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MULTIPLICATIVE λ–$*$–JORDAN TRIPLE HIGHER DERIVATIONS ON STANDARD OPERATOR ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathscr{H}$ containing an identity operator $\mathscr{I}$. In this paper, it is shown that any multiplicative λ-$*$-Jordan triple higher derivation $\mathscr{D} = \{\delta_n\}_{n\in\mathbb{N}}$ from $\mathcal{A}$ to $\mathcal{B}(\mathscr{H})$ is an additive $*$-higher derivation on $\mathcal{A}$. In particular, $\mathscr{D}$ is inner.

1. Introduction and results

Let $\mathcal{A}$ be an algebra over a commutative ring $\mathscr{R}$. Recall that an $\mathscr{R}$-linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a derivation if $\delta(\mathscr{A}\mathscr{B}) = \delta(\mathscr{A})\mathscr{B} + \mathscr{A}\delta(\mathscr{B})$ for all $\mathscr{A},\mathscr{B} \in \mathcal{A}$. In particular, a derivation δ is called an inner derivation if there exists some $\mathscr{X} \in \mathcal{A}$ such that $\delta(\mathscr{A}) = \mathscr{A}\mathscr{X} - \mathscr{X}\mathscr{A}$ for all $\mathscr{A} \in \mathcal{A}$. For $\mathscr{A},\mathscr{B} \in \mathcal{A}$, denote by $\mathscr{A} \diamond \mathscr{B} = \mathscr{A}\mathscr{B} + \mathscr{B}\mathscr{A}^*$ and $[\mathscr{A},\mathscr{B}]_* = \mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A}^*$, the $*$-Jordan product and the $*$-Lie product, respectively. Such kind of products play a key role in the problem of representability of quadratic functionals by sesqui-linear functionals on left-modules over $*$-algebras (see [7, 8, 9, 10, 15, 17, 18, 22]). Special attention has been paid to understand mappings which preserve Jordan $*$-products between $*$-algebras, see [1, 2, 4, 5, 6].

We define λ-$*$-Jordan product by $\mathscr{A} \diamond_\lambda \mathscr{B} = \mathscr{A}\mathscr{B} + \lambda\mathscr{B}\mathscr{A}^*$ and say that the map ϕ (not necessarily linear) with the property $\phi(\mathscr{A} \diamond_\lambda \mathscr{B}) = \phi(\mathscr{A}) \diamond_\lambda \mathscr{B} + \mathscr{A} \diamond_\lambda \phi(\mathscr{B})$ is a λ-$*$-Jordan derivation map. It is clear that for $\lambda = -1$ and $\lambda = 1$, the λ-Jordan $*$-derivation is a $*$-Lie derivation and a $*$-Jordan derivation, respectively. In [5], Huo et al. considered the Jordan triple λ-$*$-product (where $\eta = \lambda$) of three elements $\mathscr{A},\mathscr{B}$ and $\mathscr{C}$ in a $*$-algebra $\mathcal{A}$ and defined $\mathscr{A} \diamond_\lambda \mathscr{B} \diamond_\lambda \mathscr{C} = (\mathscr{A} \diamond_\lambda \mathscr{B}) \diamond_\lambda \mathscr{C}$ (we should be aware that $\diamond_\lambda$ is not necessarily associative). That is,

$$\mathscr{A} \diamond_\lambda \mathscr{B} \diamond_\lambda \mathscr{C} = \mathscr{A}\mathscr{B}\mathscr{C} + \lambda(\mathscr{B}\mathscr{A}^*\mathscr{C} + \mathscr{C}\mathscr{B}^*\mathscr{A}^*) + |\lambda|^2\mathscr{C}\mathscr{A}\mathscr{B}^*. \tag{1.1}$$

For more results related to λ-$*$-Jordan triple product, we refer the reader [12, 13, 19]. A mapping $\phi : \mathcal{A} \to \mathcal{A}$ (not necessarily linear) which satisfies $\phi(\mathscr{A} \diamond_\lambda \mathscr{B} \diamond_\lambda \mathscr{C}) = \phi(\mathscr{A}) \diamond_\lambda \mathscr{B} \diamond_\lambda \mathscr{C} + \mathscr{A} \diamond_\lambda \phi(\mathscr{B}) \diamond_\lambda \mathscr{C} + \mathscr{A} \diamond_\lambda \mathscr{B} \diamond_\lambda \phi(\mathscr{C})$ for all $\mathscr{A},\mathscr{B},\mathscr{C} \in \mathcal{A}$ is called

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Paper OaM-15-59

λ-$*$-Lie triple derivation. Given the consideration of λ-$*$-Jordan derivations and λ-$*$-Jordan triple derivations, Lin [10, 11] further developed them in more general way. Suppose $n \geqslant 2$ is a fixed positive integer and λ is a nonzero scalar. Let us see a sequence of polynomials with involution

$$\begin{aligned} p_1(X_1) &= X_1, \\ p_2(X_1,X_2) &= p_1(X_1) \diamond X_2 = X_1 \diamond_\lambda X_2, \\ p_3(X_1,X_2,X_3) &= p_2(X_1,X_2) \diamond_\lambda X_3 = (X_1 \diamond_\lambda X_2) \diamond_\lambda X_3 \\ &= X_1 \diamond_\lambda X_2 \diamond_\lambda X_3, \\ p_4(X_1,X_2,X_3,X_4) &= p_2(X_1,X_2,X_3) \diamond_\lambda X_4 = ((X_1 \diamond_\lambda X_2) \diamond_\lambda X_3) \diamond_\lambda X_4 \\ &= X_1 \diamond_\lambda X_2 \diamond_\lambda X_3 \diamond_\lambda X_4, \\ &\quad \ldots, \\ p_n(X_1,X_2,\ldots,X_n) &= p_{n-1}(X_1,X_2,\ldots,X_{n-1}) \diamond_\lambda X_n \\ &= (\ldots((X_1 \diamond_\lambda X_2) \diamond_\lambda X_3) \diamond_\lambda \ldots \diamond_\lambda X_{n-1}) \diamond_\lambda X_n \\ &= X_1 \diamond_\lambda X_2 \diamond_\lambda \ldots \diamond_\lambda X_{n-1} \diamond_\lambda X_n. \end{aligned}$$

A mutiplicative/nonlinear λ-$*$-Jordan n-derivation is a mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying the condition

$$\delta(p_n(\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_n)) = \sum_{k=1}^{n} p_n(\mathscr{A}_1,\ldots,\mathscr{A}_{k-1},\delta(\mathscr{A}_k),\mathscr{A}_{k+1},\ldots,\mathscr{A}_n), \quad (1.2)$$

for all $\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_n \in \mathcal{A}$. By the definition, it is clear that every λ-$*$-Jordan derivation is a λ-$*$-Jordan 2-derivation and every λ-$*$-Jordan triple derivation is a λ-$*$-Jordan 3-derivation. Let $p_n(X_1,X_2,\ldots,X_n)$ be the polynomial defined by n indeterminates $X_1,\ldots,X_n$ and their Jordan multiple λ-$*$-products. Let $\mathbb{N}$ be the set of non-negative integers and $\mathscr{D} = \{\delta_n\}_{n\in\mathbb{N}}$ be a family of nonlinear mappings $\delta_n : \mathcal{A} \to \mathcal{A}$ such that $\delta_0 = id_{\mathcal{A}}$, the identity mapping on $\mathcal{A}$. Then $\mathscr{D}$ is called a multiplicative/nonlinear λ-$*$-Jordan n higher derivation if $\mathscr{D}$ satisfies the condition

$$\delta_m(p_n(\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_n)) = \sum_{i_1+i_2+\ldots+i_n=m} p_n(\delta_{i_1}(\mathscr{A}_1),\delta_{i_2}(\mathscr{A}_2),\ldots,\delta_{i_n}(\mathscr{A}_n)) \quad (1.3)$$

for all $\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_n \in \mathcal{A}$. In the case of $\lambda = 1$, δ_m is called a $*$-Jordan higher derivation whenever $n = 2$, and is called a $*$-Jordan triple higher derivation whenever $n = 3$.

In [3], Daif initially proved that each multiplicative derivation is additive on a 2-torsion free prime ring containing a nontrivial idempotent. In [14], Li et. al showed that if $\mathcal{A} \subseteq \mathcal{A}(\mathscr{H})$ is a von Neumann algebra without central abelian projections, then $\delta : \mathcal{A} \to \mathcal{B}(\mathscr{H})$ is a nonlinear λ-$*$-Jordan derivation if and only if δ is an additive $*$-derivation and $\delta(\lambda\mathscr{A}) = \lambda\delta(\mathscr{A})$ for all $\mathscr{A} \in \mathcal{A}$. More recently, this result is generalized to the case of nonlinear 1-$*$-Jordan triple derivations by Zhao and Li [21].

In this article, we investigate that when a multiplicative λ-$*$-Jordan triple higher derivation on standard operator algebra of infinite dimensional Hilbert space $\mathscr{H}$ is additive. Following main theorem is the generalization of [19]:

MAIN THEOREM.

Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathscr{H}$ containing the identity operator $\mathscr{I}$. If λ is a non-zero scalar such that $\lambda \neq 1,2$ and $\mathcal{A}$ is closed under the adjoint operation, then every multiplicative λ-$*$-Jordan triple higher derivation $\mathscr{D} = \{\delta_n\}_{n\in\mathbb{N}}$ from $\mathcal{A}$ to $\mathcal{B}(\mathscr{H})$ is an additive $*$-higher derivation on $\mathcal{A}$.

To be in a position to prove our main theorem, we need to prove the following result.

THEOREM 1.1. *Let $\mathcal{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathscr{H}$ containing the identity operator $\mathscr{I}$. If λ is a non-zero scalar such that $\lambda \neq 1,2$ and $\mathcal{A}$ is closed under the adjoint operation, then every multiplicative λ-$*$-Jordan triple higher derivation $\mathscr{D} = \{\delta_n\}_{n\in\mathbb{N}}$ from $\mathcal{A}$ to $\mathcal{B}(\mathscr{H})$ is additive.*

Proof. Take a projection $\mathscr{P}_1 \in \mathcal{A}$ and let $\mathscr{P}_2 = \mathscr{I} - \mathscr{P}_1$. We write $\mathcal{A}_{jk} = \mathscr{P}_j\mathcal{A}\mathscr{P}_k$ for $j,k = 1,2$. Then by Peire's decomposition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Note that any operator $\mathscr{A} \in \mathcal{A}$ can be written as $\mathscr{A} = \mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}$. In view of the above facts, the proof of the theorem is given in the series of the following lemmas:

LEMMA 1.1. *$\delta_n(0) = 0$ for each $n \in \mathbb{N}$.*

Proof. Proceeding by induction on $n \in \mathbb{N}$ with $n \geqslant 1$. If $n = 1$, by [19, Claim 2.2], the result is true. Assume that the result holds for $k < n$, i.e., $\delta_k(0) = 0$. Our goal is to prove that δ_n satisfies the same property. Obverve that

$$\begin{aligned}\delta_n(0) &= \sum_{r+s+t=n} \delta_r(0) \diamond_\lambda \delta_s(0) \diamond_\lambda \delta_t(0)\\ &= \delta_n(0) \diamond_\lambda 0 \diamond_\lambda 0 + 0 \diamond_\lambda \delta_n(0) \diamond_\lambda 0 + 0 \diamond_\lambda 0 \diamond_\lambda \delta_n(0).\end{aligned}$$

This gives $\delta_n(0) = 0$ for each $n \in \mathbb{N}$. □

LEMMA 1.2. *For any $\mathscr{A}_{12} \in \mathcal{A}_{12}$ and $\mathscr{A}_{21} \in \mathcal{A}_{21}$, we have*

$$\delta_n(\mathscr{A}_{12} + \mathscr{A}_{21}) = \delta_n(\mathscr{A}_{12}) + \delta_n(\mathscr{A}_{21}).$$

Proof. Using induction on $n \in \mathbb{N}$ with $n \geqslant 1$. By [19, Claim 2.3] result holds true for $n = 1$. Assume that it is true for $k < n$, i.e., $\delta_k(\mathscr{A}_{12} + \mathscr{A}_{21}) = \delta_k(\mathscr{A}_{12}) + \delta_k(\mathscr{A}_{21})$. Let us set $\Phi = \delta_n(\mathscr{A}_{12} + \mathscr{A}_{21}) - \delta_n(\mathscr{A}_{12}) - \delta_n(\mathscr{A}_{21})$. We now show that $\Phi = 0$.

By using induction hypothesis, we first compute

$$\begin{aligned}&\delta_n(\mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda (\mathscr{A}_{12} + \mathscr{A}_{21}))\\ &= \delta_n(\mathscr{I}) \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda (\mathscr{A}_{12} + \mathscr{A}_{21}) + \mathscr{I} \diamond_\lambda \delta_n(\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda (\mathscr{A}_{12} + \mathscr{A}_{21})\\ &\quad + \mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \delta_n(\mathscr{A}_{12} + \mathscr{A}_{21})\\ &\quad + \sum_{\substack{r+s+t=n\\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{I}) \diamond_\lambda \delta_s(\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \delta_t(\mathscr{A}_{12} + \mathscr{A}_{21})\end{aligned}$$

$$
\begin{aligned}
&= \delta_n(\mathscr{I}) \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda (\mathscr{A}_{12} + \mathscr{A}_{21}) + \mathscr{I} \diamond_\lambda \delta_n(\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda (\mathscr{A}_{12} + \mathscr{A}_{21}) \\
&\quad + \mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \delta_n(\mathscr{A}_{12} + \mathscr{A}_{21}) \\
&\quad + \sum_{\substack{r+s+t=n \\ 0 \leqslant r,s,t \leqslant (n-1)}} \delta_r(\mathscr{I}) \diamond_\lambda \delta_s(\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda (\delta_t(\mathscr{A}_{12}) + \delta_t(\mathscr{A}_{21})).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\delta_n\big(\mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda (\mathscr{A}_{12} + \mathscr{A}_{21})\big) \\
&= \delta_n\big(\mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \mathscr{A}_{21}\big) \\
&= \delta_n\big(\mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \mathscr{A}_{12}\big) + \delta_n\big(\mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \mathscr{A}_{21}\big) \\
&= \delta_n(\mathscr{I}) \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \mathscr{A}_{12} + \mathscr{I} \diamond_\lambda \delta_n(\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \mathscr{A}_{12} \\
&\quad + \mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \delta_n(\mathscr{A}_{12}) \\
&\quad + \sum_{\substack{r+s+t=n \\ 0 \leqslant r,s,t \leqslant (n-1)}} \delta_r(\mathscr{I}) \diamond_\lambda \delta_s(\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \delta_t(\mathscr{A}_{12}) \\
&\quad + \delta_n(\mathscr{I}) \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \mathscr{A}_{21} + \mathscr{I} \diamond_\lambda \delta_n(\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \mathscr{A}_{21} \\
&\quad + \mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \delta_n(\mathscr{A}_{21}) \\
&\quad + \sum_{\substack{r+s+t=n \\ 0 \leqslant r,s,t \leqslant (n-1)}} \delta_r(\mathscr{I}) \diamond_\lambda \delta_s(\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \delta_t(\mathscr{A}_{21}).
\end{aligned}
$$

From the last two expressions, we obtain

$$\mathscr{I} \diamond_\lambda (\mathscr{P}_1 - \mathscr{P}_2) \diamond_\lambda \Phi = 0.$$

Since $\Phi = \Phi_{11} + \Phi_{12} + \Phi_{21} + \Phi_{22}$, so we have

$$(1 + 2\lambda + |\lambda|^2)\Phi_{11} + (1 - |\lambda|^2)\Phi_{12} + (1 - |\lambda|^2)\Phi_{21} - (1 + 2\lambda + |\lambda|^2)\Phi_{22} = 0.$$

We know that $\lambda \neq 0, 1$. Then

$$\Phi_{11} = \Phi_{12} = \Phi_{21} = \Phi_{22} = 0.$$

Therefore $\Phi = 0$ i.e., $\delta_n(\mathscr{A}_{12} + \mathscr{A}_{21}) = \delta_n(\mathscr{A}_{12}) + \delta_n(\mathscr{A}_{21})$. $\square$

LEMMA 1.3. *For any* $\mathscr{A}_{11} \in \mathcal{A}_{11}$, $\mathscr{A}_{12} \in \mathcal{A}_{12}$ *and* $\mathscr{A}_{21} \in \mathcal{A}_{21}$, *we have*

$$\delta_n(\mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21}) = \delta_n(\mathscr{A}_{11}) + \delta_n(\mathscr{A}_{12}) + \delta_n(\mathscr{A}_{21}).$$

Proof. Using induction on n with $n \geqslant 1$. The result holds for $n = 1$ by [19, Claim 2.4]. Let the result hold for $k < n$, that is $\delta_k(\mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21}) = \delta_k(\mathscr{A}_{11}) + \delta_k(\mathscr{A}_{12}) + \delta_k(\mathscr{A}_{21})$. Take $\Delta = \delta_n(\mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21}) - \delta_n(\mathscr{A}_{11}) - \delta_n(\mathscr{A}_{12}) - \delta_n(\mathscr{A}_{21})$. Our aim is to show $\Delta=0$. To show this, first we compute

$$\begin{aligned}
&\delta_n\big(\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})\big)\\
&=\delta_n(\mathscr{I})\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})\\
&\quad+\mathscr{I}\diamond_\lambda\delta_n(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})\\
&\quad+\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\delta_n(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})\\
&\quad+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(\mathscr{I})\diamond_\lambda\delta_s(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\delta_t(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})\\
&=\delta_n(\mathscr{I})\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})\\
&\quad+\mathscr{I}\diamond_\lambda\delta_n(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})\\
&\quad+\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\delta_n(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})\\
&\quad+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(\mathscr{I})\diamond_\lambda\delta_s(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda(\delta_t(\mathscr{A}_{11})+\delta_t(\mathscr{A}_{12})+\delta_t(\mathscr{A}_{21})).
\end{aligned}$$

Furhtermore, since $\mathscr{I}\diamond_\lambda\mathscr{P}_2\diamond_\lambda\mathscr{A}_{11}=0$. Hence

$$\begin{aligned}
&\delta_n\big(\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})\big)\\
&=\delta_n(\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{21}))\\
&=\delta_n(\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{11})+\delta_n(\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{12})\\
&\quad+\delta_n(\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{21})\\
&=\delta_n(\mathscr{I})\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{11}+\mathscr{I}\diamond_\lambda\delta_n(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{11}\\
&\quad+\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\delta_n(\mathscr{A}_{11})+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(\mathscr{I})\diamond_\lambda\delta_s(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\delta_t(\mathscr{A}_{11})\\
&\quad+\delta_n(\mathscr{I})\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{12}+\mathscr{I}\diamond_\lambda\delta_n(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{12}\\
&\quad+\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\delta_n(\mathscr{A}_{12})+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(\mathscr{I})\diamond_\lambda\delta_s(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\delta_t(\mathscr{A}_{12})\\
&\quad+\delta_n(\mathscr{I})\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{21}+\mathscr{I}\diamond_\lambda\delta_n(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\mathscr{A}_{21}\\
&\quad+\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\delta_n(\mathscr{A}_{21})+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(\mathscr{I})\diamond_\lambda\delta_s(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\delta_t(\mathscr{A}_{21}).
\end{aligned}$$

Comparing the last two relations, we obtain

$$\mathscr{I}\diamond_\lambda(\mathscr{P}_1-\mathscr{P}_2)\diamond_\lambda\Delta=0.$$

Following the same procedure as used in the previous lemma, we see that $\Delta_{11}=\Delta_{12}=\Delta_{21}=\Delta_{22}=0$. Hence $\Delta=0$ i.e., $\delta_n(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21})=\delta_n(\mathscr{A}_{11})+\delta_n(\mathscr{A}_{12})+\delta_n(\mathscr{A}_{21})$. □

LEMMA 1.4. *For any $\mathscr{A}_{11}\in\mathcal{A}_{11}$, $\mathscr{A}_{12}\in\mathcal{A}_{12}$, $\mathscr{A}_{21}\in\mathcal{A}_{21}$ and $\mathscr{A}_{22}\in\mathcal{A}$, we have*

$$\delta_n(\mathscr{A}_{11}+\mathscr{A}_{12}+\mathscr{A}_{21}+\mathscr{A}_{22})=\delta_n(\mathscr{A}_{11})+\delta_n(\mathscr{A}_{12})+\delta_n(\mathscr{A}_{21})+\delta_n(\mathscr{A}_{22}).$$

Proof. The proof of lemma is same as that of Lemma 1.2 and Lemma 1.3. □

LEMMA 1.5. *For each $\mathscr{A}_{ij},\mathscr{B}_{ij} \in \mathcal{A}_{ij}$, we have*

$$\delta_n(\mathscr{A}_{ij}+\mathscr{B}_{ij}) = \delta_n(\mathscr{A}_{ij})+\delta_n(\mathscr{B}_{ij}).$$

Proof.
Case I. $i \neq j$. Let $\mathscr{X}_{ij},\mathscr{Y}_{ij} \in \mathcal{A}_{ij}$. It is easy to compute that

$$\begin{aligned}\mathscr{I} \diamond_\lambda (\mathscr{X}_{ij}+\mathscr{P}_i)\diamond_\lambda (\mathscr{P}_j+\mathscr{Y}_{ij}) = (1+\lambda)(\mathscr{X}_{ij}+\mathscr{Y}_{ij})+(\lambda+|\lambda|^2)\mathscr{X}_{ij}^* \\ +(\lambda+|\lambda|^2)\mathscr{Y}_{ij}\mathscr{X}_{ij}^*.\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\delta_n\Big((1+\lambda)(\mathscr{X}_{ij}+\mathscr{Y}_{ij})+(\lambda+|\lambda|^2)\mathscr{X}_{ij}^*+(\lambda+|\lambda|^2)\mathscr{Y}_{ij}\mathscr{X}_{ij}^*\Big)\\
&= \delta_n(\mathscr{I} \diamond_\lambda (\mathscr{X}_{ij}+\mathscr{P}_i)\diamond_\lambda (\mathscr{P}_j+\mathscr{Y}_{ij}))\\
&= \delta_n(\mathscr{I})\diamond_\lambda (\mathscr{X}_{ij}+\mathscr{P}_i)\diamond_\lambda (\mathscr{P}_j+\mathscr{Y}_{ij})+\mathscr{I}\diamond_\lambda \delta_n(\mathscr{X}_{ij}+\mathscr{P}_i)\diamond_\lambda (\mathscr{P}_j+\mathscr{Y}_{ij})\\
&\quad +\mathscr{I}\diamond_\lambda (\mathscr{X}_{ij}+\mathscr{P}_i)\diamond_\lambda \delta_n(\mathscr{P}_j+\mathscr{Y}_{ij})\\
&\quad + \sum_{\substack{r+s+t=n\\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{I})\diamond_\lambda \delta_s(\mathscr{X}_{ij}+\mathscr{P}_i)\diamond_\lambda \delta_t(\mathscr{P}_j+\mathscr{Y}_{ij})\\
&= \delta_n(\mathscr{I})\diamond_\lambda (\mathscr{X}_{ij}+\mathscr{P}_i)\diamond_\lambda (\mathscr{P}_j+\mathscr{Y}_{ij})+\mathscr{I}\diamond_\lambda (\delta_n(\mathscr{X}_{ij})+\delta_n(\mathscr{P}_i))\diamond_\lambda (\mathscr{P}_j+\mathscr{Y}_{ij})\\
&\quad +\mathscr{I}\diamond_\lambda (\mathscr{X}_{ij}+\mathscr{P}_i)\diamond_\lambda (\delta_n(\mathscr{P}_j)+\delta_n(\mathscr{Y}_{ij}))\\
&\quad + \sum_{\substack{r+s+t=n\\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{I})\diamond_\lambda \delta_s(\mathscr{X}_{ij})+\delta_s(\mathscr{P}_i)\diamond_\lambda (\delta_t(\mathscr{Y}_{ij})+\delta_t(\mathscr{P}_j))\\
&= \delta_n(\mathscr{I}\diamond_\lambda \mathscr{X}_{ij}\diamond_\lambda \mathscr{Y}_{ij})+\delta_n(\mathscr{I}\diamond_\lambda \mathscr{P}_i\diamond_\lambda \mathscr{P}_j)+\delta_n(\mathscr{I}\diamond_\lambda \mathscr{X}_{ij}\diamond_\lambda \mathscr{P}_j)\\
&\quad +\delta_n(\mathscr{I}\diamond_\lambda \mathscr{P}_i\diamond_\lambda \mathscr{Y}_{ij})\\
&= \delta_n((1+\lambda)\mathscr{X}_{ij})+\delta_n((1+\lambda)\mathscr{Y}_{ij})+\delta_n((\lambda+|\lambda|^2)\mathscr{X}_{ij}^*)+\delta_n((\lambda+|\lambda|^2)\mathscr{Y}_{ij}\mathscr{X}_{ij}^*).
\end{aligned}$$

Comparing the above two relations, it follows that

$$\delta_n((1+\lambda)(\mathscr{X}_{ij}+\mathscr{Y}_{ij})) = \delta_n((1+\lambda)\mathscr{X}_{ij})+\delta_n((1+\lambda)\mathscr{Y}_{ij}).$$

Hence

$$\delta_n(\mathscr{A}_{ij}+\mathscr{B}_{ij}) = \delta_n(\mathscr{A}_{ij})+\delta_n(\mathscr{B}_{ij}),$$

where $(1+\lambda)(\mathscr{X}_{ij}) = \mathscr{A}_{ij}$ and $(1+\lambda)(\mathscr{Y}_{ij}) = \mathscr{B}_{ij}$ respectively.

Case II. $i=j$. We shall show that $\Theta = \delta_n(\mathscr{A}_{ii}+\mathscr{B}_{ii})-\delta_n(\mathscr{A}_{ii})-\delta_n(\mathscr{B}_{ii}) = 0$. Let $\mathscr{X}_{ii},\mathscr{Y}_{ii}\in\mathcal{A}_{ii}$ and $k\in\{1,2\}$ with $k\neq i$. We have

$$\begin{aligned}
0 &= \delta_n(\mathscr{P}_k\diamond_\lambda \mathscr{P}_k\diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii}))\\
&= \delta_n(\mathscr{P}_k)\diamond_\lambda \mathscr{P}_k\diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii})+\mathscr{P}_k\diamond_\lambda \delta_n(\mathscr{P}_k)\diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii})\\
&\quad +\mathscr{P}_k\diamond_\lambda \mathscr{P}_k\diamond_\lambda \delta_n(\mathscr{A}_{ii}+\mathscr{B}_{ii})\\
&\quad + \sum_{\substack{r+s+t=n\\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{P}_k)\diamond_\lambda \delta_s(\mathscr{P}_k)\diamond_\lambda \delta_t(\mathscr{A}_{ii}+\mathscr{B}_{ii})
\end{aligned}$$

$$
\begin{aligned}
&= \delta_n(\mathscr{P}_k)\diamond_\lambda \mathscr{P}_k \diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii}) + \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{P}_k)\diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii}) \\
&\quad + \mathscr{P}_k \diamond_\lambda \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{A}_{ii}+\mathscr{B}_{ii}) \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{P}_k)\diamond_\lambda \delta_s(\mathscr{P}_k)\diamond_\lambda (\delta_t(\mathscr{A}_{ii})+\delta_t(\mathscr{B}_{ii})).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
0 &= \delta_n(\mathscr{P}_k \diamond_\lambda \mathscr{P}_k \diamond_\lambda \mathscr{A}_{ii}) + \delta_n(\mathscr{P}_k \diamond_\lambda \mathscr{P}_k \diamond_\lambda \mathscr{B}_{ii}) \\
&= \delta_n(\mathscr{P}_k)\diamond_\lambda \mathscr{P}_k \diamond_\lambda \mathscr{A}_{ii} + \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{P}_k)\diamond_\lambda \mathscr{A}_{ii} \\
&\quad + \mathscr{P}_k \diamond_\lambda \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{A}_{ii}) + \sum_{\substack{r+s+t=n \\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{P}_k)\diamond_\lambda \delta_s(\mathscr{P}_k)\diamond_\lambda \delta_t(\mathscr{A}_{ii}) \\
&\quad + \delta_n(\mathscr{P}_k)\diamond_\lambda \mathscr{P}_k \diamond_\lambda \mathscr{B}_{ii} + \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{P}_k)\diamond_\lambda \mathscr{B}_{ii} \\
&\quad + \mathscr{P}_k \diamond_\lambda \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{B}_{ii}) + \sum_{\substack{r+s+t=n \\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{P}_k)\diamond_\lambda \delta_s(\mathscr{P}_k)\diamond_\lambda \delta_t(\mathscr{B}_{ii}) \\
&= \delta_n(\mathscr{P}_k)\diamond_\lambda \mathscr{P}_k \diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii}) + \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{P}_k)\diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii}) \\
&\quad + \mathscr{P}_k \diamond_\lambda \mathscr{P}_k \diamond_\lambda (\delta_n(\mathscr{A}_{ii})+\delta_n(\mathscr{B}_{ii})) \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{P}_k)\diamond_\lambda \delta_s(\mathscr{P}_k)\diamond_\lambda (\delta_t(\mathscr{A}_{ii})+\delta_t(\mathscr{B}_{ii})).
\end{aligned}
$$

Observe from the above two equalities, we see that

$$
\mathscr{P}_k \diamond_\lambda \mathscr{P}_k \diamond_\lambda \Theta = 0.
$$

Thus,

$$
(1+2\lambda+|\lambda|^2)\Theta_{kk} + (1+\lambda)\Theta_{ki} + (\lambda+|\lambda|^2)\Theta_{ik} = 0.
$$

It follows that $\Theta_{kk} = \Theta_{ki} = \Theta_{ik} = 0$. Finally, we will show that $\Theta_{ii} = 0$. To prove this, for each $\mathscr{C}_{ik} \in \mathcal{A}_{ik}$, since $\mathscr{P}_k \diamond_\lambda \mathscr{A}_{ii} = \mathscr{P}_k \diamond_\lambda \mathscr{B}_{ii} = \mathscr{P}_k \diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii}) = 0$, we have

$$
\begin{aligned}
&\delta_n(\mathscr{P}_k)\diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii})\diamond_\lambda \mathscr{C}_{ik} + \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{A}_{ii}+\mathscr{B}_{ii})\diamond_\lambda \mathscr{C}_{ik} \\
&+ \mathscr{P}_k \diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii})\diamond_\lambda \delta_n(\mathscr{C}_{ik}) + \sum_{\substack{r+s+t=n \\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{P}_k)\diamond_\lambda \delta_s(\mathscr{A}_{ii}+\mathscr{B}_{ii})\diamond_\lambda \delta_t(\mathscr{C}_{ik}) \\
&= \delta_n(\mathscr{P}_k \diamond_\lambda (\mathscr{A}_{ii}+\mathscr{B}_{ii})\diamond_\lambda \mathscr{C}_{ik}) \\
&= \delta_n(\mathscr{P}_k \diamond_\lambda \mathscr{A}_{ii}\diamond_\lambda \mathscr{C}_{ik}) + \delta_n(\mathscr{P}_k \diamond_\lambda \mathscr{B}_{ii}\diamond_\lambda \mathscr{C}_{ik}) \\
&= \delta_n(\mathscr{P}_k)\diamond_\lambda \mathscr{A}_{ii}\diamond_\lambda \mathscr{C}_{ik} + \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{A}_{ii})\diamond_\lambda \mathscr{C}_{ik} + \mathscr{P}_k \diamond_\lambda \mathscr{A}_{ii}\diamond_\lambda \delta_n(\mathscr{C}_{ik}) \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{P}_k)\diamond_\lambda \delta_s(\mathscr{A}_{ii})\diamond_\lambda \delta_t(\mathscr{C}_{ik}) \\
&\quad + \delta_n(\mathscr{P}_k)\diamond_\lambda \mathscr{B}_{ii}\diamond_\lambda \mathscr{C}_{ik} + \mathscr{P}_k \diamond_\lambda \delta_n(\mathscr{B}_{ii})\diamond_\lambda \mathscr{C}_{ik} + \mathscr{P}_k \diamond_\lambda \mathscr{B}_{ii}\diamond_\lambda \delta_n(\mathscr{C}_{ik}) \\
&\quad + \sum_{\substack{r+s+t=n \\ 0\leqslant r,s,t\leqslant (n-1)}} \delta_r(\mathscr{P}_k)\diamond_\lambda \delta_s(\mathscr{B}_{ii})\diamond_\lambda \delta_t(\mathscr{C}_{ik})
\end{aligned}
$$

$$= \delta_n(\mathscr{P}_k)\diamond_\lambda(\mathscr{A}_{ii}+\mathscr{B}_{ii})\diamond_\lambda\mathscr{C}_{ik}+\mathscr{P}_k\diamond_\lambda(\delta_n(\mathscr{A}_{ii})+\delta_n(\mathscr{B}_{ii}))\diamond_\lambda\mathscr{C}_{ik}$$
$$+\mathscr{P}_k\diamond_\lambda(\delta_n(\mathscr{A}_{ii}+\delta_n\mathscr{B}_{ii}))\diamond_\lambda\delta_n(\mathscr{C}_{ik})+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(\mathscr{P}_k)\diamond_\lambda\delta_s(\mathscr{A}_{ii}+\mathscr{B}_{ii})\diamond_\lambda\delta_t(\mathscr{C}_{ik}).$$

Thus,

$$\mathscr{P}_k\diamond_\lambda\Theta\diamond_\lambda\mathscr{C}_{ik}=0.$$

After computation, we have $\Theta_{ii}=0$. □

Hence the additivity of δ_n comes from Lemmas 1.1–1.5. This completes the proof. □

2. Proof of main theorem

To complete the proof of the theorem, we need the following lemmas. Throughout, we shall use the hypothesis of Theorem 1.1 freely without any specific mention in proving the following lemmas.

LEMMA 2.1. $\delta_n(i\mathscr{I})=\delta_n(\mathscr{I})=0$.

Proof. The result is true for $n=1$ by [19, Claim 2.9]. Assume that the result holds true if $k<n$, i.e., $\delta_k(i\mathscr{I})=\delta_k(\mathscr{I})=0$. Now we compute

$$\begin{aligned}\delta_n(\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda i\mathscr{I}) &= \delta_n(\mathscr{I})\diamond_\lambda i\mathscr{I}\diamond_\lambda i\mathscr{I}+\mathscr{I}\diamond_\lambda\delta_n(i\mathscr{I})\diamond_\lambda i\mathscr{I}\\ &\quad+\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda\delta_n(i\mathscr{I})\\ &\quad+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(\mathscr{I})\diamond_\lambda\delta_s(i\mathscr{I})\diamond_\lambda\delta_t(i\mathscr{I}).\\ &= 2i\delta_n(i\mathscr{I})-|\lambda|^2 i\delta_n(i\mathscr{I})+|\lambda|^2 i\delta_n(i\mathscr{I})^*\\ &\quad+\lambda i\delta_n(i\mathscr{I})+\lambda i\delta_n(i\mathscr{I})^*-\delta_n(\mathscr{I})\\ &\quad+|\lambda|^2 i\delta_n(\mathscr{I}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\delta_n(i\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda\mathscr{I}) &= \delta_n(i\mathscr{I})\diamond_\lambda i\mathscr{I}\diamond_\lambda\mathscr{I}+i\mathscr{I}\diamond_\lambda\delta_n(i\mathscr{I})\diamond_\lambda\mathscr{I}\\ &\quad+i\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda\delta_n(\mathscr{I})\\ &\quad+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(i\mathscr{I})\diamond_\lambda\delta_s(i\mathscr{I})\diamond_\lambda\delta_t(\mathscr{I}).\\ &= -\delta_n(\mathscr{I})+|\lambda|^2 i\delta_n(\mathscr{I})+2i\delta_n(i\mathscr{I})\\ &\quad-\lambda i\delta_n(i\mathscr{I})-\lambda i\delta_n(i\mathscr{I})^*-|\lambda|^2 i\delta_n(i\mathscr{I})\\ &\quad+|\lambda|^2 i\delta_n(i\mathscr{I})^*.\end{aligned}$$

Since $\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda i\mathscr{I}=i\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda\mathscr{I}$, so comparision of the last two expression yields

$$2(\lambda i\delta_n(i\mathscr{I})+\lambda i\delta_n(i\mathscr{I})^*)=0,$$

which gives

$$\delta_n(i\mathscr{I}) = -\delta_n(i\mathscr{I})^*. \tag{2.1}$$

Additionally, compute

$$\begin{aligned}\delta_n(i\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda i\mathscr{I}) &= \delta_n(i\mathscr{I})\diamond_\lambda i\mathscr{I}\diamond_\lambda i\mathscr{I} + i\mathscr{I}\diamond_\lambda \delta_n(i\mathscr{I})\diamond_\lambda i\mathscr{I}\\ &\quad + i\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda \delta_n(i\mathscr{I})\\ &\quad + \sum_{\substack{r+s+t=n\\ 0\leqslant r,s,t\leqslant(n-1)}} \delta_r(i\mathscr{I})\diamond_\lambda \delta_s(i\mathscr{I})\diamond_\lambda \delta_t(i\mathscr{I}).\\ &= -3\delta_n(i\mathscr{I}) + \lambda\delta_n(i\mathscr{I}) + \lambda\delta_n(i\mathscr{I})^*\\ &\quad + 2|\lambda|^2\delta_n(i\mathscr{I}) - |\lambda|^2\delta_n(i\mathscr{I})^*.\end{aligned}$$

Further, we have

$$\begin{aligned}\delta_n(\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda \mathscr{I}) &= \delta_n(\mathscr{I})\diamond_\lambda i\mathscr{I}\diamond_\lambda \mathscr{I} + \mathscr{I}\diamond_\lambda \delta_n(i\mathscr{I})\diamond_\lambda \mathscr{I}\\ &\quad + \mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda \delta_n(\mathscr{I})\\ &\quad + \sum_{\substack{r+s+t=n\\ 0\leqslant r,s,t\leqslant(n-1)}} \delta_r(\mathscr{I})\diamond_\lambda \delta_s(i\mathscr{I})\diamond_\lambda \delta_t(\mathscr{I}).\\ &= -2i\delta_n(\mathscr{I}) - \lambda\delta_n(i\mathscr{I}) - \lambda\delta_n(i\mathscr{I})^*\\ &\quad + 2|\lambda|^2 i\delta_n(i\mathscr{I}) - |\lambda|^2\delta_n(i\mathscr{I})^* - \delta_n(i\mathscr{I}).\end{aligned}$$

One can observe that $i\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda i\mathscr{I} = -\mathscr{I}\diamond_\lambda i\mathscr{I}\diamond_\lambda \mathscr{I}$. Thus from above, we get

$$\begin{aligned}&-3\delta_n(i\mathscr{I}) + \lambda\delta_n(i\mathscr{I}) + \lambda\delta_n(i\mathscr{I})^* + 2|\lambda|^2\delta_n(i\mathscr{I}) - |\lambda|^2\delta_n(i\mathscr{I})^*\\ &= -2i\delta_n(\mathscr{I}) - \lambda\delta_n(i\mathscr{I}) - \lambda\delta_n(i\mathscr{I})^* + 2|\lambda|^2 i\delta_n(i\mathscr{I}) - |\lambda|^2\delta_n(i\mathscr{I})^* - \delta_n(i\mathscr{I}).\end{aligned} \tag{2.2}$$

From (2.1), we have

$$\begin{aligned}&-3\delta_n(i\mathscr{I}) + \lambda\delta_n(i\mathscr{I}) - \lambda\delta_n(i\mathscr{I}) + 2|\lambda|^2\delta_n(i\mathscr{I}) + |\lambda|^2\delta_n(i\mathscr{I})\\ &= -2i\delta_n(\mathscr{I}) - \lambda\delta_n(i\mathscr{I}) + \lambda\delta_n(i\mathscr{I}) + 2|\lambda|^2 i\delta_n(i\mathscr{I}) + |\lambda|^2\delta_n(i\mathscr{I}) - \delta_n(i\mathscr{I}).\end{aligned} \tag{2.3}$$

This further implies that

$$-2\delta_n(i\mathscr{I}) + 2|\lambda|^2\delta_n(i\mathscr{I}) + 2i\delta_n(\mathscr{I}) - 2|\lambda|^2 i\delta_n(i\mathscr{I}) = 0. \tag{2.4}$$

Taking adjoint of above expression, we obtain

$$2\delta_n(i\mathscr{I}) - 2|\lambda|^2\delta_n(i\mathscr{I}) - 2i\delta_n(\mathscr{I}) - 2|\lambda|^2 i\delta_n(i\mathscr{I}) = 0. \tag{2.5}$$

Combination of (2.4) and (2.5) gives $\delta_n(i\mathscr{I}) = 0$. This reduces (2.5) into $\delta_n(\mathscr{I}) = 0$. $\square$

LEMMA 2.2. *$\delta_n(\mathscr{A}^*) = \delta_n(\mathscr{A})^*$ and $\delta_n(i\mathscr{A}) = i\delta_n(\mathscr{A})$ for all $\mathscr{A} \in \mathcal{A}$.*

Proof. Using induction on $n \in \mathbb{N}$ with $n \geqslant 1$. By [19, Claim 2.10 and 2.11], the result holds true for $n = 1$. Assume that the result holds for $k < n$, i.e., $\delta_k(\mathscr{A}^*) = \delta_k(\mathscr{A})^*$ and $\delta_k(i\mathscr{A}) = i\delta_k(\mathscr{A})$ for all $\mathscr{A} \in \mathcal{A}$. We know from Lemma 2.1 that $\delta_n(i\mathscr{I}) = \delta_n(\mathscr{I}) = 0$ for each $n \geqslant 1$. Thus

$$\delta_n(\mathscr{A} \diamond_\lambda i\mathscr{I} \diamond_\lambda i\mathscr{I}) = \delta_n(\mathscr{A}) \diamond_\lambda i\mathscr{I} \diamond_\lambda i\mathscr{I} + \sum_{\substack{r+s+t=n \\ 0 \leqslant r,s,t \leqslant (n-1)}} \delta_r(\mathscr{A}) \diamond_\lambda \delta_s(i\mathscr{I}) \diamond_\lambda \delta_t(i\mathscr{I}).$$

It follows that

$$\delta_n(-\mathscr{A} + |\lambda|^2 \mathscr{A}) = -\delta_n(\mathscr{A}) + |\lambda|^2 \delta_n(\mathscr{A}). \tag{2.6}$$

This gives

$$\delta_n(|\lambda|^2 \mathscr{A}) = |\lambda|^2 \delta_n(\mathscr{A}). \tag{2.7}$$

Further,

$$\delta_n(\mathscr{A} \diamond_\lambda \mathscr{I} \diamond_\lambda \mathscr{I}) = \delta_n(\mathscr{A}) \diamond_\lambda \mathscr{I} \diamond_\lambda \mathscr{I} + \sum_{\substack{r+s+t=n \\ 0 \leqslant r,s,t \leqslant (n-1)}} \delta_r(\mathscr{A}) \diamond_\lambda \delta_s(\mathscr{I}) \diamond_\lambda \delta_t(\mathscr{I}).$$

This yields

$$\delta_n(\mathscr{A} + 2\lambda \mathscr{A}^* + |\lambda|^2 \mathscr{A}) = \delta_n(\mathscr{A}) + 2\lambda \delta_n(\mathscr{A})^* + |\lambda|^2 \delta_n(\mathscr{A}). \tag{2.8}$$

Hence,

$$\delta_n(2\lambda \mathscr{A}^*) = 2\lambda \delta_n(\mathscr{A})^*. \tag{2.9}$$

Furthermore

$$\delta_n(\mathscr{I} \diamond_\lambda \mathscr{I} \diamond_\lambda \mathscr{A}^*) = \delta_n(\mathscr{I}) \diamond_\lambda \mathscr{I} \diamond_\lambda \mathscr{A}^* + \sum_{\substack{r+s+t=n \\ 0 \leqslant r,s,t \leqslant (n-1)}} \delta_r(\mathscr{I}) \diamond_\lambda \delta_s(\mathscr{I}) \diamond_\lambda \delta_t(\mathscr{A}^*),$$

gives

$$\delta_n(\mathscr{A}^* + 2\lambda \mathscr{A}^* + |\lambda|^2 \mathscr{A}^*) = \delta_n(\mathscr{A}^*) + 2\lambda \delta_n(\mathscr{A}^*) + |\lambda|^2 \delta_n(\mathscr{A}^*). \tag{2.10}$$

Thus

$$\delta_n(2\lambda \mathscr{A}^*) = 2\lambda \delta_n(\mathscr{A}^*). \tag{2.11}$$

Observe from (2.9) and (2.11) that $\delta_n(\mathscr{A}^*) = \delta_n(\mathscr{A})^*$. Now, for any $\mathscr{A} \in \mathcal{A}$, we have

$$\delta_n(\mathscr{I} \diamond_\lambda i\mathscr{I} \diamond_\lambda \mathscr{A}) = \mathscr{I} \diamond_\lambda i\mathscr{I} \diamond_\lambda \delta_n(\mathscr{A}) + \sum_{\substack{r+s+t=n \\ 0 \leqslant r,s,t \leqslant (n-1)}} \delta_r(\mathscr{I}) \diamond_\lambda \delta_s(i\mathscr{I}) \diamond_\lambda \delta_t(\mathscr{A}).$$

Proceeding as above, we get $\delta_n(i\mathscr{A}) = i\delta_n(\mathscr{A})$ for all $\mathscr{A} \in \mathcal{A}$. $\square$

LEMMA 2.3. *$\mathscr{D}=\{\delta_n\}_{n\in\mathbb{N}}$ is a $*$-higher derivation.*

Proof. For any $\mathscr{A},\mathscr{B}\in\mathcal{A}$, we know $\mathscr{I}\diamond_\lambda\mathscr{A}\diamond_\lambda\mathscr{B}=\mathscr{A}\mathscr{B}+\lambda\mathscr{A}\mathscr{B}+\lambda\mathscr{B}\mathscr{A}^*+|\lambda|^2\mathscr{B}\mathscr{A}^*$. Hence

$$\begin{aligned}\delta_n(\mathscr{I}\diamond_\lambda\mathscr{A}\diamond_\lambda\mathscr{B}) &= \mathscr{I}\diamond_\lambda\delta_n(\mathscr{A})\diamond_\lambda\mathscr{B}+\mathscr{I}\diamond_\lambda\mathscr{A}\diamond_\lambda\delta_n(\mathscr{B})\\ &\quad+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(\mathscr{I})\diamond_\lambda\delta_s(\mathscr{A})\diamond_\lambda\delta_t(\mathscr{B})\\ &= \mathscr{I}\diamond_\lambda\delta_n(\mathscr{A})\diamond_\lambda\mathscr{B}+\mathscr{I}\diamond_\lambda\mathscr{A}\diamond_\lambda\delta_n(\mathscr{B})\\ &\quad+\sum_{\substack{s+t=n\\0\leqslant s,t\leqslant(n-1)}}\mathscr{I}\diamond_\lambda\delta_s(\mathscr{A})\diamond_\lambda\delta_t(\mathscr{B})\\ &= \delta_n(\mathscr{A})\mathscr{B}+\lambda\delta_n(\mathscr{A})\mathscr{B}+\lambda\mathscr{A}\delta_n(\mathscr{B})^*\\ &\quad+|\lambda|^2\mathscr{B}\delta_n(\mathscr{A})^*+\mathscr{A}\delta_n(\mathscr{B})+\lambda\mathscr{A}\delta_n(\mathscr{B})\\ &\quad+\lambda\delta_n(\mathscr{B})\mathscr{A}^*+|\lambda|^2\delta_n(\mathscr{B})\mathscr{A}^*\\ &\quad+\sum_{\substack{s+t=n\\0\leqslant s,t\leqslant(n-1)}}\mathscr{I}\diamond_\lambda\delta_s(\mathscr{A})\diamond_\lambda\delta_t(\mathscr{B}).\end{aligned}$$

Thus,

$$\begin{aligned}&\delta_n(\mathscr{A}\mathscr{B}+\lambda\mathscr{A}\mathscr{B}+\lambda\mathscr{B}\mathscr{A}^*+|\lambda|^2\mathscr{B}\mathscr{A}^*)\\ &=\delta_n(\mathscr{A})\mathscr{B}+\lambda\delta_n(\mathscr{A})\mathscr{B}+\lambda\mathscr{A}\delta_n(\mathscr{B})^*\\ &\quad+|\lambda|^2\mathscr{B}\delta_n(\mathscr{A})^*+\mathscr{A}\delta_n(\mathscr{B})+\lambda\mathscr{A}\delta_n(\mathscr{B})\\ &\quad+\lambda\delta_n(\mathscr{B})\mathscr{A}^*+|\lambda|^2\delta_n(\mathscr{B})\mathscr{A}^*\\ &\quad+\sum_{\substack{s+t=n\\0\leqslant s,t\leqslant(n-1)}}\mathscr{I}\diamond_\lambda\delta_s(\mathscr{A})\diamond_\lambda\delta_t(\mathscr{B}).\end{aligned}\tag{2.12}$$

On the other hand, since $\mathscr{A}\mathscr{B}+\lambda\mathscr{A}\mathscr{B}-\lambda\mathscr{B}\mathscr{A}^*-|\lambda|^2\mathscr{B}\mathscr{A}^*=\mathscr{I}\diamond_\lambda i\mathscr{A}\diamond_\lambda(-i\mathscr{B})$, for any $\mathscr{A},\mathscr{B}\in\mathcal{A}$, so we get

$$\begin{aligned}\delta_n(\mathscr{I}\diamond_\lambda i\mathscr{A}\diamond_\lambda(-i\mathscr{B})) &= \mathscr{I}\diamond_\lambda\delta_n(i\mathscr{A})\diamond_\lambda(-i\mathscr{B})+\mathscr{I}\diamond_\lambda i\mathscr{A}\diamond_\lambda\delta_n(-i\mathscr{B})\\ &\quad+\sum_{\substack{r+s+t=n\\0\leqslant r,s,t\leqslant(n-1)}}\delta_r(\mathscr{I})\diamond_\lambda\delta_s(i\mathscr{A})\diamond_\lambda\delta_t(-i\mathscr{B})\\ &= \mathscr{I}\diamond_\lambda\delta_n(i\mathscr{A})\diamond_\lambda(-i\mathscr{B})+\mathscr{I}\diamond_\lambda i\mathscr{A}\diamond_\lambda\delta_n(-i\mathscr{B})\\ &\quad+\sum_{\substack{s+t=n\\0\leqslant s,t\leqslant(n-1)}}\mathscr{I}\diamond_\lambda\delta_s(i\mathscr{A})\diamond_\lambda\delta_t(-i\mathscr{B})\\ &= \delta_n(\mathscr{A})\mathscr{B}+\lambda\delta_n(\mathscr{A})\mathscr{B}-\lambda\mathscr{A}\delta_n(\mathscr{B})^*\\ &\quad-|\lambda|^2\mathscr{B}\delta_n(\mathscr{A})^*+\mathscr{A}\delta_n(\mathscr{B})+\lambda\mathscr{A}\delta_n(\mathscr{B})\\ &\quad-\lambda\delta_n(\mathscr{B})\mathscr{A}^*-|\lambda|^2\delta_n(\mathscr{B})\mathscr{A}^*\\ &\quad+\sum_{\substack{s+t=n\\0\leqslant s,t\leqslant(n-1)}}\mathscr{I}\diamond_\lambda\delta_s(i\mathscr{A})\diamond_\lambda\delta_t(-i\mathscr{B}).\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathscr{A}\mathscr{B}+\lambda\mathscr{A}\mathscr{B}-\lambda\mathscr{B}\mathscr{A}^*-|\lambda|^2\mathscr{B}\mathscr{A}^* \\
&=\delta_n(\mathscr{A})\mathscr{B}+\lambda\delta_n(\mathscr{A})\mathscr{B}-\lambda\mathscr{A}\delta_n(\mathscr{B})^* \\
&\quad -|\lambda|^2\mathscr{B}\delta_n(\mathscr{A})^*+\mathscr{A}\delta_n(\mathscr{B})+\lambda\mathscr{A}\delta_n(\mathscr{B}) \\
&\quad -\lambda\delta_n(\mathscr{B})\mathscr{A}^*-|\lambda|^2\delta_n(\mathscr{B})\mathscr{A}^* \\
&\quad +\sum_{\substack{s+t=n\\ 0\leqslant s,t\leqslant(n-1)}}\mathscr{I}\diamond_\lambda\delta_s(i\mathscr{A})\diamond_\lambda\delta_t(-i\mathscr{B}).
\end{aligned} \tag{2.13}$$

Equivalantly, we obtain from (2.12) and (2.13) that

$$\begin{aligned}
\delta_n((1+\lambda)\mathscr{A}\mathscr{B}) &= (1+\lambda)\delta_n(\mathscr{A})\mathscr{B}+(1+\lambda)\mathscr{A}\delta_n(\mathscr{B}) \\
&\quad +(1+\lambda)\sum_{\substack{s+t=n\\ 0\leqslant s,t\leqslant(n-1)}}\delta_s(\mathscr{A})\delta_t(\mathscr{B}).
\end{aligned}$$

Observe from (2.11) that

$$\begin{aligned}
(1+\lambda)\delta_n(\mathscr{A}\mathscr{B}) &= (1+\lambda)\delta_n(\mathscr{A})\mathscr{B}+(1+\lambda)\mathscr{A}\delta_n(\mathscr{B}) \\
&\quad +(1+\lambda)\sum_{\substack{s+t=n\\ 0\leqslant s,t\leqslant(n-1)}}\delta_s(\mathscr{A})\delta_t(\mathscr{B}).
\end{aligned}$$

Finally, we get

$$\delta_n(\mathscr{A}\mathscr{B}) = \delta_n(\mathscr{A})\mathscr{B}+\mathscr{A}\delta_n(\mathscr{B})+\sum_{\substack{s+t=n\\ 0\leqslant s,t\leqslant(n-1)}}\delta_s(\mathscr{A})\delta_t(\mathscr{B}). \quad \square$$

LEMMA 2.4. *$\mathscr{D}=\{\delta_n\}_{n\in\mathbb{N}}$ is inner.*

Proof. We know that every additive derivation $\delta:\mathcal{A}\to\mathcal{B}(\mathscr{H})$ is an inner derivation by [16]. Therefore, it follows from [20, Proposition 2.6] that $\mathscr{D}=\{\delta_n\}_{n\in\mathbb{N}}$ is inner. □

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