

BILINEAR CALDERÓN–ZYGmund OPERATORS ON TWO WEIGHT HERZ SPACES WITH VARIABLE EXPONENTS

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Abstract. In this paper, we obtain the boundedness of bilinear Calderón-Zygmund operators on two weight Herz spaces with variable exponents.

1. Introduction

We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of all Schwartz functions on \mathbb{R}^n and by $\mathcal{S}'(\mathbb{R}^n)$ the space of all tempered distributions on \mathbb{R}^n . Let T be a bilinear operator, which is originally defined on the 2-fold of Schwartz function space $\mathcal{S}(\mathbb{R}^n)$, and its value belongs to $\mathcal{S}'(\mathbb{R}^n)$:

$$T : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n).$$

T is called a bilinear Calderón-Zygmund operator, if it extends to a bounded bilinear operator from $L^{p_1} \times L^{p_2}$ to L^p with $1/p_1 + 1/p_2 = 1/p$, and for $f_1, f_2 \in L_C^\infty(\mathbb{R}^n)$ (the space of compactly supported bounded functions), $x \notin \text{supp}(f_1) \cap \text{supp}(f_2)$

$$T(f_1, f_2)(x) := \int_{\mathbb{R}^{2n}} K(x, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2,$$

where the kernel K is a function in \mathbb{R}^{3n} off from the diagonal $x = y_1 = y_2$ and there exist positive constants ε, A such that

$$|K(x, y_1, y_2)| \leq \frac{A}{(|x - y_1| + |x - y_2| + |y_1 - y_2|)^{2n}}$$

and

$$|K(x, y_1, y_2) - K(x', y_1, y_2)| \leq \frac{A|x - x'|^\varepsilon}{(|x - y_1| + |x - y_2| + |y_1 - y_2|)^{2n+\varepsilon}}$$

whenever $|x - x'| \leq \frac{1}{2} \max\{|x - y_1|, |x - y_2|\}$, and the two analogous difference estimates with respect to the variables y_1 and y_2 hold.

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Bilinear Calderón-Zygmund operators are a special example of multilinear Calderón-Zygmund operators which were introduced by Coifman and Meyer in 1975 in [2]. In recent decades, multilinear Calderón-Zygmund operators have attracted more and more attention from many authors. Indeed, in [7, 8], Grafakos and Torres systematically investigated the boundedness of multilinear Calderón-Zygmund operators on the products of Lebesgue spaces, such as multilinear strong type and endpoint weak type estimation, multilinear interpolation, multilinear $T1$ theorem. Huang and the second author of the article obtained boundedness of multilinear Calderón-Zygmund operators and their commutators on products of variable Lebesgue spaces in [10]. Lu and Zhu in [13] proved the boundedness of multilinear Calderón-Zygmund operators on Herz-Morrey space $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\lambda}(\mathbb{R}^n)$ with variable exponents.

In [9], Grafakos and Torres obtained the boundedness of the maximal operator associated to multilinear Calderón-Zygmund operators on products of weighted Lebesgue spaces. Bui and Duong studied boundedness of multilinear Calderón-Zygmund operators on products of weighted Lebesgue spaces in [1]. Pérez and Torres gave a weighted norm inequality for multilinear Calderón-Zygmund operators by sharp maximal function in [15]. The authors of the paper proved weighted norm inequality of bilinear Calderón-Zygmund operators on weighted Herz-Morrey spaces $M\dot{K}_{p(\cdot),\lambda}^{\alpha(\cdot),q(\cdot)}(w)$ with variable exponents in [18].

Recently, Izuki and Noi introduced two weight Herz spaces with variable exponents in [12]. Motivated by the mentioned works, we will prove the boundedness of bilinear Calderón-Zygmund operators on two weight Herz spaces with variable exponents. The plan of the paper is as follows. In Section 2, we collect some notations and state the main result. The proof of the main result will be given in Section 3.

2. Notations and main result

In this section, we firstly recall some definitions and notations, then we state our result. Let $p(\cdot)$ be a measurable function on \mathbb{R}^n taking values in $[1, \infty)$, the Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) := \left\{ f \text{ is measurable on } \mathbb{R}^n : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx < \infty \text{ for some } \lambda > 0 \right\}.$$

Then $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}.$$

The space $L_{loc}^{p(\cdot)}(\mathbb{R}^n)$ is defined by $L_{loc}^{p(\cdot)}(\mathbb{R}^n) := \{f : f\chi_K \in L^{p(\cdot)}(\mathbb{R}^n) \text{ for all compact subsets } K \subset \mathbb{R}^n\}$, where and what follows, χ_S denotes the characteristic function of a measurable set $S \subset \mathbb{R}^n$. Let $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$, we denote $p^- := \text{ess inf}_{x \in \mathbb{R}^n} p(x)$, $p^+ := \text{ess sup}_{x \in \mathbb{R}^n} p(x)$. The set $\mathcal{P}(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p^- > 1$ and $p^+ < \infty$; $\mathcal{P}_0(\mathbb{R}^n)$ consists of all $p(\cdot)$ satisfying $p^- > 0$ and $p^+ < \infty$. $L^{p(\cdot)}$ can be

similarly defined as above for $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. Let $p'(\cdot)$ be the conjugate exponent of $p(\cdot)$, that means $1/p(\cdot) + 1/p'(\cdot) = 1$.

DEFINITION 1. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight. The weight variable Lebesgue space $L^{p(\cdot)}(w)$ is defined by

$$L^{p(\cdot)}(w) := \left\{ f \text{ is measurable on } \mathbb{R}^n \text{ and } \|f\|_{L^{p(\cdot)}(w)} := \|fw\|_{L^{p(\cdot)}(\mathbb{R}^n)} < \infty \right\}.$$

Then $L^{p(\cdot)}(w)$ is a Banach space.

Let $f \in L^1_{loc}(\mathbb{R}^n)$. Then the standard Hardy-Littlewood maximal function of f is defined by

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy, \quad \forall x \in \mathbb{R}^n,$$

where the supremum is taken over all balls containing x in \mathbb{R}^n . In general, the Hardy-Littlewood maximal operator is not bounded on weighted variable Lebesgue spaces. But if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and satisfies the following global log-Hölder continuity condition and $w \in A_{p(\cdot)}$, then M is bounded on $L^{p(\cdot)}(w)$.

DEFINITION 2. Let $\alpha(\cdot)$ be a real-valued measurable function on \mathbb{R}^n .

(i) The function $\alpha(\cdot)$ is locally log-Hölder continuous if there exists a constant C_1 such that

$$|\alpha(x) - \alpha(y)| \leq \frac{C_1}{\log(e + 1/|x - y|)}, \quad x, y \in \mathbb{R}^n, \quad |x - y| < \frac{1}{2}.$$

(ii) The function $\alpha(\cdot)$ is log-Hölder continuous at the origin if there exists a constant C_2 such that

$$|\alpha(x) - \alpha(0)| \leq \frac{C_2}{\log(e + 1/|x|)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}_0^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at the origin.

(iii) The function $\alpha(\cdot)$ is log-Hölder continuous at infinity if there exists $\alpha_\infty \in \mathbb{R}$ and a constant C_3 such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{C_3}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n.$$

Denote by $\mathcal{P}_\infty^{\log}(\mathbb{R}^n)$ the set of all log-Hölder continuous functions at infinity.

(iv) The function $\alpha(\cdot)$ is global log-Hölder continuous if $\alpha(\cdot)$ are both locally log-Hölder continuous and log-Hölder continuous at infinity. Denote by $\mathcal{P}^{\log}(\mathbb{R}^n)$ the set of all global log-Hölder continuous functions.

DEFINITION 3. Fix $p \in (1, \infty)$. A positive measurable function w is said to be in the Muckenhoupt class A_p , if there exists a positive constant C for all balls B in \mathbb{R}^n such that

$$\left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} \leq C.$$

We say $w \in A_1$, if $Mw(x) \leq Cw(x)$ for a.e. x . If $1 \leq p < q < \infty$, then $A_p \subset A_q$. We denote $A_\infty = \cup_{p>1} A_p$. The Muckenhoupt A_p class with constant exponent $p \in (1, \infty)$ was firstly proposed by Muckenhoupt in [14].

DEFINITION 4. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, a positive measurable function w is said to be in $A_{p(\cdot)}$, if exists a positive constant C for all balls B in \mathbb{R}^n such that

$$\frac{1}{|B|} \|w\chi_B\|_{L^{p(\cdot)}} \|w^{-1}\chi_B\|_{L^{p'(\cdot)}} < C.$$

REMARK 1. In [3], Cruz-Uribe, Fiorenza and Neugebauer showed that $w \in A_{p(\cdot)}$ if and only if the Hardy-Littlewood maximal operator M is bounded on the space $L^{p(\cdot)}(w)$.

To give the definitions of the weighted Herz space with variable exponents, we use the following notations. For each $k \in \mathbb{Z}$ we define $B_k := \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $D_k := B_k \setminus B_{k-1} = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$, $\chi_k := \chi_{D_k}$, $\tilde{\chi}_m = \chi_m$, $m \geq 1$, $\tilde{\chi}_0 = \chi_{B_0}$. We also need the notation of the variable mixed sequence space $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$. Let w be a nonnegative measurable function. Given a sequence of functions $\{f_j\}_{j \in \mathbb{Z}}$, define the modular

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}(\{f_j\}_j) := \sum_{j \in \mathbb{Z}} \inf \left\{ \lambda_j : \int_{\mathbb{R}^n} \left(\frac{|f_j(x)w(x)|}{\lambda_j^{\frac{1}{q(x)}}} \right)^{p(x)} dx \leq 1 \right\},$$

where $\lambda^{1/\infty} = 1$. If $q^+ < \infty$ or $q(\cdot) \leq p(\cdot)$, the above can be written as

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}(\{f_j\}_j) = \sum_{j \in \mathbb{Z}} \| |f_j w|^{q(\cdot)} \|_{L^{\frac{p(\cdot)}{q(\cdot)}}}.$$

The norm is

$$\| \{f_j\}_j \|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} := \inf \{ \mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))}(\{f_j/\mu\}_j) \leq 1 \}.$$

DEFINITION 5. Let w_1, w_2 be weights on \mathbb{R}^n , $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$. Let $\alpha(\cdot)$ be a bounded real-valued measurable function on \mathbb{R}^n . The homogeneous two weight Herz space $\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q(\cdot)}(w_1, w_2)$ and non-homogeneous two weight Herz space $K_{p(\cdot), \lambda}^{\alpha(\cdot), q(\cdot)}(w_1, w_2)$ are defined respectively by

$$\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(w_1, w_2) := \left\{ f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \{0\}, w_2) : \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(w_1, w_2)} < \infty \right\},$$

and

$$K_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(w_1, w_2) := \left\{ f \in L_{\text{loc}}^{p(\cdot)}(\mathbb{R}^n, w_2) : \|f\|_{K_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(w_1, w_2)} < \infty \right\},$$

where

$$\|f\|_{K_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(w_1, w_2)} := \| \{w_1(B_k)^{\alpha(\cdot)/n} f \chi_k\}_{k \in \mathbb{Z}} \|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w_2))}$$

and

$$\|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2)} := \|\{w_1(B_k)^{\alpha(\cdot)/n} f\tilde{\chi}_k\}_{k \geq 0}\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w_2))}.$$

For any quantities A and B , $A \lesssim B$ represents that there exists a constant $C > 0$ such that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, we denote $A \approx B$.

LEMMA 1. (see [12, Theorem 3]) *Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, $w_1 \in A_r$ for some $r \in [1, \infty)$ and w_2 be a weight. If $\alpha(\cdot)$ and $q(\cdot)$ are log-Hölder continuous at infinity, then*

$$K_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2) = K_{p(\cdot)}^{\alpha_\infty,q_\infty}(w_1,w_2).$$

Additionally, if $\alpha(\cdot)$ and $q(\cdot)$ are log-Hölder continuous at the origin, then

$$\begin{aligned} \|f\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot),q(\cdot)}(w_1,w_2)} &\approx \left(\sum_{k \leq 0} w_1(B_k)^{\alpha(0)/n} \|f\tilde{\chi}_k\|_{L^{p(\cdot)}(w_2)}^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\quad + \left(\sum_{k > 0} w_1(B_k)^{\alpha_\infty/n} \|f\tilde{\chi}_k\|_{L^{p(\cdot)}(w_2)}^{q_\infty} \right)^{\frac{1}{q_\infty}}. \end{aligned}$$

The following lemma 2 is well known, for example, see [17, 5, 6].

LEMMA 2. *Let w be a weight on \mathbb{R}^n . If $r \in [1, \infty)$ and $w \in A_r$, then there exist constants $\delta \in (0, 1)$ and $C > 0$ such that for all balls B in \mathbb{R}^n and all measurable subsets $S \subset B$,*

$$\frac{w(B)}{w(S)} \leq C \left(\frac{B}{S}\right)^r, \tag{1}$$

$$\frac{w(S)}{w(B)} \leq C \left(\frac{S}{B}\right)^\delta. \tag{2}$$

If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then $p'(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w^{-1} \in A_{p'(\cdot)}$. Therefore the Hardy-Littlewood maximal operator is bounded both on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-1})$. Hence, by Lemma 6 in [11], we have the following lemma.

LEMMA 3. *If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then there exist constants $\delta_1, \delta_2 \in (0, 1)$ and $C > 0$ such that for all balls B in \mathbb{R}^n and all measurable subsets $S \subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}(w)}}{\|\chi_B\|_{L^{p(\cdot)}(w)}} \leq C \left(\frac{|S|}{|B|}\right)^{\delta_1}, \tag{3}$$

$$\frac{\|\chi_S\|_{L^{p'(\cdot)}(w^{-1})}}{\|\chi_B\|_{L^{p'(\cdot)}(w^{-1})}} \leq C \left(\frac{|S|}{|B|}\right)^{\delta_2}. \tag{4}$$

Our main result is as follows.

THEOREM 1. *Assume that T is a bilinear Calderón-Zygmund operator and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Let w_1, w_2 be weights, $w = w_1 w_2$, $w_i \in A_{p_i(\cdot)}$, and $v \in A_r$ for some $r \in [1, \infty)$, $i = 1, 2$. Suppose that $p_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ satisfying $1/p(x) = 1/p_1(x) + 1/p_2(x)$ for $x \in \mathbb{R}^n$, $\alpha_i(\cdot) \in L^\infty(\mathbb{R}^n) \cap \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $q_i(\cdot) \in \mathcal{P}_0^{\log}(\mathbb{R}^n) \cap \mathcal{P}_\infty^{\log}(\mathbb{R}^n)$, $i = 1, 2$. $\alpha(x) = \alpha_1(x) + \alpha_2(x)$, $1/q(x) = 1/q_1(x) + 1/q_2(x)$, $\delta_{i2} \in (0, 1)$ are the constants in Lemma 3 for exponents $p_i(\cdot)$ and weights w_i , $i = 1, 2$, respectively. Let $\delta \in (0, 1)$ be the constant in Lemma 2. Set*

$$w_i^- = \begin{cases} \delta & \text{if } \alpha_i^- \geq 0 \\ r & \text{if } \alpha_i^- < 0 \end{cases} \quad \text{and} \quad w_i^+ = \begin{cases} r & \text{if } \alpha_i^+ \geq 0 \\ \delta & \text{if } \alpha_i^+ < 0 \end{cases} . \tag{5}$$

If $-n\delta_{i1} < w_i^- \alpha_i^-$ and $w_i^+ \alpha_i^+ < n\delta_{i2}$, $i = 1, 2$, then there is a constant C such that for each $f_i \in \dot{K}_{p_i(\cdot)}^{\alpha_i(\cdot), q_i(\cdot)}(v, w_i)$, $i = 1, 2$,

$$\|T(f_1, f_2)\|_{\dot{K}_{p(\cdot)}^{\alpha(\cdot), q(\cdot)}(v, w)} \leq C \prod_{i=1}^2 \|f_i\|_{\dot{K}_{p_i(\cdot)}^{\alpha_i(\cdot), q_i(\cdot)}(v, w_i)}.$$

3. Proof of Theorem 1

To prove Theorem 1, we need a series of lemmas.

LEMMA 4. (see [12, Lemma 7]) *Let $k, l \in \mathbb{Z}$, $w \in A_q$ with $q \in [1, \infty)$, $\delta \in (0, 1)$ be the constant in Lemma 2, If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and is log-Hölder continuous both at the origin and infinity, then for any $x \in D_k$ and $y \in D_l$,*

$$w(B_k)^{\alpha(x)} \leq C w(B_l)^{\alpha(y)} \times \begin{cases} 2^{(k-l)nw^+ \alpha^+} & \text{if } l \leq k - 1 \\ 1 & \text{if } k - 1 < l \leq k + 1, \\ 2^{(k-l)nw^- \alpha^-} & \text{if } l > k + 1 \end{cases}$$

where α^- and α^+ are similar to (5).

LEMMA 5. (see [12, Lemma 8]) *If $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and is log-Hölder continuous both at the origin and infinity. then for all $k \in \mathbb{Z}$ and $x \in D_k$,*

$$\begin{aligned} w(D_k)^{\alpha(x)} &\approx w(D_k)^{\alpha_\infty}, \text{ if } k \geq 0, \\ w(D_k)^{\alpha(x)} &\approx w(D_k)^{\alpha(0)}, \text{ if } k \leq -1. \end{aligned}$$

LEMMA 6. (see [10, Theorem 2.3]) *Let $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ such that $1/p(x) = 1/p_1(x) + 1/p_2(x)$ for $x \in \mathbb{R}^n$. Then there exists a constant C_{p, p_1} independent of functions f and g such that*

$$\|fg\|_{L^{p(\cdot)}} \leq C_{p, p_1} \|f\|_{L^{p_1(\cdot)}} \|g\|_{L^{p_2(\cdot)}}$$

holds for every $f \in L^{p_1(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$.

If $p \in \mathcal{P}(\mathbb{R}^n)$, $w \in A_{p(\cdot)}$ with $w = w_1 w_2$, then by the Hölder inequality, we have

$$\|fg\|_{L^{p(\cdot)}(w)} \leq C_{p,p_1} \|f\|_{L^{p_1(\cdot)}(w_1)} \|g\|_{L^{p_2(\cdot)}(w_2)}.$$

The following lemma is a corollary of Theorem 2.8 in [4].

LEMMA 7. Let $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $1 < (p_i)^- \leq (p_i)^+$ and $p_i(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n) \cap \mathcal{P}(\mathbb{R}^n)$ satisfying $1/p(x) = 1/p_1(x) + 1/p_2(x)$ for $x \in \mathbb{R}^n$, $i = 1, 2$. Let $w_1 \in A_{p_1(\cdot)}, w_2 \in A_{p_2(\cdot)}$ and $w = w_1 w_2$. If T is a bilinear Calderón-Zygmund operator, then

$$\|T(f_1, f_2)\|_{L^{p(\cdot)}(w)} \lesssim \|f_1\|_{L^{p_1(\cdot)}(w_1)} \|f_2\|_{L^{p_2(\cdot)}(w_2)}.$$

LEMMA 8. (see [16, Proposition 1.2]) Let $0 < p < \infty$, $\delta > 0$. Then there is a positive constant C such that

$$\left(\sum_{j=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} 2^{-|k-j|\delta} a_k \right)^p \right)^{1/p} \leq C \left(\sum_{j=-\infty}^{\infty} a_j^p \right)^{1/p} \tag{6}$$

for non-negative sequences $\{a_j\}_{j=-\infty}^{\infty}$.

This is the position to state the proof of Theorem 1.

Proof of Theorem 1. Let f_1 and f_2 be bounded functions with compact support and write

$$f_i = \sum_{l=-\infty}^{\infty} f_i \chi_l =: \sum_{l=-\infty}^{\infty} f_{il}, \quad i = 1, 2.$$

By Lemmas 1 and 5, we have

$$\begin{aligned} \|T(f_1, f_2)\|_{\dot{K}_{p(\cdot), \lambda}^{\alpha(\cdot), q(\cdot)}(v, w)} &\approx \left(\sum_{k=-\infty}^{-1} \|v(B_k)^{\alpha(0)/n} T(f_1, f_2) \chi_k\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\quad + \left(\sum_{k=0}^{\infty} \|v(B_k)^{\alpha_{\infty}/n} T(f_1, f_2) \chi_k\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}} \\ &=: E + F. \end{aligned}$$

We decompose f_i into three parts as

$$f_i = \sum_{l=-\infty}^{k-2} f_{il} + \sum_{l=k-1}^{k+1} f_{il} + \sum_{l=k+2}^{\infty} f_{il}.$$

Therefore we decompose $T(f_1, f_2)$ into nine parts. Then we have

$$E \leq C \sum_{i=1}^9 E_i, \quad F \leq C \sum_{i=1}^9 F_i,$$

where

$$E_1 := \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)/n} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=-\infty}^{k-2} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}},$$

$$E_2 := \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)/n} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}},$$

$$E_3 := \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)/n} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}},$$

$$E_4 := \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=-\infty}^{k-2} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}},$$

$$E_5 := \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}},$$

$$E_6 := \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}},$$

$$E_7 := \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)/n} \left\| \sum_{l=k+2}^{\infty} \sum_{j=-\infty}^{k-2} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}},$$

$$E_8 := \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)/n} \left\| \sum_{l=k+2}^{\infty} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}},$$

$$E_9 := \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)/n} \left\| \sum_{l=k+2}^{\infty} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}},$$

$$F_1 := \left(\sum_{k=0}^{\infty} v(B_k)^{\alpha_{\infty}/n} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=-\infty}^{k-2} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}},$$

$$F_2 := \left(\sum_{k=0}^{\infty} v(B_k)^{\alpha_{\infty}/n} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}},$$

$$F_3 := \left(\sum_{k=0}^{\infty} v(B_k)^{\alpha_{\infty}/n} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}},$$

$$F_4 := \left(\sum_{k=0}^{\infty} v(B_k)^{\alpha_{\infty}/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=-\infty}^{k-2} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}},$$

$$F_5 := \left(\sum_{k=0}^{\infty} v(B_k)^{\alpha_{\infty}/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}},$$

$$\begin{aligned}
 F_6 &:= \left(\sum_{k=0}^{\infty} v(B_k)^{\alpha_{\infty}/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}}, \\
 F_7 &:= \left(\sum_{k=0}^{\infty} v(B_k)^{\alpha_{\infty}/n} \left\| \sum_{l=k+2}^{\infty} \sum_{j=-\infty}^{k-2} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}}, \\
 F_8 &:= \left(\sum_{k=0}^{\infty} v(B_k)^{\alpha_{\infty}/n} \left\| \sum_{l=k+2}^{\infty} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}}, \\
 F_9 &:= \left(\sum_{k=0}^{\infty} v(B_k)^{\alpha_{\infty}/n} \left\| \sum_{l=k+2}^{\infty} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}}.
 \end{aligned}$$

We shall use the following estimates. If $l \leq k - 1$, then by Hölder’s inequality, Definition 4 and Lemma 3, we have

$$\begin{aligned}
 \left\| 2^{-kn} \int_{\mathbb{R}^n} f_{il} dy_i \chi_k \right\|_{L^{p_i(\cdot)}(w_i)} &\lesssim 2^{-kn} \|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)} \|f_i w_i \chi_l\|_{L^{p_i(\cdot)}} \|\chi_l w_i^{-1}\|_{L^{p'_i(\cdot)}} \\
 &\lesssim 2^{-kn} |B_k| \|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i^{-1})}^{-1} \|\chi_{B_l}\|_{L^{p'_i(\cdot)}(w_i^{-1})} \|f_i \chi_l\|_{L^{p_i(\cdot)}(w_i)} \\
 &\lesssim 2^{(l-k)n\delta_2} \|f_{il}\|_{L^{p_i(\cdot)}(w_i)}.
 \end{aligned} \tag{7}$$

If $l = k$, then

$$\begin{aligned}
 \left\| 2^{-kn} \int_{\mathbb{R}^n} f_{il} dy_i \chi_k \right\|_{L^{p_i(\cdot)}(w_i)} &\lesssim 2^{-kn} \|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)} \|f_{il} w_i \chi_l\|_{L^{p_i(\cdot)}} \|\chi_l w_i^{-1}\|_{L^{p'_i(\cdot)}} \\
 &\lesssim 2^{-kn} \|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)} \|\chi_{B_l}\|_{L^{p'_i(\cdot)}(w_i^{-1})} \|f_i \chi_l\|_{L^{p_i(\cdot)}(w_i)} \\
 &\lesssim \|f_{il}\|_{L^{p_i(\cdot)}(w_i)}.
 \end{aligned} \tag{8}$$

If $l \geq k + 1$, then

$$\begin{aligned}
 \left\| 2^{-kn} \int_{\mathbb{R}^n} f_{il} dy_i \chi_k \right\|_{L^{p_i(\cdot)}(w_i)} &\lesssim 2^{-kn} \|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)} \|f_i w_i \chi_l\|_{L^{p_i(\cdot)}} \|\chi_l w_i^{-1}\|_{L^{p'_i(\cdot)}} \\
 &\lesssim 2^{-kn} \|\chi_{B_k}\|_{L^{p_i(\cdot)}(w_i)} \|\chi_{B_l}\|_{L^{p_i(\cdot)}(w_i)} \|\chi_{B_l}\|_{L^{p_i(\cdot)}(w_i)}^{-1} \\
 &\quad \times \|\chi_{B_l}\|_{L^{p'_i(\cdot)}(w_i^{-1})} \|f_i \chi_l\|_{L^{p_i(\cdot)}(w_i)} \\
 &\lesssim 2^{(l-k)n(1-\delta_{i1})} \|f_{il}\|_{L^{p_i(\cdot)}(w_i)}.
 \end{aligned} \tag{9}$$

By the interchange of f_1 and f_2 , we see that the estimates of E_2, E_3 and E_6 are similar to E_4, E_7 and E_8 , respectively. Thus we only need to estimate E_1, E_2, E_3, E_5, E_6 , and E_9 .

Estimate E_1 . Since $l, j \leq k - 2$, we deduce that for $i = 1, 2$,

$$|x - y_i| \geq |x| - |y_i| > 2^{k-1} - 2^{\min\{l,j\}} \geq 2^{k-2}, \quad x \in D_k, y_1 \in D_l, y_2 \in D_j.$$

Therefore, for $x \in D_k$, we have

$$|K(x, y_1, y_2)| \leq C(|x - y_1| + |x - y_2|)^{-2n} \leq C2^{-2kn}.$$

Thus, $\forall x \in D_k$ and $l, j \leq k - 2$, we have

$$\begin{aligned} |T(f_{1l}, f_{2j})(x)| &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_{1l}(y_1)| |f_{2j}(y_2)|}{(|x - y_1| + |x - y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim 2^{-2kn} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f_{1l}(y_1)| |f_{2j}(y_2)| dy_1 dy_2. \end{aligned}$$

Since $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ and $w = w_1 w_2$, by Hölder’s inequality, we obtain

$$\begin{aligned} &\left\| \sum_{l=-\infty}^{k-2} \sum_{j=-\infty}^{k-2} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim 2^{-2kn} \left\| \sum_{l=-\infty}^{k-2} \sum_{j=-\infty}^{k-2} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad \times \left\| \sum_{j=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{10}$$

Since $\alpha(0) = \alpha_1(0) + \alpha_2(0)$ and $1/q(0) = 1/q_1(0) + 1/q_2(0)$, by Hölder’s inequality, we have

$$\begin{aligned} E_1 &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k) \alpha^{(0)q(0)/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \right. \\ &\quad \left. \times \left\| \sum_{j=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k) \alpha_1^{(0)q_1(0)/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{\frac{1}{q_1(0)}} \\ &\quad \times \left(\sum_{k=-\infty}^{-1} v(B_k) \alpha_2^{(0)q_2(0)/n} \left\| \sum_{j=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\ &:= E_{1,1} \times E_{1,2}, \end{aligned}$$

where

$$E_{1,i} := \left(\sum_{k=-\infty}^{-1} v(B_k) \alpha_i^{(0)q_i(0)/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{il}(y_i)| dy_i \chi_k \right\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)} \right)^{\frac{1}{q_i(0)}}.$$

Since $n\delta_{i2} - w_i^+ \alpha_i^+ > 0$, by (7), Lemmas 4, 5 and 8 we obtain

$$\begin{aligned} E_{1,i} &\lesssim \left\{ \sum_{k=-\infty}^{-1} v(B_k)^{\alpha_i(0)q_i(0)/n} \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)n\delta_{i2}} \|f_{il}\|_{L^{p_i(\cdot)}(w_i)} \right)^{q_i(0)} \right\}^{\frac{1}{q_i(0)}} \\ &\lesssim \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{l=-\infty}^{k-2} 2^{(l-k)(n\delta_{i2} - w_i^+ \alpha_i^+)} \|v(B_l)^{\alpha_i(\cdot)/n} f_{il}\|_{L^{p_i(\cdot)}(w_i)} \right)^{q_i(0)} \right\}^{\frac{1}{q_i(0)}} \\ &\lesssim \left\{ \sum_{l=-\infty}^{-3} \|v(B_l)^{\alpha_i(\cdot)/n} f_{il}\|_{L^{p_i(\cdot)}(w_i)}^{q_i(0)} \right\}^{\frac{1}{q_i(0)}} \\ &\lesssim \|f_i\|_{\dot{K}_{p_i(\cdot)}^{\alpha_i(\cdot), q_i(\cdot)}(v, w_i)}. \end{aligned}$$

Thus, we obtain

$$E_1 \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot), \lambda_1}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot), \lambda_2}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.$$

Estimate E_2 . Since $l \leq k - 2$, $k - 1 \leq j \leq k + 1$ for $i = 1, 2$, then we have

$$|x - y_1| \geq |x| - |y_1| \geq 2^{k-2}, \quad x \in D_k, \quad y_1 \in D_l.$$

Therefore, for $x \in D_k$, we obtain

$$\begin{aligned} |T(f_{1l}, f_{2j})(x)| &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_{1l}(y_1)| |f_{2j}(y_2)|}{(|x - y_1| + |x - y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim 2^{-2kn} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f_{1l}(y_1)| |f_{2j}(y_2)| dy_1 dy_2. \end{aligned}$$

Since $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ and $w = w_1 w_2$, by Hölder’s inequality, we obtain

$$\begin{aligned} &\left\| \sum_{l=-\infty}^{k-2} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad \times \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{11}$$

Since $\alpha(0) = \alpha_1(0) + \alpha_2(0)$ and $1/q(0) = 1/q_1(0) + 1/q_2(0)$, by Hölder’s inequality, we have

$$E_2 \lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)q(0)/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \right)^{\frac{1}{q(0)}}$$

$$\begin{aligned}
 & \times \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \Big)^{\frac{1}{q(0)}} \\
 \lesssim & \left(\sum_{k=-\infty}^{-1} v(B_k) \alpha_1(0) q_1(0)/n \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{\frac{1}{q_1(0)}} \\
 & \times \left(\sum_{k=-\infty}^{-1} v(B_k) \alpha_2(0) q_2(0)/n \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\
 := & E_{2,1} \times E_{2,2}.
 \end{aligned}$$

It is obvious that

$$E_{2,1} = E_{1,1} \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)}.$$

Since $n\delta_{22} - w_2^+ \alpha_2^+ > 0$, taking (7), (8) and (9) together, then by Lemmas 4 and 5, we have

$$\begin{aligned}
 & v(B_k) \alpha_2(0)/n \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)} \\
 & \lesssim v(B_k) \alpha_2(0)/n 2^{-(n\delta_{22} - w_2^+ \alpha_2^+)} \|f_2 \chi_{k-1}\|_{L^{p_2(\cdot)}(w_2)} + v(B_k) \alpha_2(0)/n \|f_2 \chi_k\|_{L^{p_2(\cdot)}(w_2)} \\
 & \quad + v(B_k) \alpha_2(0)/n 2^{n(1-\delta_{21})} \|f_2 \chi_{k+1}\|_{L^{p_2(\cdot)}(w_2)} \\
 & \lesssim 2^{-(n\delta_{22} - w_2^+ \alpha_2^+)} \|v(B_{k-1}) \alpha_2(\cdot)/n f_2 \chi_{k-1}\|_{L^{p_2(\cdot)}(w_2)} + \|v(B_k) \alpha_2(\cdot)/n f_2 \chi_k\|_{L^{p_2(\cdot)}(w_2)} \\
 & \quad + 2^{n(1-\delta_{21})} \|v(B_{k+1}) \alpha_2(\cdot)/n f_2 \chi_{k+1}\|_{L^{p_2(\cdot)}(w_2)} \\
 & \lesssim \sum_{j=k-1}^{k+1} 2^{(j-k)n} \|v(B_j) \alpha_2(\cdot)/n f_{2j}\|_{L^{p_2(\cdot)}(w_2)}. \tag{12}
 \end{aligned}$$

Estimate $E_{2,2}$. By (12), we have

$$\begin{aligned}
 E_{2,2} & \lesssim \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{j=k-1}^{k+1} 2^{(j-k)n} \|v(B_j) \alpha_2(\cdot)/n f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_2(0)} \right\}^{\frac{1}{q_2(0)}} \\
 & \lesssim \left\{ \sum_{k=-\infty}^{-1} \|v(B_k) \alpha_2(\cdot)/n f_2 \chi_k\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right\}^{\frac{1}{q_2(0)}} \\
 & \lesssim \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.
 \end{aligned}$$

Thus, we obtain

$$E_2 \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.$$

Estimate E_3 . Since $l \leq k - 2, j \geq k + 2$, then we have

$$|x - y_1| \geq |x| - |y_1| \geq 2^{k-2}, |x - y_2| \geq |y_2| - |x| > 2^{j-2}, x \in D_k, y_1 \in D_l, y_2 \in D_j.$$

Therefore, $\forall x \in D_k, l \leq k - 2, j \geq k + 2$, we obtain

$$\begin{aligned} |T(f_{1l}, f_{2j})(x)| &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_{1l}(y_1)| |f_{2j}(y_2)|}{(|x - y_1| + |x - y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim 2^{-kn} 2^{-jn} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f_{1l}(y_1)| |f_{2j}(y_2)| dy_1 dy_2. \end{aligned}$$

Since $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ and $w = w_1 w_2$, by Hölder’s inequality, we have

$$\begin{aligned} &\sum_{l=-\infty}^{k-2} \sum_{j=k+2}^{\infty} \left\| T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=-\infty}^{k-2} \sum_{j=k+2}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{13}$$

Since $\alpha(0) = \alpha_1(0) + \alpha_2(0)$ and $1/q(0) = 1/q_1(0) + 1/q_2(0)$, by Hölder’s inequality, we have

$$\begin{aligned} E_3 &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)q(0)/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \right. \\ &\quad \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_1(0)q_1(0)/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{\frac{1}{q_1(0)}} \\ &\quad \times \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_2(0)q_2(0)/n} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\ &:= E_{3,1} \times E_{3,2}. \end{aligned}$$

It is obvious that

$$E_{3,1} = E_{1,1} \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)}.$$

Estimate $E_{3,2}$. Since $w_2^- \alpha_2^- + n\delta_{21} > 0$, by (9), Lemmas 4, 5 and 8, we obtain

$$\begin{aligned} E_{3,2} &\lesssim \left\{ \sum_{k=-\infty}^{-1} v(B_k)^{\alpha_2(0)q_2(0)/n} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)n\delta_{21}} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_2(0)} \right\}^{\frac{1}{q_2(0)}} \\ &\lesssim \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(w_2^- \alpha_2^- + n\delta_{21})} \|v(B_j)^{\alpha_2(\cdot)/n} f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_2(0)} \right\}^{\frac{1}{q_2(0)}} \end{aligned}$$

$$\lesssim \left\{ \sum_{j=-\infty}^1 \|v(B_j)^{\alpha_2(\cdot)/n} f_{2j}\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right\}^{\frac{1}{q_2(0)}}$$

$$\lesssim \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.$$

Thus, we have

$$E_3 \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.$$

Since $\alpha(0) = \alpha_1(0) + \alpha_2(0)$ and $n\delta_{i2} - w_i^+ \alpha_i^+ > 0, i = 1, 2$, by Lemmas 4 and 5 and 7, we obtain

$$\begin{aligned} & v(B_k)^{\alpha(0)/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)} \\ & \lesssim v(B_k)^{\alpha(0)/n} \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} \|f_{1l}\|_{L^{p_1(\cdot)}(w_1)} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \\ & \lesssim v(B_k)^{\alpha_2(0)/n} \sum_{j=k-1}^{k+1} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \left(2^{-(n\delta_{12} - w_1^+ \alpha_1^+)} \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l} \chi_{k-1}\|_{L^{p_1(\cdot)}(w_1)} \right. \\ & \quad \left. + \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l} \chi_k\|_{L^{p_1(\cdot)}(w_1)} + \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l} \chi_{k+1}\|_{L^{p_1(\cdot)}(w_1)} \right) \\ & \lesssim v(B_k)^{\alpha_2(0)/n} \sum_{j=k-1}^{k+1} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \sum_{l=k-1}^{k+1} \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l}\|_{L^{p_1(\cdot)}(w_1)} \\ & \lesssim \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l}\|_{L^{p_1(\cdot)}(w_1)} \|v(B_j)^{\alpha_2(\cdot)/n} f_{2j}\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{14}$$

Estimate E_5 . Since $1/q(0) = 1/q_1(0) + 1/q_2(0)$, by (14) and Hölder’s inequality, we have

$$\begin{aligned} E_5 & \lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)q(0)/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q(0)} \right)^{\frac{1}{q(0)}} \\ & \lesssim \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{l=k-1}^{k+1} \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l}\|_{L^{p_1(\cdot)}(w_1)} \right)^{q_1(0)} \right\}^{\frac{1}{q_1(0)}} \\ & \quad \times \left\{ \sum_{k=-\infty}^{-1} \left(\sum_{j=k-1}^{k+1} \|v(B_j)^{\alpha_2(\cdot)/n} f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_2(0)} \right\}^{\frac{1}{q_2(0)}} \\ & \lesssim \left\{ \sum_{k=-\infty}^{-1} \|v(B_k)^{\alpha_1(\cdot)/n} f_1 \chi_k\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right\}^{\frac{1}{q_1(0)}} \\ & \quad \times \left\{ \sum_{k=-\infty}^{-1} \|v(B_k)^{\alpha_2(\cdot)/n} f_2 \chi_k\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right\}^{\frac{1}{q_2(0)}} \\ & \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}. \end{aligned}$$

Estimate E_6 . Since $k - 1 \leq l \leq k + 1$ and $j \geq k + 2$, then we obtain

$$|x - y_1| > 2^{k-2}, |x - y_2| > 2^{j-2}, x \in D_k, y_1 \in D_l, y_2 \in D_j.$$

Thus, $\forall x \in D_k, k - 1 \leq l \leq k + 1$ and $j \geq k + 2$, we obtain

$$\begin{aligned} |T(f_{1l}, f_{2j})(x)| &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_{1l}(y_1)| |f_{2j}(y_2)|}{(|x - y_1| + |x - y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim 2^{-kn} 2^{-jn} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f_{1l}(y_1)| |f_{2j}(y_2)| dy_1 dy_2. \end{aligned}$$

Since $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ and $w = w_1 w_2$, by Hölder's inequality, we obtain

$$\begin{aligned} &\left\| \sum_{l=k-1}^{k+1} \sum_{j=k+2}^{\infty} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=k-1}^{k+1} \sum_{j=k+2}^{\infty} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{15}$$

Since $\alpha(0) = \alpha_1(0) + \alpha_2(0)$ and $1/q(0) = 1/q_1(0) + 1/q_2(0)$, by Hölder's inequality, we have

$$\begin{aligned} E_6 &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)q(0)/n} \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \right. \\ &\quad \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_1(0)q_1(0)/n} \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{\frac{1}{q_1(0)}} \\ &\quad \times \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_2(0)q_2(0)/n} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\ &:= E_{6,1} \times E_{6,2}. \end{aligned}$$

By the interchange of f_1 and f_2 , we see that the estimate of $E_{6,1}$ is similar to the estimate of $E_{2,2}$ and $E_{6,2} = E_{3,2}$.

Estimate E_9 , since $l, j \geq k + 2$, then we get

$$|x - y_i| > 2^{k-2}, x \in D_k, y_1 \in D_l, y_2 \in D_j.$$

Therefore, $\forall x \in D_k, l, j \geq k + 2$, we have

$$\begin{aligned} |T(f_{1l}, f_{2j})(x)| &\lesssim \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f_{1l}(y_1)||f_{2j}(y_2)|}{(|x - y_1| + |x - y_2|)^{2n}} dy_1 dy_2 \\ &\lesssim 2^{-ln} 2^{-jn} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f_{1l}(y_1)||f_{2j}(y_2)| dy_1 dy_2. \end{aligned}$$

Since $1/p(\cdot) = 1/p_1(\cdot) + 1/p_2(\cdot)$ and $w = w_1 w_2$, by Hölder's inequality, we have

$$\begin{aligned} &\sum_{l=k+2}^{\infty} \sum_{j=k+2}^{\infty} \|T(f_{1l}, f_{2j})\chi_k\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=k+2}^{\infty} \sum_{j=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim \left\| \sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)} \\ &\quad \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}. \end{aligned} \tag{16}$$

Since $\alpha(0) = \alpha_1(0) + \alpha_2(0)$ and $1/q(0) = 1/q_1(0) + 1/q_2(0)$, by Hölder's inequality, we have

$$\begin{aligned} E_9 &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha(0)q(0)/n} \left\| \sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q(0)} \right. \\ &\quad \left. \times \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q(0)} \right)^{\frac{1}{q(0)}} \\ &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_1(0)q_1(0)/n} \left\| \sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_1(0)} \right)^{\frac{1}{q_1(0)}} \\ &\quad \times \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_2(0)q_2(0)/n} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_2(0)} \right)^{\frac{1}{q_2(0)}} \\ &:= E_{9,1} \times E_{9,2}. \end{aligned}$$

Obviously, the estimate $E_{9,i}$ is similar to that of $E_{3,2}$ for $i = 1, 2$.

Taking all estimates for E_i together, $i = 1, 2, \dots, 9$, we obtain

$$E \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(w_2)}.$$

Finally, we estimate F . By the interchange of f_1 and f_2 , we see that the estimates of F_2, F_3 and F_6 are similar to those of F_4, F_7 and F_8 , respectively. Thus we only necessary to estimate F_1, F_2, F_3, F_5, F_6 , and F_9 .

Estimate F_1 , since $l, j \leq k-2$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$ and $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, by (10) and Hölder's inequality, we have

$$\begin{aligned} F_1 &\lesssim \left(\sum_{k=0}^\infty v(B_k)^{\alpha_\infty q_\infty/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty} \right. \\ &\quad \times \left. \left\| \sum_{j=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty} \right)^{\frac{1}{q_\infty}} \\ &\lesssim \left(\sum_{k=0}^\infty v(B_k)^{\alpha_{1\infty} q_{1\infty}/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{\frac{1}{q_{1\infty}}} \\ &\quad \times \left(\sum_{k=0}^\infty v(B_k)^{\alpha_{2\infty} q_{2\infty}/n} \left\| \sum_{j=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{\frac{1}{q_{2\infty}}} \\ &:= F_{1,1} \times F_{1,2}. \end{aligned}$$

where

$$F_{1,i} := \left(\sum_{k=0}^\infty v(B_k)^{\alpha_{i\infty} q_{i\infty}/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{il}(y_i)| dy_i \chi_k \right\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}} \right)^{\frac{1}{q_{i\infty}}}.$$

Estimate $F_{1,i}$. By (7), Lemmas 4 and 5, we obtain

$$\begin{aligned} F_{1,i} &\lesssim \left\{ \sum_{k=0}^\infty \left(\sum_{l=-\infty}^{-1} \|v(B_l)^{\alpha_i(\cdot)/n} f_{il}\|_{L^{p_i(\cdot)}(w_i)} 2^{(k-l)(w_i^+ \alpha_i^+ - n\delta_{i2})} \right. \right. \\ &\quad \left. \left. + \sum_{l=0}^k \|v(B_l)^{\alpha_i(\cdot)/n} f_{il}\|_{L^{p_i(\cdot)}(w_i)} 2^{(k-l)(w_i^+ \alpha_i^+ - n\delta_{i2})} \right)^{q_{i\infty}} \right\}^{\frac{1}{q_{i\infty}}} \\ &\lesssim \left\{ \sum_{k=0}^\infty \left(\sum_{l=-\infty}^{-1} \|v(B_l)^{\alpha_i(\cdot)/n} f_{il}\|_{L^{p_i(\cdot)}(w_i)} 2^{(k-l)(w_i^+ \alpha_i^+ - n\delta_{i2})} \right)^{q_{i\infty}} \right\}^{\frac{1}{q_{i\infty}}} \\ &\quad + \left\{ \sum_{k=0}^\infty \left(\sum_{l=0}^k \|v(B_l)^{\alpha_i(\cdot)/n} f_{il}\|_{L^{p_i(\cdot)}(w_i)} 2^{(k-l)(w_i^+ \alpha_i^+ - n\delta_{i2})} \right)^{q_{i\infty}} \right\}^{\frac{1}{q_{i\infty}}} \\ &=: I_4 + I_5. \end{aligned}$$

Estimate I_4 . Since $w_i^+ \alpha_i^+ - n\delta_{i2} < 0$, we have

$$\begin{aligned} I_4 &= \left\{ \sum_{k=0}^\infty \left(\sum_{l=-\infty}^{-1} \|v(B_l)^{\alpha_i(\cdot)/n} f_{il}\|_{L^{p_i(\cdot)}(w_i)} 2^{(k-l)(w_i^+ \alpha_i^+ - n\delta_{i2})} \right)^{q_{i\infty}} \right\}^{\frac{1}{q_{i\infty}}} \\ &\lesssim \|f_i\|_{\dot{K}_{p_i(\cdot)}^{\alpha_i(\cdot), q_i(\cdot)}(v, w_i)} \left\{ \sum_{k=0}^\infty \left(\sum_{l=-\infty}^{-1} 2^{(k-l)(w_i^+ \alpha_i^+ - n\delta_{i2})} \right)^{q_{i\infty}} \right\}^{\frac{1}{q_{i\infty}}} \\ &\lesssim \|f_i\|_{\dot{K}_{p_i(\cdot)}^{\alpha_i(\cdot), q_i(\cdot)}(v, w_i)}. \end{aligned}$$

Estimate I_5 . Since $w_i^+ \alpha_i^+ - n\delta_{i2} < 0$, then by Lemma 8, we have

$$\begin{aligned} I_5 &= \left\{ \sum_{k=0}^{\infty} \left(\sum_{l=0}^k \|v(B_l)^{\alpha_i(\cdot)/n} f_{il}\|_{L^{p_i(\cdot)}(w_i)} 2^{(k-l)(w_i^+ \alpha_i^+ - n\delta_{i2})} \right)^{q_{i\infty}} \right\}^{\frac{1}{q_{i\infty}}} \\ &\lesssim \left\{ \sum_{l=0}^k \|v(B_l)^{\alpha_i(\cdot)/n} f_{il}\|_{L^{p_i(\cdot)}(w_i)}^{q_{i\infty}} \right\}^{\frac{1}{q_{i\infty}}} \\ &\lesssim \|f_i\|_{\dot{K}_{p_i(\cdot)}^{\alpha_i(\cdot), q_i(\cdot)}(v, w_i)}. \end{aligned}$$

Thus, we get

$$F_1 \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.$$

Estimate F_2 . Since $l \leq k - 2$, $k - 1 \leq j \leq k + 1$, $\alpha_{\infty} = \alpha_{1\infty} + \alpha_{2\infty}$ and $1/q_{\infty} = 1/q_{1\infty} + 1/q_{2\infty}$, by (11) and Hölder’s inequality, we have

$$\begin{aligned} F_2 &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_{\infty} q_{\infty}/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{\infty}} \right. \\ &\quad \times \left. \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{\infty}} \right)^{\frac{1}{q_{\infty}}} \\ &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_{1\infty} q_{1\infty}/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{\frac{1}{q_{1\infty}}} \\ &\quad \times \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_{2\infty} q_{2\infty}/n} \left\| \sum_{j=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{\frac{1}{q_{2\infty}}} \\ &:= F_{2,1} \times F_{2,2}. \end{aligned}$$

It is obvious that

$$F_{2,1} = F_{1,1} \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)}.$$

Estimate $F_{2,2}$. By (12), we have

$$\begin{aligned} F_{2,2} &\lesssim \left\{ \sum_{k=0}^{\infty} \left(\sum_{j=k-1}^{k+1} 2^{(j-k)n} \|v(B_j)^{\alpha_2(\cdot)/n} f_{2j} \chi_j\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\ &\lesssim \left\{ \sum_{k=0}^{\infty} \|v(B_k)^{\alpha_2(\cdot)/n} f_2 \chi_k\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\ &\lesssim \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}. \end{aligned}$$

Thus, we obtain

$$F_2 \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.$$

Estimate F_3 . Since $l \leq k - 2$, $j \geq k + 2$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$ and $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, by (13) and Hölder's inequality, we have

$$\begin{aligned} F_3 &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_\infty q_\infty/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty} \right. \\ &\quad \times \left. \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty} \right)^{\frac{1}{q_\infty}} \\ &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_{1\infty} q_{1\infty}/n} \left\| \sum_{l=-\infty}^{k-2} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{\frac{1}{q_{1\infty}}} \\ &\quad \times \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_{2\infty} q_{2\infty}/n} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{\frac{1}{q_{2\infty}}} \\ &:= F_{3,1} \times F_{3,2}. \end{aligned}$$

It is obvious that

$$F_{3,1} = F_{1,1} \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)}.$$

Estimate $F_{3,2}$. Since $w_2^- \alpha_2^- + n\delta_{21} > 0$, by (9) and Lemma 8, we obtain

$$\begin{aligned} F_{3,2} &\lesssim \left\{ \sum_{k=0}^{\infty} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(w_2^- \alpha_2^- + n\delta_{21})} \|v(B_j)^{\alpha_2(\cdot)/n} f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\ &\lesssim \left\{ \sum_{j=0}^{\infty} \|v(B_j)^{\alpha_2(\cdot)/n} f_{2j}\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\ &\lesssim \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}. \end{aligned}$$

Thus, we get

$$F_3 \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.$$

Estimate F_5 . Since $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$ and $n\delta_{i2} - w_i^+ \alpha_i^+ > 0$, $i = 1, 2$, by Lemmas 4 and 5 and 7, we obtain

$$\begin{aligned} &v(B_k)^{\alpha_\infty/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)} \\ &\lesssim v(B_k)^{\alpha_\infty/n} \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} \|f_{1l}\|_{L^{p_1(\cdot)}(w_1)} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \\ &\lesssim v(B_k)^{\alpha_{2\infty}/n} \sum_{j=k-1}^{k+1} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \left(2^{-(n\delta_{12} - w_1^+ \alpha_1^+)} \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l} \chi_{k-1}\|_{L^{p_1(\cdot)}(w_1)} \right. \\ &\quad \left. + \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l} \chi_k\|_{L^{p_1(\cdot)}(w_1)} + \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l} \chi_{k+1}\|_{L^{p_1(\cdot)}(w_1)} \right) \end{aligned}$$

$$\begin{aligned}
 &\lesssim v(B_k)^{\alpha_{2\infty}/n} \sum_{j=k-1}^{k+1} \|f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \sum_{l=k-1}^{k+1} \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l}\|_{L^{p_1(\cdot)}(w_1)} \\
 &\lesssim \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l}\|_{L^{p_1(\cdot)}(w_1)} \|v(B_j)^{\alpha_2(\cdot)/n} f_{2j}\|_{L^{p_2(\cdot)}(w_2)}. \tag{17}
 \end{aligned}$$

Since $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, by (17) and Hölder’s inequality, we have

$$\begin{aligned}
 F_5 &\lesssim \left(\sum_{k=0}^\infty v(B_k)^{\alpha_\infty q_\infty/n} \left\| \sum_{l=k-1}^{k+1} \sum_{j=k-1}^{k+1} T(f_{1l}, f_{2j}) \chi_k \right\|_{L^{p(\cdot)}(w)}^{q_\infty} \right)^{\frac{1}{q_\infty}} \\
 &\lesssim \left\{ \sum_{k=0}^\infty \left(\sum_{l=k-1}^{k+1} \|v(B_l)^{\alpha_1(\cdot)/n} f_{1l}\|_{L^{p_1(\cdot)}(w_1)} \right)^{q_{1\infty}} \right\}^{\frac{1}{q_{1\infty}}} \\
 &\quad \times \left\{ \sum_{k=0}^\infty \left(\sum_{j=k-1}^{k+1} \|v(B_j)^{\alpha_2(\cdot)/n} f_{2j}\|_{L^{p_2(\cdot)}(w_2)} \right)^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\
 &\lesssim \left\{ \sum_{k=0}^\infty \|v(B_k)^{\alpha_1(\cdot)/n} f_1 \chi_k\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right\}^{\frac{1}{q_{1\infty}}} \\
 &\quad \times \left\{ \sum_{k=0}^\infty \|v(B_k)^{\alpha_2(\cdot)/n} f_2 \chi_k\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right\}^{\frac{1}{q_{2\infty}}} \\
 &\lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.
 \end{aligned}$$

Estimate F_6 . Since $k - 1 \leq l \leq k + 1$ and $j \geq k + 2$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, by (15) and Hölder’s inequality, we have

$$\begin{aligned}
 F_6 &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_\infty q_\infty/n} \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty} \right. \\
 &\quad \times \left. \left\| \sum_{j=k+2}^\infty 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty} \right)^{\frac{1}{q_\infty}} \\
 &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_{1\infty} q_{1\infty}/n} \left\| \sum_{l=k-1}^{k+1} 2^{-kn} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{\frac{1}{q_{1\infty}}} \\
 &\quad \times \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_{2\infty} q_{2\infty}/n} \left\| \sum_{j=k+2}^\infty 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{\frac{1}{q_{2\infty}}} \\
 &:= F_{6,1} \times F_{6,2}.
 \end{aligned}$$

By the symmetry of f_1 and f_2 , we can know that the estimate of $E_{6,1}$ is similar to that of $E_{1,2}$ and $E_{6,2} = E_{3,2}$.

Estimate F_9 , since $l, j \geq k + 2$, $\alpha_\infty = \alpha_{1\infty} + \alpha_{2\infty}$, $1/q_\infty = 1/q_{1\infty} + 1/q_{2\infty}$, by (16) and Hölder's inequality, we have

$$\begin{aligned} F_9 &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_\infty q_\infty/n} \left\| \sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_\infty} \right. \\ &\quad \times \left. \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_\infty} \right)^{\frac{1}{q_\infty}} \\ &\lesssim \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_{1\infty} q_{1\infty}/n} \left\| \sum_{l=k+2}^{\infty} 2^{-ln} \int_{\mathbb{R}^n} |f_{1l}(y_1)| dy_1 \chi_k \right\|_{L^{p_1(\cdot)}(w_1)}^{q_{1\infty}} \right)^{\frac{1}{q_{1\infty}}} \\ &\quad \times \left(\sum_{k=-\infty}^{-1} v(B_k)^{\alpha_{2\infty} q_{2\infty}/n} \left\| \sum_{j=k+2}^{\infty} 2^{-jn} \int_{\mathbb{R}^n} |f_{2j}(y_2)| dy_2 \chi_k \right\|_{L^{p_2(\cdot)}(w_2)}^{q_{2\infty}} \right)^{\frac{1}{q_{2\infty}}} \\ &:= F_{9,1} \times F_{9,2}. \end{aligned}$$

Obviously, the estimate of $F_{9,i}$ is similar to that of $F_{3,2}$ for $i = 1, 2$.

Taking all estimates for F_i together, $i = 1, 2, \dots, 9$, we obtain

$$F \lesssim \|f_1\|_{\dot{K}_{p_1(\cdot)}^{\alpha_1(\cdot), q_1(\cdot)}(v, w_1)} \|f_2\|_{\dot{K}_{p_2(\cdot)}^{\alpha_2(\cdot), q_2(\cdot)}(v, w_2)}.$$

This completes the proof. \square

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