Operators
and
Matrices
Volume 15, Number 3 (2021), 853–870
doi:10.7153/oam-2021-15-57

ESSENTIAL NORM OF WEIGHTED COMPOSITION FOLLOWED AND PROCEEDED BY DIFFERENTIATION OPERATOR FROM BLOCH–TYPE INTO BERS–TYPE SPACES

HAMID VAEZI AND MOHAMAD NAGHLISAR*

(*Communicated by G. Misra*)

Abstract. We consider the weighted composition followed and proceeded by differentiation operator $DC_{\varphi}^{u}D$ from Bloch-type space B^{α} into Bers-type space H_{β}^{∞}. First, we give necessary and sufficient conditions for boundedness and compactness of this operator. Then, we obtain the essential norm estimate of such an operator in terms of u and φ.

1. Introduction

Denote by $H(\mathbb{D})$ the space of all analytic functions on open unit disc $\mathbb{D}$ in the complex plane. An analytic function f on $\mathbb{D}$ belongs to the Bloch-type space B^{α}, $(0<\alpha<\infty)$ if

$$||f||_{B^{\alpha}} = \sup_{w\in\mathbb{D}}(1-|w|^2)^{\alpha}|f'(w)| < \infty.$$

The norm $||f|| = ||f||_{B^{\alpha}} + |f(0)|$ makes B^{α} into a Banach space.

Let B_0^{α} be the subspace of B^{α} which consisting of all $f\in B^{\alpha}$ satisfying

$$(1-|w|^2)^{\alpha}|f'(w)| \to 0 \qquad \text{as} \qquad |w|\to 1.$$

This space is called the little Bloch-type space.

The Bers-type space H_{β}^{∞} is the space of all $f\in H(\mathbb{D})$, $(0<\beta<\infty)$ such that

$$||f||_{H_{\beta}^{\infty}} = \sup_{w\in\mathbb{D}}(1-|w|^2)^{\beta}|f(w)| < \infty.$$

Let $H_{\beta,0}^{\infty}$ be the subspace of H_{β}^{∞} which consisting of all $f\in H_{\beta}^{\infty}$ satisfying

$$(1-|w|^2)^{\beta}|f(w)| \to 0 \qquad \text{as} \qquad |w|\to 1.$$

This space is called the little Bers-type space.

Mathematics subject classification (2020): 47B33, 30H30, 30H99.

Keywords and phrases: Weighted composition operator, Bloch-type space, Bers-type space, essential norm.

* Corresponding author.

Given a function $u \in H(\mathbb{D})$ and a nonconstant analytic self-map φ on $\mathbb{D}$, we define a linear operator C_φ^u on $H(\mathbb{D})$ by $C_\varphi^u(f) = u \cdot (f o \varphi) = u \cdot f(\varphi)$. If $u \equiv 1$, then, C_φ is called the composition operator. For more information about these operators, see [2, 16]. In 2013, S. Yamaji [19] considered composition operators on the Bergman spaces. The weighted composition operators acting on various spaces of analytic functions has been studied by many authors. For example, C_φ^u was studied by Sh. Ohno, K. Stroethoff and R. Zhao in [12], where they have studied the boundedness and compactness of C_φ^u between Bloch-type spaces. X.-C. Guo and Z.-H. Zhou provide new characterizations for the boundedness and compactness of the weighted composition operator from Zygmund-type spaces to Bloch-type spaces in [3]. M. Hassanlou, H. Vaezi and M. Wang in [4] characterized the bounded and the compact weighted compoition operators on weak vector-valued Bergman spaces and Hardy spaces. For more results in this context we refer to [1, 6, 14, 22].

The weighted composition followed by differentiation operator DC_φ^u is defined by

$$DC_\varphi^u(f) = (u \cdot f(\varphi))' = u' \cdot f(\varphi) + u \cdot f'(\varphi) \cdot \varphi',$$

where C_φ^u and D are weighted composition and differentiation operators respectively.

The operator DC_φ was first studied by R. A. Hibschweiler and N. Portnoy in [5], where the boundedess and compactness of DC_φ between Hardy and Bergman spaces are investigated. S. Li and S. Stevic in [7] characterized the boundedness and compactness of DC_φ between Bloch-type spaces.

We define a linear operator $C_\varphi^u D$ on $H(\mathbb{D})$ by

$$C_\varphi^u D(f) = u \cdot (f' o \varphi) = u \cdot f'(\varphi).$$

This operator is called weighted composition proceeded by differentation operator. The operator $C_\varphi D$ between Hardy spaces was studied in [11] by S. Ohno. J. S. Manhas and R. Zhao in [10] characterized the boundedness and compactness of $C_\varphi^u D$ between weighted Banach spaces of analytic functions and weighted Zygmund spaces or weighted Bloch spaces.

We define a linear operator $DC_\varphi^u D$ on $H(\mathbb{D})$ by

$$DC_\varphi^u Df = DC_\varphi^u f' = u' \cdot f'(\varphi) + u \cdot f''(\varphi) \cdot \varphi'.$$

We called this operator, weighted composition followed and proceeded by differentiation operator. Boundedness and compactness of the operator $DC_\varphi^u D$ from Zygmund spaces to Bloch-type spaces were described by J. Long, C. Qiu and P. Wu in [8].

Recall that the essential norm $||T||_e$ of a bounded operator T between Banach spaces X and Y is defined as the distance from T to the compact operators, that is

$$||T||_e = \inf\{||T - K|| : K \text{ is compact}\}.$$

Notic that $||T||_e = 0$ if and only if T is compact. The essential norm of the composition operator on Bloch spases was studied by A. Montes-Rodriguez in [13]. R. Zhao in [20] give estimates for the essential norms of the composition operators between Bloch-type spaces. Essential norms of the weighted composition operators between Bloch-type spaces are investigated by B. D. Macculuer and R. Zhao in [9]. In [17], S. Stevic, estimate essential norms of the weighted composition operators from Bloch-type spaces to a weighted-type space on the unit ball, and A. H. Sanatpour and M. Hassanlou in [15] were proved the lower and upper bound of the essential norms of weighted composition operators between Zygmund-type spaces and Bloch-type spaces.

In this article we characterize the boundedness and compactness of $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ in section 2, and boundedness and compactness of this operator from B_0^α into $H_{\beta,0}^\infty$ in section 3. Finally we give lower and upper bounds for the essential norm of the operator $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ in section 4.

We denote the constants by C which will differ from one appearance to the another. If there exists a positive constant C such that $A \leqslant CB$ then, we write $A \preceq B$. If $A \preceq B$ and $B \preceq A$ we denote by $A \sim B$.

2. Boundedness and compactness of $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$

The boundedness and compactness criteria for the operator $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ will be given in this section.

THEOREM 1. *For a fixed $u \in H(\mathbb{D})$, φ an analytic self-map on $\mathbb{D}$ and α and β positive real numbers, the operator $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded if and only if*

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} < \infty \tag{1}$$

and

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} < \infty. \tag{2}$$

Proof. First, we prove sufficiency. For a function $f \in B^\alpha$,

$$\begin{aligned}(1-|w|^2)^\beta |DC_\varphi^u Df(w)| &= (1-|w|^2)^\beta |DC_\varphi^u f'(w)| \\ &\leqslant (1-|w|^2)^\beta |u'(w)||f'(\varphi(w))| \\ &\quad + (1-|w|^2)^\beta |u(w)||f''(\varphi(w))||\varphi'(w)| \\ &\leqslant \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} ||f||_{B^\alpha} \\ &\quad + \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} ||f||_{B^\alpha} \\ &= C||f||_{B^\alpha}.\end{aligned}$$

We have used the following characterization of Bloch-type functions (see [7, Theorem 1] and [21, Proposition 8]):

$$\sup_{w\in\mathbb{D}} (1-|w|^2)^\alpha |f'(w)| \sim |f'(0)| + \sup_{w\in\mathbb{D}} (1-|w|^2)^{\alpha+1} |f''(w)|,$$

in the last inequality.

Using conditions (1) and (2) it follows that the operator $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded. Now, suppose that $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded. Taking $f(w) = w$ and $f(w) = w^2$ respectively, we obtain

$$\sup_{w\in\mathbb{D}}(1-|w|^2)^\beta |u'(w)| < \infty \tag{3}$$

and

$$\sup_{w\in\mathbb{D}}(1-|w|^2)^\beta |2u'(w)\varphi(w)+2u(w)\varphi'(w)| < \infty.$$

Using these facts and the boundedness of the function $\varphi(w)$, we have

$$\sup_{w\in\mathbb{D}}(1-|w|^2)^\beta |u(w)||\varphi'(w)| < \infty. \tag{4}$$

For fixed $w_o \in \mathbb{D}$, consider the function f_o defined by

$$f_o(w) = \frac{(\alpha+1)(\alpha+2)(1-|\varphi(w_o)|^2)}{(1-w\overline{\varphi(w_o)})^\alpha} - \frac{\alpha(\alpha+1)(1-|\varphi(w_o)|^2)^2}{(1-w\overline{\varphi(w_o)})^{\alpha+1}}, \tag{5}$$

for $w \in \mathbb{D}$. Then,

$$f_o'(w) = \frac{\alpha(\alpha+1)(\alpha+2)\overline{\varphi(w_o)}(1-|\varphi(w_o)|^2)}{(1-w\overline{\varphi(w_o)})^{\alpha+1}} - \frac{\alpha(\alpha+1)^2\overline{\varphi(w_o)}(1-|\varphi(w_o)|^2)^2}{(1-w\overline{\varphi(w_o)})^{\alpha+2}},$$

for $w \in \mathbb{D}$. Hence,

$$\begin{aligned}
|f_o'(w)| &\leqslant \frac{\alpha(\alpha+1)(\alpha+2)(1-|\varphi(w_o)|^2)}{(1-|w\overline{\varphi(w_o)}|)^{\alpha+1}} + \frac{\alpha(\alpha+1)^2(1-|\varphi(w_o)|^2)^2}{(1-|w\overline{\varphi(w_o)}|)^{\alpha+2}} \\
&\leqslant \frac{\alpha(\alpha+1)(\alpha+2)(1-|\varphi(w_o)|^2)}{(1-|w|)^\alpha(1-|\varphi(w_o)|)} + \frac{\alpha(\alpha+1)^2(1-|\varphi(w_o)|^2)^2}{(1-|w|)^\alpha(1-|\varphi(w_o)|)^2} \\
&\leqslant \frac{2\alpha(\alpha+1)(\alpha+2)}{(1-|w|)^\alpha} + \frac{2^2\alpha(\alpha+1)^2}{(1-|w|)^\alpha} = \frac{2\alpha(\alpha+1)(3\alpha+4)}{(1-|w|)^\alpha} \\
&\leqslant \frac{2^{\alpha+3}\alpha(\alpha+1)^2}{(1-|w|^2)^\alpha},
\end{aligned}$$

for all $w \in \mathbb{D}$. So, it follows that $f_o \in B^\alpha$. We also have

$$\begin{aligned}
f_o''(w) = &\frac{\alpha(\alpha+1)^2(\alpha+2)\overline{\varphi(w_o)}^2(1-|\varphi(w_o)|^2)}{(1-w\overline{\varphi(w_o)})^{\alpha+2}} \\
&- \frac{\alpha(\alpha+1)^2(\alpha+2)\overline{\varphi(w_o)}^2(1-|\varphi(w_o)|^2)^2}{(1-w\overline{\varphi(w_o)})^{\alpha+3}},
\end{aligned}$$

for $w \in \mathbb{D}$. It can be shown that

$$f_o'(\varphi(w_o)) = \frac{\alpha(\alpha+1)\overline{\varphi(w_o)}}{(1-|\varphi(w_o)|^2)^\alpha} \quad \text{and} \quad f_o''(\varphi(w_o)) = 0.$$

Then, for $w_o \in \mathbb{D}$,

$$\frac{\alpha(\alpha+1)|\varphi(w_o)|(1-|w_o|^2)^\beta|u'(w_o)|}{(1-|\varphi(w_o)|^2)^\alpha} = (1-|w_o|^2)^\beta|(DC_\varphi^u Df_o)(w_o)| \leqslant ||DC_\varphi^u Df_o||_{H_\beta^\infty} \leqslant C||f_o||_{B^\alpha} < \infty.$$

Since, w_o is arbitrary, hence, for any $w \in \mathbb{D}$,

$$\frac{|\varphi(w)|(1-|w|^2)^\beta|u'(w)|}{(1-|\varphi(w)|^2)^\alpha} < \infty. \tag{6}$$

For any δ, $0 < \delta < 1$, by (6), we have

$$\sup_{|\varphi(w)>\delta} \frac{(1-|w|^2)^\beta|u'(w)|}{(1-|\varphi(w)|^2)^\alpha} < \infty. \tag{7}$$

For $w \in \mathbb{D}$, such that $|\varphi(w)| \leqslant \delta$,

$$\frac{(1-|w|^2)^\beta|u'(w)|}{(1-|\varphi(w)|^2)^\alpha} \leqslant \frac{(1-|w|^2)^\beta|u'(w)|}{(1-\delta^2)^\alpha}. \tag{8}$$

From (3) and (8), it follows that

$$\sup_{|\varphi(w)\leqslant\delta} \frac{(1-|w|^2)^\beta|u'(w)|}{(1-|\varphi(w)|^2)^\alpha} < \infty. \tag{9}$$

Hence, (7) and (9) implies that

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta|u'(w)|}{(1-|\varphi(w)|^2)^\alpha} < \infty.$$

Therefore, (1) holds.

Now, for fixed $w_o \in \mathbb{D}$ consider the function g_o defined by

$$g_o(w) = \frac{\alpha(1-|\varphi(w_o)|^2)^2}{(1-w\overline{\varphi(w_o)})^{\alpha+1}} - \frac{(\alpha+1)(1-|\varphi(w_o)|^2)}{(1-w\overline{\varphi(w_o)})^\alpha}, \tag{10}$$

for $w \in \mathbb{D}$. Then,

$$g_o'(w) = \frac{\alpha(\alpha+1)\overline{\varphi(w_o)}(1-|\varphi(w_o)|^2)^2}{(1-w\overline{\varphi(w_o)})^{\alpha+2}} - \frac{\alpha(\alpha+1)\overline{\varphi(w_o)}(1-|\varphi(w_o)|^2)}{(1-w\overline{\varphi(w_o)})^{\alpha+1}},$$

for $w \in \mathbb{D}$. Hence,

$$|g_o'(w)| \leqslant \frac{\alpha(\alpha+1)(1-|\varphi(w_o)|^2)^2}{(1-|w|)^\alpha(1-|\varphi(w_o)|)^2} + \frac{\alpha(\alpha+1)(1-|\varphi(w_o)|^2)}{(1-|w|)^\alpha(1-|\varphi(w_o)|)}$$
$$\leqslant \frac{2^2\alpha(\alpha+1)}{(1-|w|)^\alpha} + \frac{2\alpha(\alpha+1)}{(1-|w|)^\alpha} \leqslant \frac{2^{\alpha+3}\alpha(\alpha+1)}{(1-|w|^2)^\alpha},$$

for all $w \in \mathbb{D}$. So, it follows that $g_o \in B^\alpha$. We also have

$$g_o''(w) = \frac{\alpha(\alpha+1)(\alpha+2)\overline{\varphi(w_o)}^2(1-|\varphi(w_o)|^2)^2}{(1-w\overline{\varphi(w_o)})^{\alpha+3}} - \frac{\alpha(\alpha+1)^2\overline{\varphi(w_o)}^2(1-|\varphi(w_o)|^2)}{(1-w\overline{\varphi(w_o)})^{\alpha+2}},$$

for $w \in \mathbb{D}$. It can be shown that

$$g_o'(\varphi(w_o)) = 0 \quad \text{and} \quad g_o''(\varphi(w_o)) = \frac{\alpha(\alpha+1)\overline{\varphi(w_o)}^2}{(1-|\varphi(w_o)|^2)^{\alpha+1}}.$$

Then, for $w_o \in \mathbb{D}$,

$$\frac{\alpha(\alpha+1)|\overline{\varphi(w_o)}|^2(1-|w_o|^2)^\beta|u(w_o)||\varphi'(w_o)|}{(1-|\varphi(w_o)|^2)^{\alpha+1}} = (1-|w_o|^2)^\beta|(DC_\varphi^u Dg_o)(w_o)|$$
$$\leqslant ||DC_\varphi^u Dg_o||_{H_\beta^\infty} \leqslant C||g_o||_{B^\alpha} < \infty.$$

Since, w_o is arbitrary, hence, for any $w \in \mathbb{D}$,

$$\frac{|\varphi(w)|^2(1-|w|^2)^\beta|u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} < \infty. \tag{11}$$

For any δ, $0 < \delta < 1$, by (11), we have

$$\sup_{|\varphi(w)|>\delta} \frac{(1-|w|^2)^\beta|u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} < \infty. \tag{12}$$

For $w \in \mathbb{D}$ such that $|\varphi(w)| \leqslant \delta$,

$$\frac{(1-|w|^2)^\beta|u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} \leqslant \frac{(1-|w|^2)^\beta|u(w)||\varphi'(w)|}{(1-\delta^2)^{\alpha+1}}. \tag{13}$$

From (4) and (13), it follows that

$$\sup_{|\varphi(w)|\leqslant\delta} \frac{(1-|w|^2)^\beta|u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} < \infty. \tag{14}$$

Hence, (12) and (14) implies that

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta|u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} < \infty.$$

This show that the condition (2) holds and the proof of the theorem is completed. □

THEOREM 2. *For a fixed $u \in H(\mathbb{D})$, φ an analytic self-map on $\mathbb{D}$ and α and β positive real numbers, if $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded, then, it is compact if and only if*

$$\lim_{|\varphi(w)|\to 1} \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} = 0 \tag{15}$$

and

$$\lim_{|\varphi(w)|\to 1} \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} = 0. \tag{16}$$

Proof. Since, $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded, from Theorem 1 (relations (3) and (4)), we have

$$L = \sup_{w\in\mathbb{D}} (1-|w|^2)^\beta |u'(w)| < \infty \quad \text{and} \quad M = \sup_{w\in\mathbb{D}} (1-|w|^2)^\beta |u(w)||\varphi'(w)| < \infty.$$

Now, suppose that (15) and (16) are true. Then, for every $\varepsilon > 0$, there exists a $\delta \in (0,1)$, such that

$$\frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} < \frac{\varepsilon}{2} \tag{17}$$

and

$$\frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} < \frac{\varepsilon}{2}, \tag{18}$$

whenever $\delta < |\varphi(w)| < 1$.

To prove the compactness of $DC_\varphi^u D$, assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in B^α, such that $||f_k||_{B^\alpha} \leqslant 1$ and converges to zero uniformly on compact subsets of $\mathbb{D}$. From Weak Convergence Theorem in [16, Section 2.4, Page 29] it is sufficient to show that $||DC_\varphi^u D f_k||_{H_\beta^\infty} \to 0$.

If $|\varphi(w)| > \delta$, then, by (17) and (18),

$$\begin{aligned}
||DC_\varphi^u D f_k||_{H_\beta^\infty} &= \sup_{w\in\mathbb{D}} (1-|w|^2)^\beta |DC_\varphi^u D f_k(w)| \\
&\leqslant \sup_{w\in\mathbb{D}} (1-|w|^2)^\beta |u'(w)||f_k'(\varphi(w))| \\
&\quad + \sup_{w\in\mathbb{D}} (1-|w|^2)^\beta |u(w)||\varphi'(w)||f_k''(\varphi(w))| \\
&\leqslant \sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} ||f_k||_{B^\alpha} \\
&\quad + \sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} ||f_k||_{B^\alpha} \\
&\leqslant \frac{\varepsilon}{2} ||f_k||_{B^\alpha} + \frac{\varepsilon}{2} ||f_k||_{B^\alpha} = \varepsilon ||f_k||_{B^\alpha} \leqslant \varepsilon,
\end{aligned}$$

in which we have used the following relation between the first and second derivative of f:

$$\sup_{w\in\mathbb{D}}(1-|w|^2)^{\alpha}|f'(w)| \sim |f'(0)| + \sup_{w\in\mathbb{D}}(1-|w|^2)^{\alpha+1}|f''(w)|.$$

Now, consider the case $|\varphi(w)| \leqslant \delta$,

$$\begin{aligned}||DC_{\varphi}^{u}Df_k||_{H_{\beta}^{\infty}} &\leqslant \sup_{w\in\mathbb{D}}(1-|w|^2)^{\beta}|u'(w)||f_k'(\varphi(w))| \\ &\quad + \sup_{w\in\mathbb{D}}(1-|w|^2)^{\beta}|u(w)||f_k''(\varphi(w))||\varphi'(w)| \\ &\leqslant L\max_{|\varphi(w)|\leqslant\delta}|f_k'(\varphi(w))| + M\max_{|\varphi(w)|\leqslant\delta}|f_k''(\varphi(w))|.\end{aligned}$$

So, $||DC_{\varphi}^{u}Df_k||_{H_{\beta}^{\infty}} \to 0$.

Now, we are going to prove that (15) and (16) are also necessary conditions for compactness of $DC_{\varphi}^{u}D$.

Suppose that $\left(w_k\right)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi(w_k)| \to 1$ as $k\to\infty$. Consider the functions f_k defined by

$$f_k(w) = \frac{(\alpha+1)(\alpha+2)(1-|\varphi(w_k)|^2)}{(1-w\overline{\varphi(w_k)})^{\alpha}} - \frac{\alpha(\alpha+1)(1-|\varphi(w_k)|^2)^2}{(1-w\overline{\varphi(w_k)})^{\alpha+1}},$$

for $w\in\mathbb{D}$. Clearly $f_k\to 0$ uniformly on compact subsets of $\mathbb{D}$. Since,

$$\begin{aligned}f_k'(w) &= \frac{\alpha(\alpha+1)(\alpha+2)\overline{\varphi(w_k)}(1-|\varphi(w_k)|^2)}{(1-w\overline{\varphi(w_k)})^{\alpha+1}} \\ &\quad - \frac{\alpha(\alpha+1)^2\overline{\varphi(w_k)}(1-|\varphi(w_k)|^2)^2}{(1-w\overline{\varphi(w_k)})^{\alpha+2}}\end{aligned}$$

and

$$\begin{aligned}f_k''(w) &= \frac{\alpha(\alpha+1)^2(\alpha+2)\overline{\varphi(w_k)}^2(1-|\varphi(w_k)|^2)}{(1-w\overline{\varphi(w_k)})^{\alpha+2}} \\ &\quad - \frac{\alpha(\alpha+1)^2(\alpha+2)\overline{\varphi(w_k)}^2(1-|\varphi(w_k)|^2)^2}{(1-w\overline{\varphi(w_k)})^{\alpha+3}}\end{aligned}$$

for $w\in\mathbb{D}$, hence, it can be shown that,

$$|f_k'(w)| \leqslant \frac{2^{\alpha+3}\alpha(\alpha+1)^2}{(1-|w|^2)^{\alpha}}.$$

So, the $(||f_k||_{B^{\alpha}})_{k\in\mathbb{N}}$ is uniformly bounded. It is clear that,

$$f_k'(\varphi(w_k)) = \frac{\alpha(\alpha+1)\overline{\varphi(w_k)}}{(1-|\varphi(w_k)|^2)^{\alpha}} \quad \text{and} \quad f_k''(\varphi(w_k)) = 0.$$

Since, $DC_\varphi^u D$ is compact, it follows that, $||DC_\varphi^u Df_k||_{H_\beta^\infty} \to 0$. Hence,

$$\frac{\alpha(\alpha+1)|\varphi(w_k)|(1-|w_k|^2)^\beta |u'(w_k)|}{(1-|\varphi(w_k)|^2)^\alpha} = (1-|w_k|^2)^\beta |DC_\varphi^u Df_k(w_k)| \leqslant ||DC_\varphi^u Df_k||_{H_\beta^\infty}.$$

So,

$$\frac{(1-|w_k|^2)^\beta |u'(w_k)|}{(1-|\varphi(w_k)|^2)^\alpha} \to 0 \quad \text{as} \quad k\to\infty.$$

Thus, the condition (15) holds.

Next, consider the functions g_k defined by

$$g_k(w) = \frac{\alpha(1-|\varphi(w_k)|^2)^2}{(1-w\overline{\varphi(w_k)})^{\alpha+1}} - \frac{(\alpha+1)(1-|\varphi(w_k)|^2)}{(1-w\overline{\varphi(w_k)})^{\alpha}},$$

for $w\in\mathbb{D}$. Clearly $g_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. Since,

$$g_k'(w) = \frac{\alpha(\alpha+1)\overline{\varphi(w_k)}(1-|\varphi(w_k)|^2)^2}{(1-w\overline{\varphi(w_k)})^{\alpha+2}} - \frac{\alpha(\alpha+1)\overline{\varphi(w_k)}(1-|\varphi(w_k)|^2)}{(1-w\overline{\varphi(w_k)})^{\alpha+1}}$$

and

$$g_k''(w) = \frac{\alpha(\alpha+1)(\alpha+2)\overline{\varphi(w_k)}^2(1-|\varphi(w_k)|^2)^2}{(1-w\overline{\varphi(w_k)})^{\alpha+3}} - \frac{\alpha(\alpha+1)^2\overline{\varphi(w_k)}^2(1-|\varphi(w_k)|^2)}{(1-w\overline{\varphi(w_k)})^{\alpha+2}}$$

for $w\in\mathbb{D}$, hence, it can be shown that,

$$|g_k'(w)| \leqslant \frac{2^{\alpha+3}\alpha(\alpha+1)}{(1-|w|^2)^\alpha}.$$

So, the $(||g_k||_{B^\alpha})_{k\in\mathbb{N}}$ is uniformly bounded. It is clear that,

$$g_k'(\varphi(w_k)) = 0 \quad \text{and} \quad g_k''(\varphi(w_k)) = \frac{\alpha(\alpha+1)\overline{\varphi(w_k)}^2}{(1-|\varphi(w_k)|^2)^{\alpha+1}}.$$

Since, $DC_\varphi^u D$ is compact, then, $||DC_\varphi^u Dg_k||_{H_\beta^\infty} \to 0$. Hence,

$$\frac{\alpha(\alpha+1)|\varphi(w_k)|^2(1-|w_k|^2)^\beta |u(w_k)||\varphi'(w_k)|}{(1-|\varphi(w_k)|^2)^{\alpha+1}} = (1-|w_k|^2)^\beta |DC_\varphi^u Dg_k(w_k)| \leqslant ||DC_\varphi^u Dg_k||_{H_\beta^\infty}.$$

So,

$$\frac{(1-|w_k|^2)^\beta |u(w_k)||\varphi'(w_k)|}{(1-|\varphi(w_k)|^2)^{\alpha+1}} \to 0 \quad \text{as} \quad k\to\infty.$$

Thus, the condition (16) holds and the proof is completed. □

3. Boundedness and compactness of $DC_\varphi^u D : B_0^\alpha \to H_{\beta,0}^\infty$

The boundedness and compactness criteria for the operator $DC_\varphi^u D : B_0^\alpha \to H_{\beta,0}^\infty$ will be given in this section.

THEOREM 3. *For a fixed $u \in H(\mathbb{D})$, φ an analytic self-map on $\mathbb{D}$ and α and β positive real numbers, the operator $DC_\varphi^u D : B_0^\alpha \to H_{\beta,0}^\infty$ is bounded if and only if $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded and $u', u\varphi' \in H_{\beta,0}^\infty$*

Proof. Suppose that $DC_\varphi^u D$ maps B_0^α boundedly into $H_{\beta,0}^\infty$. First, taking $f(w) = w \in B_0^\alpha$, since $DC_\varphi^u Df$ belongs to $H_{\beta,0}^\infty$, we obtain

$$u' = DC_\varphi^u Dw \in H_{\beta,o}^\infty,$$

so,

$$\lim_{|w|\to 1} (1-|w|^2)^\beta |u'(w)| = 0.$$

Next, taking $f(w) = w^2 \in B_0^\alpha$, we obtain

$$\lim_{|w|\to 1} (1-|w|^2)^\beta |2u'(w)\varphi(w) + 2u(w)\varphi'(w)| = 0.$$

Thus,

$$\lim_{|w|\to 1} (1-|w|^2)^\beta |u(w)||\varphi'(w)| = 0,$$

then, $u\varphi' \in H_{\beta,0}^\infty$. For fixed $w_o \in \mathbb{D}$, the functions defined in (5) and (10) are in fact in B_0^α, so, the proof of Theorem 1 shows that, if $DC_\varphi^u D$ maps B_0^α bundedly into $H_{\beta,0}^\infty$, then,

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} < \infty$$

and

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} < \infty.$$

Thus, again from Theorem 1, $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded.

Conversely, suppose that u and φ are such that $u', u\varphi' \in H_{\beta,0}^\infty$ and $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded. We will show that $DC_\varphi^u D : B_0^\alpha \to H_{\beta,0}^\infty$ is bounded. We only need to prove that $DC_\varphi^u Df \in H_{\beta,o}^\infty$ for any $f \in B_o^\alpha$.

Since, $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded then, Theorem 1 shows that

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} = C$$

and

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} = C.$$

Let $f \in B_o^\alpha$, then, there exists $\delta \in (0,1)$, such that

$$(1-|\varphi(w)|^2)^\alpha |f'(\varphi(w))| < \frac{\varepsilon}{2C} \qquad \text{as} \qquad \delta < |\varphi(w)| < 1$$

and

$$(1-|\varphi(w)|^2)^{\alpha+1} |f''(\varphi(w))| < \frac{\varepsilon}{2C} \qquad \text{as} \qquad \delta < |\varphi(w)| < 1.$$

We consider two cases, $\delta < |\varphi(w)| < 1$ and $|\varphi(w)| \leqslant \delta$.

First, consider $\delta < |\varphi(w)| < 1$. Then,

$$\begin{aligned}
(1-|w|^2)^\beta |DC_\varphi^u Df(w)| &= (1-|w|^2)^\beta |DC_\varphi^u f'(w)| \\
&\leqslant (1-|w|^2)^\beta |u'(w)||f'(\varphi(w))| \\
&\quad +(1-|w|^2)^\beta |u(w)||f''(\varphi(w))||\varphi'(w)| \\
&= \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} (1-|\varphi(w)|^2)^\alpha |f'(\varphi(w))| \\
&\quad + \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} (1-|\varphi(w)|^2)^{\alpha+1} |f''(\varphi(w))| \\
&< C \cdot \frac{\varepsilon}{2C} + C \cdot \frac{\varepsilon}{2C} = \varepsilon.
\end{aligned}$$

So, $DC_\varphi^u Df \in H_{\beta,0}^\infty$. Next, consider $|\varphi(w)| \leqslant \delta$. Then,

$$\begin{aligned}
(1-|w|^2)^\beta |DC_\varphi^u Df(w)| &\leqslant (1-|w|^2)^\beta |u'(w)||f'(\varphi(w))| \\
&\quad +(1-|w|^2)^\beta |u(w)||f''(\varphi(w))||\varphi'(w)| \\
&\quad + \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} ||f||_{B^\alpha} \\
&\leqslant (1-|w|^2)^\beta |u'(w)| \frac{||f||_{B^\alpha}}{(1-\delta^2)^\alpha} \\
&\quad +(1-|w|^2)^\beta |u(w)||\varphi'(w)| \frac{||f||_{B^\alpha}}{(1-\delta^2)^{\alpha+1}}.
\end{aligned}$$

Taking the limit from both sides of the above inequality, since $u', u\varphi' \in H_{\beta,0}^\infty$, so,

$$\lim_{|w|\to 1} (1-|w|^2)^\beta |DC_\varphi^u Df(w)| = 0.$$

Thus, it follows from the Closed Graph Theorem that, $DC_\varphi^u D$ maps B_0^α boundedly into $H_{\beta,0}^\infty$. □

Next, we characterize the compactness of $DC_\varphi^u D : B_0^\alpha \to H_{\beta,0}^\infty$. For this purpose we need the following Lemma.

LEMMA 1. [18, Lemma 2.1] *Let $\beta > 0$. A closed set K in $H_{\beta,0}^\infty$ is compact if and only if it is bounded and satisfies*

$$\lim_{|w|\to 1} \sup_{f\in K} (1-|w|^2)^\beta |f(w)| = 0.$$

THEOREM 4. *For a fixed* $u \in H(\mathbb{D})$, φ *an analytic self-map on* $\mathbb{D}$ *and* α *and* β *positive real numbers, if* $DC_\varphi^u D : B_0^\alpha \to H_{\beta,0}^\infty$ *is bounded, then, it is compact if and only if*

$$\lim_{|w|\to 1} \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} = 0 \tag{19}$$

and

$$\lim_{|w|\to 1} \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} = 0. \tag{20}$$

Proof. Assume that (19) and (20) are true, then, we prove that $DC_\varphi^u D : B_0^\alpha \to H_{\beta,0}^\infty$ is compact. Suppose that $f \in B_0^\alpha$ is such that $||f||_{B^\alpha} \leqslant 1$, then,

$$\begin{aligned}(1-|w|^2)^\beta |DC_\varphi^u Df(w)| &\leqslant (1-|w|^2)^\beta |u'(w)||f'(\varphi(w))| \\ &\quad +(1-|w|^2)^\beta |u(w)||\varphi'(w)||f''(\varphi(w))| \\ &\leqslant \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} ||f||_{B^\alpha} \\ &\quad + \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)}{(1-|\varphi(w)|^2)^{\alpha+1}} ||f||_{B^\alpha}.\end{aligned}$$

Thus,

$$\begin{aligned}&\sup\{(1-|w|^2)^\beta |DC_\varphi^u Df(w)| : f \in B_0^\alpha, ||f||_{B^\alpha} \leqslant 1\} \\ &\leqslant \frac{(1-|w|^2)^\beta |u'(w)|}{(1-|\varphi(w)|^2)^\alpha} + \frac{(1-|w|^2)^\beta |u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}}.\end{aligned}$$

It follows that

$$\lim_{|w|\to 1} \sup\{(1-|w|^2)^\beta |DC_\varphi^u Df(w)| : f \in B_0^\alpha, ||f||_{B^\alpha} \leqslant 1\} = 0,$$

so, by Lemma 1, $DC_\varphi^u D : B_0^\alpha \to H_{\beta,0}^\infty$ is compact.

Conversely, suppose that $DC_\varphi^u D$ is compact, then, the set

$$\{DC_\varphi^u Df : f \in B_0^\alpha, ||f||_{B^\alpha} \leqslant 1\}$$

has compact closure in $H_{\beta,0}^\infty$ and with using Lemma 1,

$$\lim_{|w|\to 1} \sup\{(1-|w|^2)^\beta |DC_\varphi^u Df(w)| : f \in B_0^\alpha, ||f||_{B^\alpha} \leqslant C\} = 0, \tag{21}$$

for some $C > 0$. If (21) is satisfied, then, it followes by the proof of the Theorem 1 and the fact that the functions given in (5) and (10) are in B_0^α and have norms bounded independently of w, that (19) and (20) are true and the proof of the theorem is completed. □

Putting $u \equiv 1$, theorems 1 and 2, implies the following corollaries about the boundedness and compactness of the operator $DC_\varphi D : B^\alpha \to H_\beta^\infty$.

COROLLARY 1. *For an analytic self-map φ on $\mathbb{D}$ and α and β positive real numbers, the operator $DC_\varphi D : B^\alpha \to H^\infty_\beta$ is bounded if and only if*

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta |\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} < \infty.$$

COROLLARY 2. *For an analytic self-map φ on $\mathbb{D}$ and α and β positive real numbers. If $DC_\varphi D : B^\alpha \to H^\infty_\beta$ is bounded, then, it is compact if and only if*

$$\lim_{|\varphi(w)|\to 1} \frac{(1-|w|^2)^\beta |\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} = 0.$$

4. Essential norm of $DC^u_\varphi D : B^\alpha \to H^\infty_\beta$

The essential norm estimate of the operator $DC^u_\varphi D : B^\alpha \to H^\infty_\beta$ will be given in this section. We begin with the following two Lemmas.

LEMMA 2. [20, Lemma 2.2] *Let $\alpha > 0$, $n \in \mathbb{N}$, $0 \leqslant x \leqslant 1$ and $H_{n,\alpha}(x) = x^{n-1}(1-x^2)^\alpha$. Then, $H_{n,\alpha}$ has the following properties.*

(i)

$$\max_{0\leqslant x\leqslant 1} H_{n,\alpha}(x) = H_{n,\alpha}(r_n) = \begin{cases} 1\ , & as \quad n=1 \\ \left(\dfrac{2\alpha}{n-1+2\alpha}\right)^\alpha \left(\dfrac{n-1}{n-1+2\alpha}\right)^{\frac{n-1}{2}}, & as \quad n \geqslant 2 \end{cases}$$

where

$$r_n = \begin{cases} 0\ , & as \ \ n=1 \\ \left(\dfrac{n-1}{n-1+2\alpha}\right)^{\frac{1}{2}}, & as \ \ n \geqslant 2. \end{cases}$$

(ii) For $n \geqslant 1$, $H_{n,\alpha}$ is increasing on $[0, r_n]$ and decreasing on $[r_n, 1]$.

(iii) For $n \geqslant 1$, $H_{n,\alpha}$ is decreasing on $[r_n, r_{n+1}]$ and so,

$$\min_{x\in[r_n,r_{n+1}]} H_{n,\alpha}(x) = H_{n,\alpha}(r_{n+1}) = \left(\frac{2\alpha}{n+2\alpha}\right)^\alpha \left(\frac{n}{n+2\alpha}\right)^{\frac{(n-1)}{2}}.$$

Consequently,

$$\lim_{n\to\infty} n^\alpha \min_{x\in[r_n,r_{n+1}]} H_{n,\alpha}(x) = \left(\frac{2\alpha}{e}\right)^\alpha.$$

We need the following Lemma to obtain the upper estimates of essential norm. For $r \in (0,1)$, let $K_r f(w) = f(rw)$. Then, K_r is a compact operator on the space B^α (or

B_0^α) for any positive number α (see for example [6, 9, 20]), with $||K_r|| \leqslant 1$. Indeed, $K_r f(w) = f(rw)$ implies that

$$\begin{aligned} ||K_r|| = \sup_{||f||\leqslant 1} ||K_r f|| &= \sup_{||f||\leqslant 1} \left(\sup_{w\in\mathbb{D}} (1-|w|^2)^\alpha |rf'(rw)| + |K_r f(0)|\right) \\ &\leqslant \sup_{||f||\leqslant 1} \left(\sup_{w\in\mathbb{D}} (1-|rw|^2)^\alpha |f'(rw)| r + |f(0)|\right) \\ &\leqslant \sup_{||f||\leqslant 1} ||f|| = 1. \end{aligned}$$

LEMMA 3. [20, Lemma 4.1] *Let $0 < \alpha \leqslant 1$. Then, there is a sequence $\{r_k\}$, $0 < r_k < 1$, tending to 1, such that the compact operator $L_n = \frac{1}{n}\sum_{k=1}^{n} K_{r_k}$ on B_0^α satisfies*

(i) For any $t \in [0,1)$, $\lim_{n\to\infty} \sup_{||f||_{B^\alpha}\leqslant 1} \sup_{|w|\leqslant t} |\big((I-L_n)f\big)'(w)| = 0$.

(ii) $\lim_{n\to\infty} \sup_{||f||_{B^\alpha}\leqslant 1} \sup_{w\in\mathbb{D}} |(I-L_n)f(w)| = 0$.

(iii) $\limsup_{n\to\infty} ||I-L_n|| \leqslant 1$.

THEOREM 5. *For a fixed $u \in H(\mathbb{D})$, φ an analytic self-map on $\mathbb{D}$, α and β positive real numbers with $0 < \alpha \leqslant 1$ and $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$ is bounded, then,*

$$||DC_\varphi^u D||_e = \lim_{t\to 1} \sup_{|\varphi(w)|>t} \frac{|u(w)||\varphi'(w)|(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+1}}.$$

Proof. We first give the lower estimate. Let $n \in \mathbb{N}$, consider the function w^n, by Lemma 2,

$$||w^n||_{B^\alpha} = \max_{w\in\mathbb{D}} n|w|^{n-1}(1-|w|^2)^\alpha = n\Big(\frac{2\alpha}{n-1+2\alpha}\Big)^\alpha \Big(\frac{n-1}{n-1+2\alpha}\Big)^{\frac{n-1}{2}},$$

where the maximum is attained at any point on the circle with radius

$$r_n = \Big(\frac{n-1}{n-1+2\alpha}\Big)^{\frac{1}{2}}.$$

Let $f_n(w) = \frac{w^n}{n||w^n||_{B^\alpha}}$. Then, $||f_n||_{B^\alpha} = \frac{1}{n}$ and $f_n \to 0$ weakly in B^α. This follows since a bounded sequence contained in B_0^α which tends to 0 uniformly on compact subsets of $\mathbb{D}$ converges weakly to 0 in B^α. In particular, if K is any compact operator from B^α to H_β^∞, then, $\lim_{n\to\infty} ||Kf_n||_{H_\beta^\infty} = 0$.

Let $A_n = \{w \in \mathbb{D} : \ r_n \leqslant |w| \leqslant r_{n+1}\}$. Then,

$$\begin{aligned}\min_{w\in A_n} |f_n''(w)|(1-|w|^2)^\alpha &= \min_{w\in A_n} \frac{(n-1)|w|^{n-2}}{||w^n||_{B^\alpha}}(1-|w|^2)^\alpha \\ &= \left(\frac{n-1}{n}\right)\left(\frac{n-1+2\alpha}{n+2\alpha}\right)^\alpha \left(\frac{n}{n+2\alpha}\right)^{\frac{n-2}{2}} \left(\frac{n-1+2\alpha}{n-1}\right)^{\frac{n-1}{2}}.\end{aligned}$$

Simple calculation shows that this minimum tends to 1 as $n \to \infty$. For any compact operator K from B^α to H_β^∞,

$$||DC_\varphi^u D - K|| \geqslant \lim_{n\to\infty} \sup ||(DC_\varphi^u D - K)f_n||_{H_\beta^\infty} \geqslant \lim_{n\to\infty} \sup ||DC_\varphi^u D f_n||_{H_\beta^\infty}.$$

Thus, for $DC_\varphi^u D : B^\alpha \to H_\beta^\infty$,

$$\begin{aligned}||DC_\varphi^u D||_e &\geqslant \limsup_{n\to\infty} ||DC_\varphi^u D f_n||_{H_\beta^\infty} \\ &\geqslant \limsup_{n\to\infty} \sup_{w\in\mathbb{D}} (1-|w|^2)^\beta |DC_\varphi^u D f_n(w)| \\ &\geqslant \lim_{n\to\infty} \sup_{\varphi(w)\in A_n} |u(w)||\varphi'(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+1}} (1-|\varphi(w)|^2)^{\alpha+1} |f_n''(\varphi(w))| \\ &\quad - \lim_{n\to\infty} \sup_{\varphi(w)\in A_n} |u'(w)|(1-|w|^2)^\beta |f_n'(\varphi(w))|.\end{aligned}$$

We know that $u' \in H_\beta^\infty$, then, for $0 < \alpha < 1$,

$$\begin{aligned}&\lim_{n\to\infty} \sup_{\varphi(w)\in A_n} |u'(w)|(1-|w|^2)^\beta |f_n'(\varphi(w))| \\ &\leqslant ||u'||_{H_\beta^\infty} \lim_{n\to\infty} \sup_{\varphi(w)\in A_n} |f_n'(\varphi(w))| \\ &= ||u'||_{H_\beta^\infty} \lim_{n\to\infty} \frac{\left(\frac{n}{n+2\alpha}\right)^{\frac{n-1}{2}}}{n\left(\frac{2\alpha}{n-1+2\alpha}\right)^\alpha \left(\frac{n-1}{n-1+2\alpha}\right)^{\frac{n-1}{2}}} \\ &= 0.\end{aligned}$$

When $\alpha = 1$, we have

$$\begin{aligned}\lim_{n\to\infty} \sup_{\varphi(w)\in A_n} |u'(w)|(1-|w|^2)^\beta |f_n'(\varphi(w))| &\leqslant C \lim_{n\to\infty} \sup_{\varphi(w)\in A_n} (1-|\varphi(w)|^2)^\alpha |f_n'(\varphi(w))| \\ &\leqslant \frac{C}{2} \lim_{n\to\infty} \sup_{\varphi(w)\in A_n} (1-|\varphi(w)|^2)^\alpha = 0.\end{aligned}$$

Therefore, for $0 < \alpha \leqslant 1$,

$$\begin{aligned}||DC_\varphi^u D||_e \geqslant &\lim_{n\to\infty} \sup_{\varphi(w)\in A_n} |u(w)||\varphi'(w)| \frac{(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+1}} \\ &\times \min_{\varphi(w)\in A_n} (1-|\varphi(w)|^2)^{\alpha+1} |f_n''(\varphi(w))|,\end{aligned}$$

where the minimum is attained at anypoint on the circle with radius r_{n+1}. Because

$$\lim_{n\to\infty} \sup_{\varphi(w)\in A_n} \min_{\varphi(w)\in A_n} (1-|\varphi(w)|^2)^{\alpha+1}|f_n''(\varphi(w))| = 1,$$

we get

$$||DC_\varphi^u D||_e \geqslant \lim_{t\to 1} \sup_{|\varphi(w)|>t} \frac{|u(w)||\varphi'(w)|(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+1}}.$$

Now, we are going to give the upper estimate. Let $\{L_n\}$ be the sequence of operators given in Lemma 3. Since each L_n is compact as an operator from B^α to B^α, $DC_\varphi^u DL_n : B^\alpha \to H_\beta^\infty$ is also compact and we have

$$\begin{aligned} ||DC_\varphi^u D||_e &\leqslant ||DC_\varphi^u D - DC_\varphi^u DL_n|| = ||DC_\varphi^u D(I-L_n)|| \\ &= \sup_{||f||_{B^\alpha}\leqslant 1} ||DC_\varphi^u D(I-L_n)f||_{H_\beta^\infty} \\ &\leqslant \sup_{||f||_{B^\alpha}\leqslant 1} \sup_{w\in\mathbb{D}} |u'(w)||((I-L_n)f)'(\varphi(w))|(1-|w|^2)^\beta \\ &\quad + \sup_{||f||_{B^\alpha}\leqslant 1} \sup_{w\in\mathbb{D}} |u(w)||((I-L_n)f)''(\varphi(w))||\varphi'(w)|(1-|w|^2)^\beta, \end{aligned}$$

using Lemma 3,

$$\sup_{||f||_{B^\alpha}\leqslant 1} \sup_{w\in\mathbb{D}} |u'(w)||((I-L_n)f)'(\varphi(w))|(1-|w|^2)^\beta = 0.$$

Now, we need only consider the term

$$\sup_{||f||_{B^\alpha}\leqslant 1} \sup_{w\in\mathbb{D}} |u(w)||((I-L_n)f)''(\varphi(w))||\varphi'(w)|(1-|w|^2)^\beta.$$

For arbitrary $0<t<1$, consider

$$\sup_{||f||_{B^\alpha}\leqslant 1} \sup_{|\varphi(w)|\leqslant t} |u(w)|(1-|w|^2)^\beta \big|\big((I-L_n)f\big)''(\varphi(w))\big||\varphi'(w)| \tag{22}$$

and

$$\sup_{||f||_{B^\alpha}\leqslant 1} \sup_{|\varphi(w)|>t} |u(w)|(1-|w|^2)^\beta \big|\big((I-L_n)f\big)''(\varphi(w))\big||\varphi'(w)|. \tag{23}$$

Since, $DC_\varphi^u D$ is bounded from B^α into H_β^∞, by Theorem 1,

$$\sup_{w\in\mathbb{D}} \frac{(1-|w|^2)^\beta|u(w)||\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}} < \infty.$$

Hence,

$$\sup_{w\in\mathbb{D}} (1-|w|^2)^\beta|u(w)||\varphi'(w)| < \infty.$$

Thus, from (22) and using Cauchy's estimate in proof of Lemma 3,

$$\sup_{\|f\|_{B^\alpha}\leqslant 1}\sup_{|\varphi(w)|\leqslant t}|u(w)|(1-|w|^2)^\beta\left|\big((I-L_n)f\big)''(\varphi(w))\right||\varphi'(w)|=0. \tag{24}$$

From (23),

$$\sup_{\|f\|_{B^\alpha}\leqslant 1}\sup_{|\varphi(w)|>t}|u(w)|(1-|w|^2)^\beta\left|\big((I-L_n)f\big)''(\varphi(w))\right||\varphi'(w)|$$
$$\leqslant \|I-L_n\|\sup_{|\varphi(w)|>t}|u(w)|\frac{(1-|w|^2)^\beta|\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}}.$$

Thus, by (iii) of Lemma 3,

$$\limsup_{n\to\infty}\sup_{\|f\|_{B^\alpha}\leqslant 1}\sup_{|\varphi(w)|>t}(1-|w|^2)^\beta\left|\big((I-L_n)f\big)'(\varphi(w))\right||\varphi'(w)|$$
$$\leqslant \sup_{|\varphi(w)|>t}|u(w)|\frac{(1-|w|^2)^\beta|\varphi'(w)|}{(1-|\varphi(w)|^2)^{\alpha+1}}. \tag{25}$$

By using (24) and (25) as $n\to\infty$, we obtain

$$\|DC_\varphi^u D\|_e \leqslant \sup_{|\varphi(w)|>t}\frac{|u(w)||\varphi'(w)|(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+1}}.$$

Since, t was arbitrary, so,

$$\|DC_\varphi^u D\|_e \leqslant \lim_{t\to 1}\sup_{|\varphi(w)|>t}\frac{|u(w)||\varphi'(w)|(1-|w|^2)^\beta}{(1-|\varphi(w)|^2)^{\alpha+1}}.$$

The proof of the theorem is completed. □

Acknowledgements. The authors would like to express their sincere gratitude to the referee for a very careful reading of the article and for all the valuable comments and suggestions, which led to improvement in this article.

REFERENCES

[1] F. COLONNA AND S. LI, *Weighted composition operator from the Besov spaces to the Bloch spaces*, Bull Malaysian Sci. Soc. **36**, 4 (2013), 1027–1039.

[2] C. C. COWEN AND B. D. MACCLUER, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1995.

[3] X. C. GUI AND Z. H. ZHOU, *New characterizations for weighted composition operator from Zygmund type spaces to Bloch type spaces*, Czechoslovak Math. J. **65**, 2 (2015), 331–346.

[4] M. HASSANLOU, H. VAEZI AND M. WANG, *Weighted composition operators on weak vector-valued Bergman spaces and Hardy spaces*, Banach J. Math. Anal. **9**, 2 (2015), 35–43.

[5] R. A. HIBSCHWEILER AND N. PORTNOY, *Composition by differentiation between Bergman and Hardy spaces*, Can. J. Math. **38** (1986), 878–906.

[6] S. LI, E. ABBASI AND H. VAEZI, *Weighted composition operators from Bloch-type spaces to nth weighted-type spaces*, Ann. Polon. Math. **124**, 1 (2020), 93–107.

[7] S. LI AND S. STEVIC, *Composition followed by differeniation between Bloch-type spaces*, J. Comput. Anal. and Appl. **9**, 2 (2007), 195–205.
[8] J. LONG, C. QIU AND P. WU, *Weighted composition followed and proceeded by differentiation operators from Zygmund spaces to Bloch-type spaces*, J. Inequal. Appl. **2014** (2014), 12 pages.
[9] B. D. MACCLUER AND R. ZHAO, *Essential norms of weighted composition operators between Bloch-type spaces*, Rocky mountain J. Math. **33**, 4 (2003), 1437–1458.
[10] J. S. MANHAS AND R. ZHAO, *Products of weighted composition and differentiation operators into weighted Zygmund and Bloch spaces*, Acta Math. Sci. **38**, 4 (2018), 1105–1120.
[11] S. OHNO, *Products of composition and differentiation between Hardy spaces*, Bull. Austral. Math. Soc. **73**, 2 (2006), 235–243.
[12] S. OHNO, K. STROETHOFF AND R. ZHAO, *Weighted composition operators between Bloch-type spaces*, Rocky mountain J. Math. **33**, 1 (2003), 191–215.
[13] A. M. RODRIGUEZ, *The essential norm of a composition operator on Bloch spaces*, Pacific J. Math. **188**, 2 (1999), 339–351.
[14] A. SALARYAN AND H. VAEZI, *Adjoints of rationally induced weighted composition operators on the Hardy, Bergman and Dirichlet spaces*, Period. Math. Hung. **72** (2016), 76–89.
[15] A. H. SANATPOUR AND M. HASSANLOU, *Essential norms of weighted composition operators between Zygmund-type spaces and Bloch-type spaces*, Turk. J. Math. **38** (2014), 872–882.
[16] J. H. SHAPIRO, *Composition operators and classical function theory*, New York, Springer Verlag, 1993.
[17] S. STEVIC, *Essential norms of weighted composition operators from the α-Bloch space to a weighted-type space on the unit ball*, Abstr. Appl. Anal. **2008** (2008), 11 pages.
[18] H. WEIXIANG AND J. LIJIAN, *Composition operator on Bers-type spaces*, Acta. Math. Sci. **22B**, 3 (2002), 404–412.
[19] S. YAMAJI, *Composition operator on the Bergman spaces of a minimal bounded homogeneous domain*, Hiroshima Math. J. **43**, 1 (2013), 107–128.
[20] R. ZHAO, *Essential norms of composition operators between Bloch-type spaces*, Proc. Amer. Math. Soc. **138**, 7 (2010), 2537–2546.
[21] X. ZHU, *Bloch type spaces of analytic functions*, Rocky mountain J. Math. **23**, 3 (1993), 1143–1177.
[22] X. ZHU, *Generalized weighted composition operators from Bloch spaces into Bers-type spaces*, Filomat **26**, 6 (2012), 1163–1169.

(Received June 4, 2019)

Hamid Vaezi
Department of pure Mathematics
Faculty of Mathematical Sciences
University of Tabriz, Tabriz, Iran
e-mail: hvaezi@tabrizu.ac.ir

Mohamad Naghlisar
Department of pure Mathematics
Faculty of Mathematical Sciences
University of Tabriz
Tabriz, Iran
e-mail: m.naghlisar@tabrizu.ac.ir