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CHARACTERIZATIONS AND REPRESENTATIONS FOR THE DRAZIN INVERSE OF ANTI–TRIANGULAR BLOCK OPERATOR MATRICES WITH INDEX LESS THAN TWO

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Abstract. In the 1980s, Campbell proposed a problem to find an expression of the Drazin inverse for the block matrix $\begin{pmatrix} A & B \\ -I & 0 \end{pmatrix}$ to research on singular differential equations. In this paper, some characterizations and detail representations for the Drazin inverse of anti-triangular block operator matrices $M = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}$ with $ind(M) = 1$ (resp. $ind(M) = 2$) are given.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be infinite dimensional complex Hilbert spaces. We denote the set of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$ by $\mathscr{B}(\mathscr{H},\mathscr{K})$ and by $\mathscr{B}(\mathscr{H})$ when $\mathscr{H} = \mathscr{K}$. For $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$, let A^*, $\mathscr{R}(A)$ and $\mathscr{N}(A)$ be the adjoint, the range and the null space of A, respectively. An operator $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is regular if there is an operator $X \in \mathscr{B}(\mathscr{K},\mathscr{H})$ such that $AXA = A$. It is well-know that A is regular if and only if A has closed range. The notation of $\oplus$ is used in this paper with following means. For any Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, we let

$$\mathscr{H}_1 \oplus \mathscr{H}_2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} | x_i \in \mathscr{H}_i, i = 1,2 \right\}.$$

The Moore-Penrose inverse of $A \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is the operator $X \in \mathscr{B}(\mathscr{K},\mathscr{H})$ (unique when it exists) satisfying the following equations:

$$(i)\ AXA = A, \quad (ii)\ XAX = X, \quad (iii)\ (AX)^* = AX, \quad (iv)\ (XA)^* = XA.$$

Denote by $A^\dagger = X$ and $A^\dagger$ exists if and only if $\mathscr{R}(A)$ is closed. An operator $A \in \mathscr{B}(\mathscr{H})$ is called Drazin invertible, if there exist $X \in \mathscr{B}(\mathscr{H})$ and non-negative integer k satisfying:

$$(i)\ XAX = X, \quad (ii)\ AX = XA, \quad (iii)\ A^{k+1}X = A^k.$$

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The index of A, denoted by $ind(A)$, is the smallest k such that $A^{k+1}A^D = A^k$, in the case when such k exists. $ind(A) = 0$ if and only if A is invertible. The Drazin inverse X of A is unique (if it exists) and is denoted by $X = A^D$ [2]. When $ind(A) = 1$, the Drazin inverse X is called the group inverse of A and is denoted by $X = A^{\#}$.

The Drazin inverse has widely applications in many fields such as singular differential equations, singular difference equations, Markov chains, iterative methods, structured matrices and perturbation bounds for the relative eigenvalue problem can be found in [2–7, 11, 15, 17, 21, 28].

In 1979, Campbell and Meyer in [5] proposed an open problem to find an explicit expression for the Drazin inverse of block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where the blocks A and D are assumed to be square. But until now this open problem has not been solved yet even for the case $D = 0$. Owing to the difficulty of the problem itself, there were few results given with some conditions by many authors, In 2005, Castro-González in [8] gave a representation of $(A+B)^D$ with $A^D B = 0$, $AB^D = 0$ and $B^{\pi}ABA^{\pi} = 0$. Castro-González and Dopazo in [9] gave the expression of the Drazin inverse for $\begin{pmatrix} I & I \\ E & 0 \end{pmatrix}$ with E square. Deng and Wei in [14] gave some results on the Drazin inverse of an anti-triangular matrix on Banach spaces. In 2011, Bu et al. [3] gave a representation of Drazin inverse of $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ under the condition that $EF = FE$. In 2018, Xu et al. [27] gave the expression of the Drazin inverse of $M = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}$ with B having closed range and $ind(M) \leqslant 2$. It is very difficult to give the representations of M^D with $ind(M) \leqslant k$.

In this paper, we only consider the cases that $ind(M) = 1$ or $ind(M) = 2$. We used the methods of space decomposition to obtain the necessary and sufficient conditions for the existence of M^D and obtain the detail expressions of M^D. The main results of the paper are Theorems 2.4, 2.6, 3.1 and 3.2. Let

$$M = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H}), \tag{1}$$

where $A, B \in \mathscr{B}(\mathscr{H})$ and I being the identity operator on $\mathscr{B}(\mathscr{H})$.

2. Some lemmas and basic propositions

LEMMA 2.1. *If $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ has closed range, then by considering*

$$\mathscr{H} = \mathscr{R}(A^*) \oplus \mathscr{N}(A), \quad \mathscr{K} = \mathscr{R}(A) \oplus \mathscr{N}(A^*),$$

we obtain

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{pmatrix}, \tag{2}$$

where $A_1 : \mathscr{R}(A^*) \to \mathscr{R}(A)$ is invertible. Therefore, the Moore-Penrose inverse of A

can be represented as

$$A^{\dagger} = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(A) \\ \mathscr{N}(A^*) \end{pmatrix} \rightarrow \begin{pmatrix} \mathscr{R}(A^*) \\ \mathscr{N}(A) \end{pmatrix}. \tag{3}$$

LEMMA 2.2. [1, Proposition 2.3] *Let A, B, S and $T \in \mathscr{B}(\mathscr{H})$ such that S, T are invertible and $A = SBT$. Then the following assertions hold.*

(i) *$\mathscr{R}(A)$ is closed $\Longleftrightarrow \mathscr{R}(B)$ is closed.*

(ii) *$\mathscr{N}(A) = \mathscr{N}(BT)$ and $\mathscr{R}(A) = \mathscr{R}(SB)$.*

(iii) *$\dim \mathscr{N}(A) = \dim \mathscr{N}(B)$, $\operatorname{codim}\mathscr{R}(A) = \operatorname{codim}\mathscr{R}(B)$ and $ind(A) = ind(B)$.*

(iv) *A is injective (resp. surjective) $\Longleftrightarrow B$ is injective (resp. surjective).*

THEOREM 2.1. *Let M be defined by (1). If $M^{\#}$ exists, then $\mathscr{R}(B)$ is closed. If M is Drazin invertible with $ind(M) \leqslant 2$, then the range of $T =: \begin{pmatrix} B & 0 \\ A & B \end{pmatrix}$ is closed. If $\mathscr{R}(B)$ is closed, then $\mathscr{R}(T)$ is closed if and only if $\mathscr{R}((I - BB^{\dagger})A(I - B^{\dagger}B))$ is closed.*

Proof. If $M^{\#}$ exists, then $\mathscr{R}(M)$ is closed. Since

$$\begin{pmatrix} 0 & I \\ I & -A \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix},$$

by Lemma 2.2, we know that $\mathscr{R}(M)$ is closed if and only if $\mathscr{R}(B)$ is closed. If M^D exists with $ind(M) \leqslant 2$, then $\mathscr{R}(M^2)$ is closed. Since

$$\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^2 = \begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} \begin{pmatrix} A^2 + B & AB \\ A & B \end{pmatrix} = \begin{pmatrix} B & 0 \\ A & B \end{pmatrix},$$

by Lemma 2.2, we know that $\mathscr{R}(M^2)$ is closed if and only if $\mathscr{R}\left(\begin{pmatrix} B & 0 \\ A & B \end{pmatrix}\right)$ is closed. If $\mathscr{R}(B)$ is closed, then $B^{\dagger}$ exists. Similarly, it is easy to get $\mathscr{R}(M^2)$ is closed if and only if $\mathscr{R}\left((I - BB^{\dagger})A(I - B^{\dagger}B)\right)$ is closed. $\square$

Theorem 2.1 shows that $\mathscr{R}(B)$ is closed, which is a necessary condition for existence of group inverse of operator M. Hence, we always suppose that $\mathscr{R}(B)$ is closed in further research. The following result can be verified directly.

THEOREM 2.2. *Let M be defined by (1), $s, t, x, y \in \mathscr{H}$. Then the following statements hold:*

(i) *$(x \quad y)^T \in \mathscr{N}(M^3)$ if and only if*

$$B(Ax + By) = 0, \quad A(Ax + By) + Bx = 0. \tag{4}$$

(ii) *$(x \quad y)^T \in \mathscr{N}(M^2)$ if and only if*

$$Bx = 0, \quad Ax + By = 0. \tag{5}$$

(iii) $M^3(s \quad t)^T = M^2(x \quad y)^T$ *if and only if*

$$B(As+Bt) = Bx, \quad A(As+Bt)+Bs = Ax+By. \tag{6}$$

Let

$$N = \begin{pmatrix} A & I \\ B & 0 \end{pmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H}), \tag{7}$$

where $A, B \in \mathscr{B}(\mathscr{H})$ and I being the identity operator on $\mathscr{B}(\mathscr{H})$.

THEOREM 2.3. *Let M and N be defined by (1) and (7), respectively. Then M is Drazin invertible if and only if N is Drazin invertible. In this case, $ind(M) = ind(N)$ and*

$$N^D = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \left(M^D\right)^2 \begin{pmatrix} A & I \\ I & 0 \end{pmatrix}.$$

Proof. Clearly,

$$M = \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \quad N = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & I \\ I & 0 \end{pmatrix}.$$

By Cline formula [12], if M (resp. N) is Drazin invertible, then N (resp. M) is Drazin invertible and

$$N^D = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \left(M^D\right)^2 \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} \quad \left(resp.\ M^D = \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} \left(N^D\right)^2 \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}\right).$$

If $ind(M) = k$, then

$$\begin{aligned} N^{k+2}N^D &= N^{k+2} \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \left(M^D\right)^2 \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} M^{k+2} \left(M^D\right)^2 \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} M^k \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} = N^{k+1}. \end{aligned}$$

Hence, $ind(N) \leqslant k+1$. Note that

$$M \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} A & I \\ I & 0 \end{pmatrix} N = \begin{pmatrix} A^2+B & A \\ A & I \end{pmatrix}.$$

M and N are similar and $ind(M) = ind(N) = k$. $\square$

Let M be defined by (1). It is clear that M is invertible if B is invertible.

THEOREM 2.4. *Let M be defined by (1) with B being not invertible. Then $ind(M) = 1$ if and only if $\mathscr{R}(B)$ is closed,*

$$\mathscr{N}(B) = \mathscr{R}\left((I - B^\dagger B)A^*\right) \quad \textit{and} \quad \mathscr{N}(B^*) = \mathscr{R}\left((I - BB^\dagger)A\right).$$

Proof. Necessity. By Theorems 2.1 and 2.3, if $ind(M) = 1$, then $\mathscr{R}(B)$ is closed and M, N are group invertible simultaneously. The group invertibility of N implies that

$$\begin{aligned}\mathscr{R}(N) = \mathscr{R}(N^2) &= \mathscr{R}\left(N\begin{pmatrix} I & 0 \\ 0 & B\end{pmatrix}\begin{pmatrix} A & I \\ I & 0\end{pmatrix}\right) = \mathscr{R}\left(N\begin{pmatrix} I & 0 \\ 0 & B\end{pmatrix}\right)\\ &= \mathscr{R}\left(\begin{pmatrix} A & B \\ B & 0\end{pmatrix}\right) = \mathscr{R}\left(\begin{pmatrix} A & I \\ B & 0\end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B\end{pmatrix}\right) \subset \mathscr{R}(N).\end{aligned}$$

Thus,

$$\mathscr{R}(N) = \mathscr{R}\left(\begin{pmatrix} A & B \\ B & 0\end{pmatrix}\right) = \mathscr{R}\left(\begin{pmatrix} B & A \\ 0 & B\end{pmatrix}\right) = \mathscr{R}(B) \oplus \mathscr{R}\left((I - BB^\dagger)A\right) \oplus \mathscr{R}(B).$$

Note that

$$\mathscr{R}(N) = \mathscr{R}\left(N\begin{pmatrix} 0 & I \\ I & -A\end{pmatrix}\right) = \mathscr{R}\left(\begin{pmatrix} I & 0 \\ 0 & B\end{pmatrix}\right) = \mathscr{H} \oplus \mathscr{R}(B)$$

and $\mathscr{H} = \mathscr{R}(B) \oplus \mathscr{N}(B^*)$ one derives that $\mathscr{N}(B^*) = \mathscr{R}\left((I - BB^\dagger)A\right)$. By the same way one can show that $\mathscr{N}(B) = \mathscr{R}\left((I - B^\dagger B)A^*\right)$.

Sufficiency. By first step, if $\mathscr{N}(B^*) = \mathscr{R}\left((I - BB^\dagger)A\right)$, then

$$\mathscr{R}(N) = \mathscr{R}\left(\begin{pmatrix} A & B \\ B & 0\end{pmatrix}\right) = \mathscr{R}\left(N\begin{pmatrix} I & 0 \\ 0 & B\end{pmatrix}\right).$$

Hence,

$$\begin{aligned}\mathscr{R}(M) &= \mathscr{R}\left(M\begin{pmatrix} A & I \\ I & 0\end{pmatrix}\right) = \mathscr{R}\left(\begin{pmatrix} A & I \\ I & 0\end{pmatrix}N\right) = \begin{pmatrix} A & I \\ I & 0\end{pmatrix}\mathscr{R}(N)\\ &= \begin{pmatrix} A & I \\ I & 0\end{pmatrix}\mathscr{R}\left(N\begin{pmatrix} I & 0 \\ 0 & B\end{pmatrix}\right) = \mathscr{R}\left(M^2\right).\end{aligned}$$

On the other hand,

$$\mathscr{R}(M^*) = \mathscr{R}\left(M^*\begin{pmatrix} 0 & I \\ I & -A^*\end{pmatrix}\right) = \mathscr{R}\left(\begin{pmatrix} I & 0 \\ 0 & B^*\end{pmatrix}\right) = \mathscr{H} \oplus \mathscr{R}(B^*)$$

and

$$\begin{aligned}\mathscr{R}\left(M^*\begin{pmatrix} I & 0 \\ 0 & B^*\end{pmatrix}\right) &= \mathscr{R}\left(\begin{pmatrix} A^* & B^* \\ B^* & 0\end{pmatrix}\right) = \mathscr{R}\left(\begin{pmatrix} B^* & A^* \\ 0 & B^*\end{pmatrix}\right)\\ &= \mathscr{R}(B^*) \oplus \mathscr{R}\left((I - B^\dagger B)A^*\right) \oplus \mathscr{R}(B^*).\end{aligned}$$

If $\mathscr{N}(B) = \mathscr{R}\left((I - B^\dagger B)A^*\right)$, then

$$\begin{aligned}\mathscr{R}(M^*) &= \mathscr{R}\left(M^*\begin{pmatrix} I & 0 \\ 0 & B^*\end{pmatrix}\right) = \begin{pmatrix} I & 0 \\ 0 & B^*\end{pmatrix}\mathscr{R}(N^*)\\ &= \begin{pmatrix} I & 0 \\ 0 & B^*\end{pmatrix}\mathscr{R}\left(N^*\begin{pmatrix} A^* & I \\ I & 0\end{pmatrix}\right) = \mathscr{R}((M^*)^2).\end{aligned}$$

Thus $\mathscr{N}(M) = \mathscr{R}(M^*)^\perp = \mathscr{R}((M^*)^2)^\perp = \mathscr{N}(M^2)$ which implies $ind(M) \leqslant 1$. Since B is not invertible, M is not invertible and $ind(M) = 1$. $\square$

Let M be defined by (1) such that $\mathscr{R}(B)$ is closed. For convenience, put

$$\begin{aligned} F &= (I - BB^\dagger)A + B, \quad F_2 = (I - BB^\dagger)A(I - B^\dagger B) + B, \\ F_1 &= A(I - B^\dagger B) + B, \quad F_3 = (I - BB^\dagger)A(I - B^\dagger B), \\ R &= [BB^\dagger A - (I - BB^\dagger)](I - F^\dagger F). \end{aligned} \tag{8}$$

If M is Drazin invertible with $ind(M) \leqslant 2$, then we denote M^D by

$$M^D = \begin{pmatrix} P & Q \\ X & Y \end{pmatrix} \tag{9}$$

and

$$M^D M = \begin{pmatrix} PA+Q & PB \\ XA+Y & XB \end{pmatrix} = \begin{pmatrix} AP+BX & AQ+BY \\ P & Q \end{pmatrix} = MM^D. \tag{10}$$

From $M^2(M^D M) = M^2$ ($ind(M) \leqslant 2$) and $M^2 M^\# = M$ ($ind(M) = 1$) one gets

$$\begin{cases} \left(A^2+B\right)(PA+Q) + ABP = A^2 + B, \\ \left(A^2+B\right)PB + ABQ = AB, \\ A(PA+Q) + BP = A, \\ APB + BQ = B \end{cases} \quad \text{and} \quad \begin{cases} \left(A^2+B\right)P + ABX = A, \\ \left(A^2+B\right)Q + ABY = B, \\ AP + BX = I, \\ AQ + BY = 0, \end{cases} \tag{11}$$

respectively. From $M^D M M^D = M^D$ one gets

$$\begin{cases} (PA+Q)P + PBX = P, \\ (PA+Q)Q + PBY = Q, \\ P^2 + QX = X, \\ PQ + QY = Y. \end{cases} \tag{12}$$

THEOREM 2.5. *Let $A, B \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and B have closed range, let F, F_1 and F_2 be defined by (8). Then the following statements are equivalent:*

(i) *F is invertible (resp. has closed range),*

(ii) *F_1 is invertible (resp. has closed range),*

(iii) *F_2 is invertible (resp. has closed range).*

Proof. Since $B \in \mathscr{B}(\mathscr{H})$ has closed range, by (2), B has following form:

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{pmatrix}, \tag{13}$$

where $B_1 : \mathscr{R}(B^*) \to \mathscr{R}(B)$ is invertible. Let

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{pmatrix}. \tag{14}$$

Then

$$F = \begin{pmatrix} B_1 & 0 \\ A_3 & A_4 \end{pmatrix}, \quad F_1 = \begin{pmatrix} B_1 & A_2 \\ 0 & A_4 \end{pmatrix}, \quad F_2 = \begin{pmatrix} B_1 & 0 \\ 0 & A_4 \end{pmatrix}. \tag{15}$$

Obviously, F is invertible (resp. has closed range) is equivalent to A_4 is invertible (resp. has closed range). Therefore, (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii). $\square$

Let V_1 and V_2 be orthogonal projections. We say $V_1 \leqslant V_2$ if $V_1V_2 = V_2V_1 = V_1$.

THEOREM 2.6. *Let M be defined by (1) such that $\mathscr{R}(B)$ is closed and M^D exists, let F, F_3 and M^D be defined as (8) and (9), respectively.*

(i) *If $ind(M) = 1$, then $BP = PB = QP = 0$, $BQ = B$ and $Q^2 = Q$.*

(ii) *If $ind(M) \leqslant 2$, then $APB = BPA$, $BPB = 0$, $\mathscr{R}(F_3)$ and $\mathscr{R}(F)$ are closed. Moreover,*

$$I - F^\dagger F = (I - B^\dagger B)(I - F_3^\dagger F_3) = (I - F_3^\dagger F_3)(I - B^\dagger B) \leqslant I - B^\dagger B$$

and

$$\begin{aligned} (P - QB^\dagger A)(I - F^\dagger F) = 0, \quad (X - YB^\dagger A)(I - F^\dagger F) = 0, \\ Q(I - B^\dagger B) = 0, \quad Y(I - B^\dagger B) = 0. \end{aligned}$$

Proof. (i) If $ind(M) = 1$, from $MM^\# = M^\# M$ and $M^2M^\# = M$, we get $AP + BX = I$, $PB = AQ + BY = 0$ by (10) and (11). So,

$$(A^2 + B)P + ABX = A \Longleftrightarrow A(AP + BX) + BP = A \Longleftrightarrow BP = 0$$

and

$$(A^2 + B)Q + ABY = B \Longleftrightarrow A(AQ + BY) + BQ = B \Longleftrightarrow BQ = B.$$

Thus, by (10),

$$QP = XBP = 0, \quad Q^2 = XBQ = XB = Q.$$

(ii) If $ind(M) \leqslant 2$, by (10) and (11), one derives

$$\begin{cases} (A^2 + B)PB + ABQ = AB \\ APB + BQ = B \end{cases} \Rightarrow BPB = 0$$

and

$$\begin{cases} (A^2 + B)(PA + Q) + ABP = A^2 + B \\ A(PA + Q) + BP = A \\ APB + BQ = B \end{cases} \Rightarrow APB = BPA.$$

Since $\mathscr{R}(B)$ is closed and $ind(M) \leqslant 2$, by Theorem 2.1, one gets $F_3 = (I - BB^\dagger)A(I - B^\dagger B)$ has closed range. Hence, $\mathscr{R}(F)$ is also closed by Theorem 2.5. Using [13, Theorem 6],

$$F^\dagger = \begin{pmatrix} \Delta B_1^* & \Delta A_3^*(I - A_4A_4^\dagger) \\ -A_4^\dagger A_3 \Delta B_1^* & A_4^\dagger - A_4^\dagger A_3 \Delta A_3^*(I - A_4A_4^\dagger) \end{pmatrix},$$

where $\Delta = [B_1^*B_1 + A_3^*(I - A_4A_4^\dagger)A_3]^{-1}$. Thus

$$I - F^\dagger F = \begin{pmatrix} 0 & 0 \\ 0 & I - A_4^\dagger A_4 \end{pmatrix} = (I - B^\dagger B)(I - F_3^\dagger F_3) = (I - F_3^\dagger F_3)(I - B^\dagger B) \leqslant I - B^\dagger B.$$

By (10) one derives that $Q = XB = XBB^\dagger B = QB^\dagger B$ and $AQ + BY = PB$. We get

$$BY(I - B^\dagger B) = (PB - AQ)(I - B^\dagger B) = 0, \quad QY(I - B^\dagger B) = QB^\dagger BY(I - B^\dagger B) = 0.$$

Since $PQ + QY = Y$ by (12), one has $Y(I - B^\dagger B) = 0$. Note that $B(I - F^\dagger F) = 0$ and $(I - BB^\dagger)A(I - F^\dagger F) = 0$ implies $A(I - F^\dagger F) = BB^\dagger A(I - F^\dagger F)$, hence, by (10),

$$P(I - F^\dagger F) = (XA + Y)(I - F^\dagger F) = XA(I - F^\dagger F) = XBB^\dagger A(I - F^\dagger F) = QB^\dagger A(I - F^\dagger F)$$

and

$$\begin{aligned} &QYB^\dagger A(I - F^\dagger F) = QB^\dagger BYB^\dagger A(I - F^\dagger F) = QB^\dagger(PB - AQ)B^\dagger A(I - F^\dagger F) \\ =&QB^\dagger PA(I - F^\dagger F) - QB^\dagger AP(I - F^\dagger F) = QB^\dagger(BX - Q)(I - F^\dagger F) \\ =&QX(I - F^\dagger F). \end{aligned}$$

Hence, applying (12), we get

$$\begin{aligned} YB^\dagger A(I - F^\dagger F) &= (PQ + QY)B^\dagger A(I - F^\dagger F) \\ &= (P^2 + QX)(I - F^\dagger F) = X(I - F^\dagger F). \quad \square \end{aligned}$$

Let B and A be given by (13) and (14), respectively. Then

$$BB^\dagger A(I - F^\dagger F) = \begin{pmatrix} 0 & A_2(I - A_4^\dagger A_4) \\ 0 & 0 \end{pmatrix}.$$

By (8), we know $BB^\dagger R = BB^\dagger A(I - F^\dagger F)$. Suppose that

$$R = \begin{pmatrix} 0 & A_2(I - A_4^\dagger A_4) \\ R_3 & R_4 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{pmatrix}.$$

Observing that

$$R = R(I - F^\dagger F) = \begin{pmatrix} 0 & A_2(I - A_4^\dagger A_4) \\ R_3 & R_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I - A_4^\dagger A_4 \end{pmatrix} = \begin{pmatrix} 0 & A_2(I - A_4^\dagger A_4) \\ 0 & R_4(I - A_4^\dagger A_4) \end{pmatrix}$$

one gets

$$R = \begin{pmatrix} 0 & A_2(I - A_4^\dagger A_4) \\ 0 & R_4(I - A_4^\dagger A_4) \end{pmatrix}, \quad F + R = \begin{pmatrix} B_1 & A_2(I - A_4^\dagger A_4) \\ A_3 & A_4 + R_4(I - A_4^\dagger A_4) \end{pmatrix}. \tag{16}$$

THEOREM 2.7. *Let M (defined by (1)) be Drazin invertible with $\mathscr{R}(B)$ closed and $ind(M) \leqslant 2$, let F and R be defined by (8). If $F+R$ is invertible, then*

$$PB = (I - F^{\dagger}F)(F+R)^{-1}B, \quad PA + Q = I + (I - F^{\dagger}F)(F+R)^{-1}(I - BB^{\dagger}).$$

Proof. According to (16) and Banachiewicz-Schur form we obtain

$$(F+R)^{-1} = \begin{pmatrix} B_1^{-1} + B_1^{-1}A_2(I - A_4^{\dagger}A_4)J^{-1}A_3B_1^{-1} & -B_1^{-1}A_2(I - A_4^{\dagger}A_4)J^{-1} \\ -J^{-1}A_3B_1^{-1} & J^{-1} \end{pmatrix},$$

where $J = A_4 + R_4(I - A_4^{\dagger}A_4) - A_3B_1^{-1}A_2(I - A_4^{\dagger}A_4)$ is Schur complement of B_1. If M is Drazin invertible with $\mathscr{R}(B)$ closed and $ind(M) \leqslant 2$, then P and Q in (9) are unique. Let

$$P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{pmatrix}.$$

Then P_i are unique, $i = 1,2,3,4$. By Theorem 2.6, $BPB = 0$ and $BPA = APB$ imply that $P_1 = 0$ and

$$\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & P_2 \\ P_3 & P_4 \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} 0 & P_2 \\ P_3 & P_4 \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} B_1P_2A_3 & B_1P_2A_4 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_2P_3B_1 & 0 \\ A_4P_3B_1 & 0 \end{pmatrix}.$$

Thus $B_1P_2A_3 = A_2P_3B_1$. From (11) and Theorem 2.6, one has

$$(BQB^{\dagger} + BP)(F+R) = BPA + BQ = B$$

and so

$$BQ = BQB^{\dagger}B = B(F+R)^{-1}B, \quad BP = B(F+R)^{-1}(I - BB^{\dagger}).$$

By $BP = B(F+R)^{-1}(I - BB^{\dagger})$, one gets

$$P_2 = -B_1^{-1}A_2(I - A_4^{\dagger}A_4)J^{-1}.$$

So, $A_2P_3B_1 = B_1P_2A_3 = -A_2(I - A_4^{\dagger}A_4)J^{-1}A_3$. Since P_3 is unique,

$$P_3 = -(I - A_4^{\dagger}A_4)J^{-1}A_3B_1^{-1}.$$

Therefore,

$$PB = \begin{pmatrix} 0 & 0 \\ P_3B_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -(I - A_4^{\dagger}A_4)J^{-1}A_3 & 0 \end{pmatrix} = (I - F^{\dagger}F)(F+R)^{-1}B.$$

By (10) and (11), we obtain $A(PA+Q) = A - BP$ and $B(PA+Q) = (PA+Q)B = B$. So we might suppose

$$PA + Q = I + (I - B^{\dagger}B)\alpha(I - BB^{\dagger}), \quad \text{for some } \alpha \in \mathscr{B}(\mathscr{H}).$$

Thus,

$$A(PA+Q) = A + A(I-B^{\dagger}B)\alpha(I-BB^{\dagger}) = A - BP = A - B(F+R)^{-1}(I-BB^{\dagger})$$

implies

$$A(I-B^{\dagger}B)\alpha(I-BB^{\dagger}) = -B(F+R)^{-1}(I-BB^{\dagger}).$$

Let

$$\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^*) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(B^*) \\ \mathscr{N}(B) \end{pmatrix}.$$

By above the equation, we deduce that

$$\begin{pmatrix} 0 & A_2\alpha_4 \\ 0 & A_4\alpha_4 \end{pmatrix} = \begin{pmatrix} 0 & A_2(I-A_4^{\dagger}A_4)J^{-1} \\ 0 & 0 \end{pmatrix}.$$

Obviously, $\alpha_4 = (I-A_4^{\dagger}A_4)J^{-1}$ satisfies the above equation and

$$(I-B^{\dagger}B)\alpha(I-BB^{\dagger}) = \begin{pmatrix} 0 & 0 \\ 0 & (I-A_4^{\dagger}A_4)J^{-1} \end{pmatrix} = (I-F^{\dagger}F)(F+R)^{-1}(I-BB^{\dagger}).$$

So,

$$PA+Q = I + (I-F^{\dagger}F)(F+R)^{-1}(I-BB^{\dagger}). \quad \square$$

3. The case of $ind(M) = 1$ and $ind(M) = 2$

It is obvious that M in (1) is invertible ($ind(M) = 0$) if and only if B is invertible ($ind(B) = 0$). In [26, Theorem 2.5], the authors had proved that $M^{\#}$ exists if and only if $B \in \mathscr{B}(\mathscr{H})$ has closed range and F is invertible. The following result give a different proof and obtain the detail representation of $M^{\#}$. First we give the detail representation of $M^{\#}$.

THEOREM 3.1. *Let F, F_1 and M be defined by (8) and (1), respectively. Then $ind(M) = 1$ if and only if $B \in \mathscr{B}(\mathscr{H})$ is not invertible with closed range and F is invertible. In this case,*

$$M^{\#} = \begin{pmatrix} P & Q \\ X & Y \end{pmatrix}, \tag{17}$$

where

$$\begin{cases} P = F^{-1}(I-BB^{\dagger}), \\ Q = F^{-1}B, \\ X = P^2 + QF_1^{-1}, \\ Y = P - XA. \end{cases}$$

Proof. $\Rightarrow$ Applying Theorem 2.1, if $M^{\#}$ exists, then $B^{\dagger}$ exists and there are unique P, Q, X, Y such that $M^{\#} = \begin{pmatrix} P & Q \\ X & Y \end{pmatrix}$. According to (11), we know $AP+BX = I$. Multiplying the equation from left by $I-BB^{\dagger}$, we get $(I-BB^{\dagger})AP = I-BB^{\dagger}$. By Theorem 2.6, $BP=0$, we get

$$FP = [(I-BB^{\dagger})A+B]P = I-BB^{\dagger}.$$

By Theorem 2.4, $\mathscr{R}(F) = \mathscr{R}((I-BB^{\dagger})A+B) = \mathscr{R}(B)\oplus\mathscr{N}(B^{*}) = \mathscr{H}$. Similarly, $\mathscr{R}(F^{*}) = \mathscr{H}$. So, F is invertible and the unique solution P has the form

$$P = F^{-1}(I-BB^{\dagger}).$$

Similarly, by $AQ+BY=0$ and $BQ=B$, it imply that

$$FQ = [(I-BB^{\dagger})A+B]Q = B.$$

So Q has the form

$$Q = F^{-1}B.$$

Let A,B be given by (14),(13), respectively. Put

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^{*}) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(B^{*}) \\ \mathscr{N}(B) \end{pmatrix}. \tag{18}$$

Note that

$$P = F^{-1}(I-BB^{\dagger}) = \begin{pmatrix} 0 & 0 \\ 0 & A_4^{-1} \end{pmatrix} : \begin{pmatrix} \mathscr{R}(B) \\ \mathscr{N}(B^{*}) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(B^{*}) \\ \mathscr{N}(B) \end{pmatrix} \tag{19}$$

and

$$Q = F^{-1}B = \begin{pmatrix} I & 0 \\ -A_4^{-1}A_3 & 0 \end{pmatrix} : \begin{pmatrix} \mathscr{R}(B^{*}) \\ \mathscr{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathscr{R}(B^{*}) \\ \mathscr{N}(B) \end{pmatrix}. \tag{20}$$

From $XB=Q$ in (10) we get $\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -A_4^{-1}A_3 & 0 \end{pmatrix}$. Comparing the two sides of the above equation, we obtain $X_1 = B_1^{-1}$ and $X_3 = -A_4^{-1}A_3B_1^{-1}$. From $AP+BX=I$ in (11) one has

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & A_4^{-1} \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1^{-1} & X_2 \\ -A_4^{-1}A_3B_1^{-1} & X_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

which implies that $X_2 = -B_1^{-1}A_2A_4^{-1}$. Thus

$$\begin{aligned}
&QXA(I-B^{\dagger}B) \\
&= \begin{pmatrix} I & 0 \\ -A_4^{-1}A_3 & 0 \end{pmatrix}\begin{pmatrix} B_1^{-1} & -B_1^{-1}A_2A_4^{-1} \\ -A_4^{-1}A_3B_1^{-1} & X_4 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ -A_4^{-1}A_3 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & -A_4^{-1}A_3B_1^{-1}A_2 + X_4A_4 \end{pmatrix} \\
&= 0.
\end{aligned}$$

Since $QXB = Q^2 = Q$ by Theorem 2.6, $QXF_1 = QX[A(I-B^{\dagger}B)+B] = Q$ and $QX = QF_1^{-1}$. From (12), $P^2 + QX = X$. So,

$$X = P^2 + QF_1^{-1}.$$

At last, by (10), one has $Y = P - XA$.

$\Leftarrow$ It is clear. $\square$

REMARK. Let

$$E = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathscr{B}(\mathscr{H}_1 \oplus \mathscr{H}_2).$$

$T = (I-BB^{\dagger})A(I-C^{\dagger}C)$ and $G = [BB^{\dagger}A-(I-BB^{\dagger})](I-B^{\dagger}B)(I-T^{\dagger}T)$. In [26, Theorem 2.5], the authors state that, if $\mathscr{R}(B)$, $\mathscr{R}(C)$ and $\mathscr{R}(T)$ are closed, then E is group invertible $\Longleftrightarrow$ $F+G$ is invertible.

In fact, in the case of $C = I$ (this moment $\mathscr{H}_1 = \mathscr{H}_2$ and $G = T = 0$), according to Theorem 2.1, $M^{\#}$ exists is the sufficient condition of $\mathscr{R}(B)$ is closed. Moreover, compare Theorem 3.1 with the results of the [26, Theorem 2.5], we can see that our result is more concisely than the expression of [26, Theorem 2.5] under the condition of $C = I$. Appling Theorem 2.5, one has a different representation of $M^{\#}$.

COROLLARY 3.1. *Let M and F_2 be defined by (1) and (8), respectively. Then $ind(M) = 1$ if and only if $B \in \mathscr{B}(\mathscr{H})$ is not invertible with closed range and F_2 is invertible. In this case,*

$$M^{\#} = \begin{pmatrix} P & Q \\ X & Y \end{pmatrix},$$

where

$$\begin{cases} P = F_2^{-1}(I-BB^{\dagger}), \\ Q = F_2^{-1}[B-(I-BB^{\dagger})AB^{\dagger}B], \\ X = P^2 + QF_2^{-1}, \\ Y = P - XA. \end{cases}$$

Next we give the detail representation of M^D with $ind(M) \leqslant 2$.

THEOREM 3.2. *Let M be defined by (1) such that $\mathscr{R}(B)$ is closed, let F, F_3 and R be defined by (8). Then $ind(M) \leqslant 2$ if and only if $\mathscr{R}(F)$ is closed and $F+R$ is invertible. In this case,*

$$M^D = \begin{pmatrix} P & Q \\ X & Y \end{pmatrix},$$

where

$$\begin{cases} \Delta_1 = (I - F^\dagger F)(F+R)^{-1}B, \\ \Delta_2 = I + (I - F^\dagger F)(F+R)^{-1}(I - BB^\dagger), \\ P = [\Delta_2 + \Delta_1 B^\dagger (I - F^\dagger F - A)](F+R)^{-1}(I - BB^\dagger) + \Delta_1 B^\dagger, \\ Q = [\Delta_2 + \Delta_1 B^\dagger (I - F^\dagger F - A)](F+R)^{-1}B, \\ X = [P + QB^\dagger (I - F^\dagger F - A)](F+R)^{-1}(I - BB^\dagger) + QB^\dagger, \\ Y = P - XA. \end{cases}$$

Proof. $\Longrightarrow$ By Theorem 2.6, $\mathscr{R}(F_3)$ and $\mathscr{R}(F)$ are closed, we first prove that $F+R$ is invertible.

Injection If $x \in \mathscr{N}(F+R)$, we need to prove that $x = 0$. Assume $x = (x_1 \quad x_2)^T \in (\mathscr{R}(B^*) \quad \mathscr{N}(B))^T$. Since $(BQB^\dagger + BP)(F+R) = B$, $Bx = 0$ implies $x_1 = 0$. By (16) one derives $(F+R)x = \begin{pmatrix} B_1 & A_2(I - A_4^\dagger A_4) \\ A_3 & A_4 + R_4(I - A_4^\dagger A_4) \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_2(I - A_4^\dagger A_4)x_2 \\ [A_4 + R_4(I - A_4^\dagger A_4)]x_2 \end{pmatrix} = 0$. Hence, $A(I - F^\dagger F)x = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I - A_4^\dagger A_4 \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_2(I - A_4^\dagger A_4)x_2 \\ 0 \end{pmatrix} = 0$ implies

$$\begin{aligned} (F+R)x &= (I - BB^\dagger)Ax - (I - BB^\dagger)(I - F^\dagger F)x \\ &= Ax + B[B^\dagger (I - F^\dagger F) - B^\dagger A]x - (I - F^\dagger F)x \\ &= 0. \end{aligned}$$

Let $y = [B^\dagger (I - F^\dagger F) - B^\dagger A]x$. We observe that $(x \quad y)^T$ satisfy (4), i.e., $(x \quad y)^T \in \mathscr{N}(M^3)$. Since $ind(M) \leqslant 2$, $(x \quad y)^T \in \mathscr{N}(M^2)$ and then $Ax + By = 0$ by (5). Therefore, $(I - F^\dagger F)x = 0$ and yield $(I - A_4^\dagger A_4)x_2 = 0$. By $[A_4 + R_4(I - A_4^\dagger A_4)]x_2 = 0$ we get $A_4 x_2 = 0$ and so $x_2 = 0$, i.e., $x = 0$.

Surjection (see the proof in [27]) Since $ind(M) \leqslant 2$, $\mathscr{R}(M^2) = \mathscr{R}(M^3)$. Thus for any $x, y \in \mathscr{H}$, there exists $s, t \in \mathscr{H}$ such that (6) holds. From $B(As + Bt) = Bx$ one has $h \in \mathscr{H}$ such that

$$As + Bt = (I - B^\dagger B)h + x. \tag{21}$$

So, $Ax + By = A(As + Bt) + Bs = A[(I - B^\dagger B)h + x] + Bs$. One gets

$$A(I - B^\dagger B)h = By - Bs. \tag{22}$$

Multiplying (22) from left by $(I - BB^\dagger)$, we get $F_3 h = (I - BB^\dagger)A(I - B^\dagger B)h = 0$. So, there exists $m \in \mathscr{H}$ such that

$$h = (I - F_3^\dagger F_3)m. \tag{23}$$

Note that $I - F^\dagger F = (I - B^\dagger B)(I - F_3^\dagger F_3)$. We combine (21) and (23) to get $As + Bt = (I - F^\dagger F)m + x$. Multiplying the equation from left by $(I - BB^\dagger)$, we obtain

$$(I - BB^\dagger)As - (I - BB^\dagger)(I - F^\dagger F)m = (I - BB^\dagger)x. \tag{24}$$

Similarly, by (22) and (23) we obtain

$$Bs + BB^{\dagger}A(I - F^{\dagger}F)m = By. \tag{25}$$

By (24) and (25), we get

$$By + (I - BB^{\dagger})x = Fs + Rm = (F + R)[F^{\dagger}Fs + (I - F^{\dagger}F)m].$$

Notice that x, y are arbitrary and $\mathscr{H} = \{By + (I - BB^{\dagger})x : \forall x, y \in \mathscr{H}\}$. So $F + R$ is surjective. In conclusion, $F + R$ is invertible.

We now giving the expression of M^D. Denote

$$\Delta_1 = PB = (I - F^{\dagger}F)(F + R)^{-1}B,$$

$$\Delta_2 = PA + Q = I + (I - F^{\dagger}F)(F + R)^{-1}(I - BB^{\dagger}).$$

By Theorem 2.6 one has

$$\begin{aligned}
&[P(I - BB^{\dagger}) + QB^{\dagger}](F + R) \\
=&P(I - BB^{\dagger})A - P(I - BB^{\dagger})(I - F^{\dagger}F) + Q + QB^{\dagger}A(I - F^{\dagger}F) \\
=&PA + Q + PBB^{\dagger}(I - F^{\dagger}F - A).
\end{aligned}$$

Therefore,

$$\begin{aligned}
P =&[PA + Q + PBB^{\dagger}(I - F^{\dagger}F - A)](F + R)^{-1}(I - BB^{\dagger}) + PBB^{\dagger} \\
=&[\Delta_2 + \Delta_1 B^{\dagger}(I - F^{\dagger}F - A)](F + R)^{-1}(I - BB^{\dagger}) + \Delta_1 B^{\dagger}
\end{aligned}$$

and

$$\begin{aligned}
Q &= QB^{\dagger}B \\
&= [PA + Q + PBB^{\dagger}(I - F^{\dagger}F - A)](F + R)^{-1}B \\
&= [\Delta_2 + \Delta_1 B^{\dagger}(I - F^{\dagger}F - A)](F + R)^{-1}B.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&[X(I - BB^{\dagger}) + YB^{\dagger}](F + R) \\
=&X(I - BB^{\dagger})A - X(I - BB^{\dagger})(I - F^{\dagger}F) + Y + YB^{\dagger}A(I - F^{\dagger}F) \\
=&XA + Y + XBB^{\dagger}(I - F^{\dagger}F - A) \\
=&P + QB^{\dagger}(I - F^{\dagger}F - A).
\end{aligned}$$

Therefore,

$$X = [P + QB^{\dagger}(I - F^{\dagger}F - A)](F + R)^{-1}(I - BB^{\dagger}) + QB^{\dagger}.$$

By $MM^D = M^DM$, we have that

$$Y = P - XA.$$

$\Longleftarrow$ It is clear. $\square$

REMARK. Let $\mathscr{R}(B)$ be closed. By Theorem 3.2, the necessary and sufficient conditions of the M^D exists with $ind(M) \leqslant 2$ and the expression of M^D can be obtained. In[27, Theorem 3.1], authors didn't observe that $\mathscr{R}(F_3)$ is closed if $ind(M) \leqslant 2$. If F is invertible, then $I - F^\dagger F = 0$ and $R = 0$. Therefore, $F + R$ is invertible. But $ind(M) = 1$. The necessary and sufficient conditions of the M^D exists $ind(M) = 2$ and expression of M^D can be given directly following.

THEOREM 3.3. *Let M be defined by (1) such that $\mathscr{R}(B)$ is closed, let F and R be defined by (8). Then $ind(M) = 2$ if and only if $\mathscr{R}(F)$ is closed, $F + R$ is invertible but F is not invertible. In this case,*

$$M^D = \begin{pmatrix} P & Q \\ X & Y \end{pmatrix},$$

where

$$\begin{cases} \Delta_1 = (I - F^\dagger F)(F + R)^{-1} B, \\ \Delta_2 = I + (I - F^\dagger F)(F + R)^{-1}(I - BB^\dagger), \\ P = [\Delta_2 + \Delta_1 B^\dagger (I - F^\dagger F - A)](F + R)^{-1}(I - BB^\dagger) + \Delta_1 B^\dagger, \\ Q = [\Delta_2 + \Delta_1 B^\dagger (I - F^\dagger F - A)](F + R)^{-1} B, \\ X = [P + QB^\dagger (I - F^\dagger F - A)](F + R)^{-1}(I - BB^\dagger) + QB^\dagger, \\ Y = P - XA. \end{cases}$$

COROLLARY 3.2. *Let M be defined by (1) such that $\mathscr{R}(B)$ is closed, let B, A and R_4 be defied by (13), (14) and (16), respectively. Then the following statements are equivalent:*

(i) *M is Drazin invertible such that $ind(M) \leqslant 2$,*

(ii) *$\mathscr{R}(A_4)$ is closed, $\begin{pmatrix} R_4 - A_3 B_1^{-1} A_2 & A_4 \\ I & 0 \end{pmatrix}$ is group invertible.*

Proof. Obviously, (i) $\Longleftrightarrow$ $\mathscr{R}(F)$ is closed and $F + R$ is invertible $\Longleftrightarrow$ $\mathscr{R}(A_4)$ is closed and J is invertible $\Longleftrightarrow$ (ii). $\square$

We give two examples that illustrate the correctness of Theorem 3.1 and Theorem 3.3.

EXAMPLE 1. Let

$$M = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

By computation, we obtain

$$B^{\dagger} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, \quad F = (I - BB^{\dagger})A + B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

is invertible. So

$$F^{-1} = [(I - BB^{\dagger})A + B]^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix},$$

$$F_1^{-1} = [A(I - B^{\dagger}B) + B]^{-1} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Thus

$$P = F^{-1}(I - BB^{\dagger}) = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}, \quad Q = F^{-1}B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$X = P^2 + QF_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Y = P - XA = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Therefore,

$$M^{\#} = \begin{pmatrix} -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ -1 & 0 & 1 & 1 \end{pmatrix}.$$

EXAMPLE 2. Let

$$M = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} 0 & -2 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right).$$

By computation, we obtain

$$F = (I - BB^{\dagger})A + B = \begin{pmatrix} 0 & -1 \\ 0 & 3 \end{pmatrix} \text{ is not invertible,}$$

$$R = [BB^{\dagger}A - (I - BB^{\dagger})](I - F^{\dagger}F) = \begin{pmatrix} -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{pmatrix}$$

and

$$F + R = \begin{pmatrix} -\frac{1}{2} & -1 \\ \frac{1}{2} & 3 \end{pmatrix} \text{ is invertible imply } ind(M) = 2.$$

Using Theorem 3.3,

$$\Delta_1 = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}, \quad \Delta_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 9 \\ 0 & 1 \end{pmatrix}$$

and

$$X = \begin{pmatrix} 0 & 9 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -22 \\ 0 & -2 \end{pmatrix}.$$

Hence,

$$M^D = \begin{pmatrix} 0 & -4 & 0 & 9 \\ 0 & 0 & 0 & 1 \\ 0 & 9 & 0 & -22 \\ 0 & 1 & 0 & -2 \end{pmatrix}.$$

REFERENCES

[1] M. BARRAA, E. H. BENABDI, *On equivalence of linear combinations of idempotents*, Linear Multilinear Algebra 152 (2018) 38–64.

[2] A. BEN-ISRAL, T. N. E. GREVILLE, *Generalized Inverses: Theory and Applications*, 2nd ed, Springer-Verlag, 2003.

[3] C. J. BU, K. Z. ZHANG AND J. M. ZHAO, *Representations of the Drazin inverse on solution of a class singular differential equations*, Linear Multilinear Algebra 8 (2011) 863–877.

[4] S. L. CAMPBELL, *The Drazin inverse and systems of second order linear differential equations*, Linear Multilinear Algebra 14 (1983) 195–198.

[5] S. L. CAMPBELL AND C. D. MEYER, *Generalized Inverse of Linear Transformations*, London, Pitman, 1979.

[6] S. L. CAMPBELL, *Singular systems of differential equations*, London, Pitman, 1980.

[7] S. L. CAMPBELL, C. D. MEYER JR AND N. J. ROSE, *Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients*, SIAM J. Appl. Math. 31 (1976) 411–425.

[8] N. CASTRO-GONZÁLEZ, *Additive perturbation results for the Drazin inverse*, Linear Algebra Appl. 397 (2005) 279–297.

[9] N. CASTRO-GONZÁLEZ, E. DOPAZO, *Representations of the Drazin inverse for a class of block matrices*, Linear Algebra Appl. 400 (2005) 253–269.

[10] N. CASTRO-GONZÁLEZ, J. J. KOLIHA, *New additive results for the g-Drazin inverse*, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 1085–1097.

[11] J. J. CLIMENT, M. NEUMANN, AND A. SIDI, *A semi-iterative method for real spectrum singular linear systems with an arbitrary index*, J. Comput. Appl. Math. 87 (1997) 21–38.

[12] R. E. CLINE, *An application of representation of a matrix*, MRC Technical Report 592 (1965).

[13] C. Y. DENG AND H. K. DU, *Representations of the Moore-Penrose inverse for a class of 2-by-2 block operator valued partial matrices*, Linear and Multilinear Algebra 58 (2010) 15–26.

[14] C. Y. DENG AND Y. M. WEI, *A note on the Drazin inverse of an anti-triangular matrix*, Linear Algebra Appl. 431 (2009) 1910–1922.

[15] D. S. DJORDJEVIĆ, Y. M. WEI, *Additive results for the generalized Drazin inverse*, J. Aust. Math. Soc. 73 (2002) 115–125.

[16] R. G. DOUGLAS, *On majorization, factorization, and range inclusion of operators on Hilbert spaces*, Proc. Amer. Math. Soc. 17 (1966) 413–416.

[17] R. E. HARTE, *Invertibility and Singularity for Bounded Linear Operators*, Marcel Dekker, New York, 1988.

[18] R. E. HARTWIG, X. Z. LI AND Y. M. WEI, *Representations for the Drazin inverse of a 2×2 block matrix*, SIAM J. Matrix Anal. Appl. 27 (2005) 757–771.

[19] R. E. HARTWIG, G. R. WANG AND Y. M. WEI, *Some additive results on Drazin inverse*, Linear Algebra Appl. 322 (2001) 207–217.

[20] L. L. HE, C. N. SONG AND Q. X. XU, *General exact solutions of certain second-order homogeneous algebraic differential equations*, Linear Multilinear Algebra 64 (2016) 1011–1031.

[21] J. J. KOLIHA, *A generalized Drazin inverse*, Glasg. Math. J. 38 (1996) 367–381.

[22] J. J. KOLIHA, V. RAKOČEVIĆ, *Continuity of the Drazin inverse II*, Studia Math. 131 (1998) 167–177.

[23] P. KUNKEL AND V. MEHRMANN, *Differential-algebraic equations (Analysis and numerical solution)*, Zürich: European Mathematical Society Publishing House, 2006.

[24] Y. M. WEI, X. Z. LI AND F. B. BU, *A perturbation bound of the Drazin inverse of a matrix by separation of simple invariant subspaces*, SIAM J. Matrix Anal. Appl. 27 (2005) 72–81.
[25] Q. X. XU, C. N. SONG AND X. F. LIU, *General exact solutions of the second-order homogeneous algebraic differential equations*, Linear Multilinear Algebra 63 (2015) 244–263.
[26] Q. X. XU, C. N. SONG AND L. L. HE, *Representations for the group inverse of anti-triangular block operator matrices*, Linear Algebra Appl. 443 (2014) 191–203.
[27] Q. X. XU, C. N. SONG AND L. L. HE, *Representations for the Drazin inverse of an anti-triangular block operator matrix E with $ind(E) \leqslant 2$*, Linear Multilinear Algebra 66 (2018) 1026–1045.

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