(q) - \alpha I$ has not bounded inverse. Thus

$$\mathcal{III}_2\sigma(C_1(q), c) = \left\{\frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\}$$

is obtained. $\square$

COROLLARY 2.13.

$$\mathcal{III}_1\sigma(C_1(q), c) = \left\{\alpha \in \mathbb{C} : \left|\alpha - \frac{1}{1+q}\right| < \frac{q}{1+q}\right\} \setminus \left\{\frac{1}{1+q}, \frac{1}{1+q+q^2}, \ldots\right\}.$$

Proof. Since $\sigma_r(C_1(q), c) = \mathcal{III}_1\sigma(C_1(q), c) \cup \mathcal{III}_2\sigma(C_1(q), c)$, Theorems 2.9 and 2.12 give us required result. $\square$

THEOREM 2.14. $\mathcal{III}_3\sigma(C_1(q), c) = \{1\}$.

Proof. Since $\{1\} = \sigma_p(C_1(q), c) = \mathcal{I}_3\sigma(C_1(q), c) \cup \mathcal{II}_3\sigma(C_1(q), c) \cup \mathcal{III}_3\sigma(C_1(q), c)$ from Table 1 and Theorem 2.5, $\mathcal{III}_3\sigma(C_1(q), c) \subseteq \{1\}$. Also, from (2.5), we know that

$$\ker(C_1(q) - I)^* = \{(x_0, x_1, 0, 0, \ldots) : x_0, x_1 \in \mathbb{R}\} \neq \{(0, 0, \ldots)\},$$

so this means that the operator of $(C_1(q) - I)^*$ is not one to one. From Lemma 2.11, the operator of $C_1(q) - \alpha I$ does not have intense image. Consequently, $\mathcal{III}_3\sigma(C_1(q), c) = \{1\}$ is obtained. $\square$

COROLLARY 2.15. $\mathcal{I}_3\sigma(C_1(q), c) = \mathcal{II}_3\sigma(C_1(q), c) = \emptyset$.

Proof. Since

$$\sigma_p\left(C_1\left(q\right),c\right)=\mathcal{I}_3\sigma\left(C_1\left(q\right),c\right)\cup\mathcal{II}_3\sigma\left(C_1\left(q\right),c\right)\cup\mathcal{III}_3\sigma\left(C_1\left(q\right),c\right)$$

from Table 1, the required result is obtained from Theorems 2.5 and 2.14. □

In the theorems below, σ_c, σ_{ap}, σ_δ, σ_{co} spectrum types of $C_1\left(q\right)$ spectral operator on c will be examined in order.

THEOREM 2.16. $\sigma_c\left(C_1\left(q\right),c\right)=\left\{\alpha\in\mathbb{C}:\left|\alpha-\frac{1}{1+q}\right|=\frac{q}{1+q}\right\}\setminus\{1\}.$

Proof. Since $\sigma_c\left(C_1\left(q\right),c\right)=\sigma\left(C_1\left(q\right),c\right)\setminus\left[\sigma_r\left(C_1\left(q\right),c\right)\cup\sigma_p\left(C_1\left(q\right),c\right)\right]$ from Table 1, the required result is obtained from Theorems 2.5, 2.7 ve 2.9. □

THEOREM 2.17. *The following statements are hold*

(a) $\sigma_{ap}\left(C_1\left(q\right),c\right)=\left\{\alpha\in\mathbb{C}:\left|\alpha-\frac{1}{1+q}\right|=\frac{q}{1+q}\right\}\cup\left\{\frac{1}{1+q},\frac{1}{1+q+q^2},\dots\right\},$

(b) $\sigma_\delta\left(C_1\left(q\right),c\right)=\left\{\alpha\in\mathbb{C}:\left|\alpha-\frac{1}{1+q}\right|\leqslant\frac{q}{1+q}\right\},$

(c) $\sigma_{co}\left(C_1\left(q\right),c\right)=\left\{\alpha\in\mathbb{C}:\left|\alpha-\frac{1}{1+q}\right|<\frac{q}{1+q}\right\}\cup\{1\}.$

Proof. (a) Since

$$\sigma_{ap}\left(C_1\left(q\right),c\right)=\sigma\left(C_1\left(q\right),c\right)\setminus\mathcal{III}_1\sigma\left(C_1\left(q\right),c\right)$$

from Table 1, give us required result Theorem 2.7 and Corollary 2.13.

(b) Since

$$\sigma_\delta\left(C_1\left(q\right),c\right)=\sigma\left(C_1\left(q\right),c\right)\setminus\mathcal{I}_1\sigma\left(C_1\left(q\right),c\right)$$

from Table 1, give us required result Theorem 2.7 and Corollary 2.15.

(c) Since

$$\sigma_{co}\left(C_1\left(q\right),c\right)=\sigma_r\left(C_1\left(q\right),c\right)\cup\mathcal{III}_3\sigma\left(C_1\left(q\right),c\right)$$

from Table 1, give us required result Theorem 2.9 and Corollary 2.15. □

COROLLARY 2.18. *(a)* $\sigma_{ap}\left(C_1^*\left(q\right),\ell_1\right)=\left\{\alpha\in\mathbb{C}:\left|\alpha-\frac{1}{1+q}\right|\leqslant\frac{q}{1+q}\right\},$

(b) $\sigma_\delta\left(C_1^*\left(q\right),\ell_1\right)=\left\{\alpha\in\mathbb{C}:\left|\alpha-\frac{1}{1+q}\right|=\frac{q}{1+q}\right\}\cup\left\{1,\frac{1}{1+q},\frac{1}{1+q+q^2},\dots\right\}.$

Proof. Using Proposition 1.1 (c) and (d), we have

$$\sigma_{ap}(C_1^*\left(q\right),\ell_1)=\sigma_\delta(C_1\left(q\right),c)$$

and

$$\sigma_\delta(C_1^*\left(q\right),\ell_1)=\sigma_{ap}(C_1\left(q\right),c).$$

Using Theorem 2.17 (a) and (b) we get the required results. □

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