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SPECTRA OF INFINITE GRAPHS: TWO METHODS OF COMPUTATION

LEONID GOLINSKII

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Abstract. Two methods for computation of the spectra of certain infinite graphs are suggested. The first one can be viewed as a reversed Gram–Schmidt orthogonalization procedure. It relies heavily on the spectral theory of Jacobi matrices. The second method is related to the Schur complement for block matrices. A number of examples including finite graphs with tails, chains of cycles and ladders are worked out in detail.

Contents

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1. Introduction

We begin with some rudiments of the graph theory. For the sake of simplicity we restrict ourselves with simple, connected, undirected, finite or infinite (countable) weighted graphs, although the main result holds for weighted multigraphs and graphs with loops as well. We will primarily label the vertex set $\mathscr{V}(\Gamma)$ by positive integers $\mathbb{N}=\{1,2,\dots\}$, $\{v\}_{v\in\mathscr{V}}=\{j\}_{j=1}^{\omega}$, $\omega\leqslant\infty$. The symbol $i\sim j$ means that the vertices i and j are incident, i.e., $\{i,j\}$ belongs to the edge set $\mathscr{E}(\Gamma)$. A graph Γ is weighted if a positive number d_{ij} (weight) is assigned to each edge $\{i,j\}\in\mathscr{E}(\Gamma)$. When $d_{ij}=1$ for all i,j, the graph is unweighted.

The degree (valency) of a vertex $v\in\mathscr{V}(\Gamma)$ is a number $\gamma(v)$ of edges emanating from v. A graph Γ is said to be locally finite, if $\gamma(v)<\infty$ for all $v\in\mathscr{V}(\Gamma)$, and uniformly locally finite, if $\sup_{\mathscr{V}}\gamma(v)<\infty$. The latter will be the case in all our considerations below.

The spectral graph theory deals with the study of spectra and spectral properties of certain matrices related to graphs (more precisely, operators generated by such matrices in the standard basis $\{e_k\}_{k\in\mathbb{N}}$ and acting in the corresponding Hilbert spaces $\mathbb{C}^n$, $\ell^2=\ell^2(\mathbb{N})$, or, more generally, $\ell^2(\mathscr{V}(\Gamma))$. One of the most notable of them is the *adjacency matrix* $A(\Gamma)$

$$A(\Gamma)=\|a_{ij}\|_{ij=1}^{\omega},\quad a_{ij}=\begin{cases} d_{ij}, & \{i,j\}\in\mathscr{E}(\Gamma);\\ 0, & \text{otherwise.}\end{cases}\tag{1.1}$$

The corresponding adjacency operator will be denoted by the same symbol. It acts as

$$A(\Gamma)\,e_k=\sum_{j\sim k}a_{jk}\,e_j,\qquad k\in\mathbb{N}.\tag{1.2}$$

Clearly, $A(\Gamma)$ is a symmetric, densely-defined linear operator, whose domain is the set of all finite linear combinations of the basis vectors. The operator $A(\Gamma)$ is bounded and selfadjoint in ℓ^2, as long as the graph Γ is uniformly locally finite.

Under the *spectrum* $\sigma(\Gamma)$ (*resolvent set* $\rho(\Gamma)$) of the graph we always mean the spectrum (the resolvent set) of its adjacency operator $A(\Gamma)$. We stick to the following classification for the parts of the spectrum (one of the most confusing notions in the spectral theory). Under the *discrete spectrum* $\sigma_d(\Gamma)$ we mean the set of all isolated eigenvalues of $A(\Gamma)$ of finite multiplicity. The *essential spectrum* $\sigma_{ess}(\Gamma)$ is the complement $\sigma(\Gamma)\backslash\sigma_d(\Gamma)$. If $H_\Gamma\{(a,b)\}$ is the spectral subspace of $A(\Gamma)$ for the interval (a,b), a point $\lambda\in\sigma_{ess}(\Gamma)$ if and only if for each $\varepsilon>0$ the dimension $\dim H_\Gamma\{(\lambda-\varepsilon,\lambda+\varepsilon)\}=+\infty$. We denote by $\sigma_p(\Gamma)$ the *point spectrum* of Γ, i.e., the set of *all* eigenvalues of $A(\Gamma)$. Sometimes we use a notation $\sigma_h(\Gamma)$ for the set of eigenvalues of $A(\Gamma)$, lying on the essential spectrum (the so-called hidden spectrum). We will observe the situation, when $\sigma_h(\Gamma)\neq\emptyset$ in a number of subsequent examples.

The underlying Hilbert space, wherein the adjacency operator $A(\Gamma)$ acts, is ℓ^2 as soon as we use the set $\mathbb{N}$ to enumerate the vertex set $\mathscr{V}(\Gamma)$. But sometimes it is more convenient to use another set of indices. In general, the underlying Hilbert space $\ell^2(\Gamma)$ is the set of all square summable sequences defined on $\mathscr{V}(\Gamma)$. The standard basis in this space is $\{e_v(\cdot)\}_{v\in\mathscr{V}(\Gamma)}$ with

$$e_v(w) = \begin{cases} 1, & w = v; \\ 0, & w \neq v. \end{cases}$$

Relation (1.2) (for unweighted graphs) looks as

$$(A(\Gamma)e_v)(w) = \sum_{u\sim v} e_u(w), \qquad u,v \in \mathscr{V}(\Gamma). \tag{1.3}$$

Whereas the spectral theory of finite graphs is very well established (see, e.g., [1, 5, 6, 7]), the corresponding theory for infinite graphs is in its youth. We refer to [24, 25, 33] for the basics of this theory. In contrast to the general consideration in [25], the goal of these notes is to carry out a complete spectral analysis for certain classes of infinite graphs.

We suggest two methods of computation of spectra for such graphs. The first one applies to the graphs which can be called "finite graphs with tails attached to them" and some other closely related graphs. It is pursued in two stages. At the first one, we construct a canonical model for the adjacency operators of such graphs, which is an orthogonal sum of a finite dimensional operator and a Jacobi operator of finite rank (finite and Jacobi components of the graph). At the second stage, the spectrum of the Jacobi component is computed by means of the Jost solution for the corresponding recurrence relation, and the spectrum of the finite component by the standard means of linear algebra.

To be precise, we define first an operation of coupling well known for finite graphs (see, e.g., [7, Theorem 2.12]).

DEFINITION 1.1. Let Γ_k, $k=1,2$, be two weighted graphs with no common vertices, with the vertex sets and edge sets $\mathscr{V}(\Gamma_k)$ and $\mathscr{E}(\Gamma_k)$, respectively, and let $v_k \in \mathscr{V}(\Gamma_k)$. A weighted graph $\Gamma = \Gamma_1 + \Gamma_2$ will be called a *coupling by means of the bridge* $\{v_1, v_2\}$ *of weight* d if

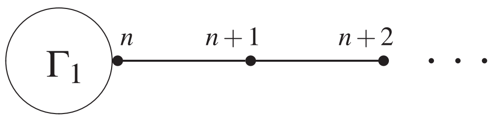
$$\mathscr{V}(\Gamma) = \mathscr{V}(\Gamma_1) \cup \mathscr{V}(\Gamma_2), \qquad \mathscr{E}(\Gamma) = \mathscr{E}(\Gamma_1) \cup \mathscr{E}(\Gamma_2) \cup \{v_1, v_2\}. \tag{1.4}$$

So, we join Γ_2 to Γ_1 by a new edge of weight d between v_2 and v_1.

If the graph Γ_1 is finite, $V(\Gamma_1) = \{1,2,\dots,n\}$, and $V(\Gamma_2) = \{j\}_{j=n+1}^{\omega}$, we can with no loss of generality put $v_1 = n$, $v_2 = n+1$, so the adjacency matrix $A(\Gamma)$ can be written as a block matrix

$$A(\Gamma) = \begin{bmatrix} A(\Gamma_1) & E_d \\ E_d^* & A(\Gamma_2) \end{bmatrix}, \qquad E_d = \begin{bmatrix} 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots \\ d & 0 & 0 & \dots \end{bmatrix}, \tag{1.5}$$

the matrix with n rows and one nonzero entry. If $\Gamma_2 = \mathbb{P}_\infty\{a_i\}$, the one-sided weighted infinite path, $a_i = d_{i,i+1}$, we can view the coupling $\Gamma = \Gamma_1 + \mathbb{P}_\infty\{a_i\}$ as a finite graph with the tail. This is the class of graphs we will primarily be dealing with here.



A special class of infinite matrices will play a crucial role in what follows.

Under *Jacobi* or *tridiagonal matrices* we mean here semi-infinite matrices of the form

$$J = J(\{b_i\},\{a_i\})_{i\in\mathbb{N}} = \begin{bmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & a_2 & b_3 & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \quad b_i \in \mathbb{R}, \quad a_i > 0. \tag{1.6}$$

They generate linear operators (called the Jacobi operators) on the Hilbert space $\ell^2(\mathbb{N})$. The matrix

$$J_0 := \begin{bmatrix} 0 & 1 & 0 & 0 & \\ 1 & 0 & 1 & 0 & \\ 0 & 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{1.7}$$

known as a *discrete Laplacian* or a *free Jacobi matrix*, is of particular interest in the sequel. We denote by $J_{0,m}$ a $m\times m$-matrix, which is a principal block of order m of J_0 (the discrete Laplacian of order m).

The Jacobi matrices arise in the spectral graph theory because of relation to the adjacency matrix $A(\mathbb{P}_\infty\{a_i\})$ of the weighted path

$$A(\mathbb{P}_\infty(\{a_i\})) := J(\{0\},\{a_i\}). \tag{1.8}$$

In the case of the unweighted path, that is, $a_i \equiv 1$, we have

$$A(\mathbb{P}_\infty) = J_0. \tag{1.9}$$

The spectrum of J_0 is $\sigma(J_0) = [-2,2]$.

Similarly, the discrete Laplacian of order m is the adjacency matrix of the path $\mathbb{P}_m$ with m vertices, $J_{0,m} = A(\mathbb{P}_m)$. It is well known [5, Section 1.4.4] that the spectrum

$$\sigma(J_{0,m}) = \left\{2\cos\frac{\pi j}{m+1}\right\}_{j=1}^{m}. \tag{1.10}$$

Sometimes two-sided Jacobi matrices $J = J(\{b_i\},\{a_i\})_{i\in\mathbb{Z}}$, acting on the Hilbert space $\ell^2(\mathbb{Z})$, show up in our consideration. The discrete Laplacian is

$$J = J_0(\mathbb{Z}) = J(\{0\},\{1\})_{i\in\mathbb{Z}}, \quad \sigma(J_0(\mathbb{Z})) = [-2,2]. \tag{1.11}$$

It follows from (1.5) that for an arbitrary finite weighted graph G

$$A(G+\mathbb{P}_\infty\{a_i\}) = \begin{bmatrix} A(G) & E_d \\ E_d^* & J(\{0\},\{a_i\}) \end{bmatrix}. \tag{1.12}$$

To proceed further, let us recall the notions of truncation and extension for Jacobi matrices.

Given two Jacobi matrices $J_k = J\left(\{b_i^{(k)}\},\{a_i^{(k)}\}\right)_{i\in\mathbb{N}}$, $k=1,2$, the matrix J_2 is called a *truncation* of J_1 (and J_1 is an *extension* of J_2) if

$$b_i^{(2)} = b_{i+q}^{(1)}, \quad a_i^{(2)} = a_{i+q}^{(1)}, \qquad i\in\mathbb{N},$$

for some $q\in\mathbb{N}$. In other words, J_2 is obtained from J_1 by deleting the first q rows and columns. The term q*-stripped matrix* is also in common usage. If the discrete Laplacian J_0 is the truncation of a Jacobi matrix J_1, the latter is called a *Jacobi matrix of finite rank* or an *eventually free Jacobi matrix*.

For Jacobi matrices J of finite rank, it is well known that

$$\sigma(J) = \sigma_{ess}(J)\cup\sigma_d(J) = [-2,2]\cup\sigma_d(J), \tag{1.13}$$

the discrete spectrum $\sigma_d(J)$ is finite, and the union is disjoint.

We suggest a "canonical" form for the block matrices (1.12) and the algorithm of their reduction to this form.

THEOREM 1.2. *Let A be a selfadjoint operator on ℓ^2, given by a block matrix*

$$A = \begin{bmatrix} \mathscr{A} & E_d \\ E_d^* & J \end{bmatrix}, \qquad E_d = \begin{bmatrix} 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots \\ d & 0 & 0 & \dots \end{bmatrix}, \quad d>0, \tag{1.14}$$

where $\mathscr{A}$ is a real, symmetric matrix of order n, $J=J(\{b_i\},\{a_i\})$ a Jacobi matrix. Then there is a unitary operator U on ℓ^2 such that

$$U^{-1}AU = \begin{bmatrix} \widehat{\mathscr{A}} & \\ & \widehat{J} \end{bmatrix} = \widehat{\mathscr{A}}\oplus\widehat{J}, \tag{1.15}$$

where $\widehat{\mathscr{A}}$ is a real, symmetric matrix of order at most $n-1$, and the Jacobi matrix $\widehat{J}$ is the extension of J.

The proof of Theorem 1.2 in [12] is somewhat cumbersome. An alternative, "coordinate-free" proof due to Kozhan [18] is based on a canonical representation of an arbitrary real, symmetric (and even Hermitian) matrix as an orthogonal sum of finite, Jacobi matrices. Precisely, given such $n\times n$ matrix $\mathscr{A}$, there is a unitary matrix V on

$\mathbb{C}^n$ so that

$$V^{-1}\mathscr{A}V = \widehat{J}_n = \begin{bmatrix} \hat{b}_1 & \hat{a}_1 & & \\ \hat{a}_1 & \hat{b}_2 & \ddots & \\ & \ddots & \ddots & \hat{a}_{n-1} \\ & & \hat{a}_{n-1} & \hat{b}_n \end{bmatrix}, \qquad \hat{b}_i \in \mathbb{R}, \quad \hat{a}_i \geqslant 0.$$

Moreover,

$$Ve_n = V^{-1}e_n = e_n, \tag{1.16}$$

e_n is the n-th standard basis vector in $\mathbb{C}^n$, and if

$$\dim \operatorname{span}\{\mathscr{A}^k e_n,\ k \geqslant 0\} = m \leqslant n, \tag{1.17}$$

then

$$\hat{a}_i > 0, \quad i = n-m+1, \dots, n-1, \quad \hat{a}_{n-m} = 0.$$

To prove Theorem 1.2, put

$$U = \begin{bmatrix} V & \\ & I \end{bmatrix},$$

V as above, so

$$U^{-1}AU = \begin{bmatrix} \widehat{J}_n & V^{-1}E_d \\ E_d^* V & J \end{bmatrix}.$$

In view of (1.16), $V^{-1}E_d = E_d$, $E_d^* V = E_d^*$, and hence

$$U^{-1}AU = \begin{bmatrix} \widehat{J}_n & E_d \\ E_d^* & J \end{bmatrix} = \begin{bmatrix} \widehat{\mathscr{A}} & \\ & \widehat{J} \end{bmatrix},$$

where the matrix $\widehat{\mathscr{A}}$ of order $n-m$ is the orthogonal sum of finite, Jacobi matrices,

$$\widehat{J} = J(\{\hat{b}_{n-m+1}, \dots, \hat{b}_n, b_1, b_2, \dots\}, \{\hat{a}_{n-m+1}, \dots, \hat{a}_{n-1}, d, a_1, a_2, \dots\})$$

is the extension of J, as claimed.

COROLLARY 1.3. *Given a finite weighted graph G, the adjacency operator of the coupling $\Gamma = G + \mathbb{P}_\infty\{a_i\}$ is unitarily equivalent to the orthogonal sum*

$$U^{-1}A(\Gamma)U = F(\Gamma) \oplus J(\Gamma) \tag{1.18}$$

of a finite-dimensional operator $F(\Gamma)$ and a Jacobi operator $J(\Gamma)$, which is an extension of $J(\{0\},\{a_i\})$.

The matrix $J(\Gamma)$ is of finite rank, as long as $\mathbb{P}_\infty$ is unweighted. We call $F(\Gamma)$ a *finite-dimensional component* of the coupling Γ, and $J(\Gamma)$ its *Jacobi component*.

Note that in this situation the vectors e_n and $A(G)e_n$ are linearly independent, so $m \geqslant 2$ in (1.17), and the dimension of the finite-dimensional component $F(\Gamma)$ is at most $n-2$. We observe the both extreme cases $\dim F(\Gamma)=n-2$ in Example 3.1, and $\dim F(\Gamma)=0$ (the finite component is missing) in Example 3.3 below.

It follows from the above canonical form that the spectrum of Γ is

$$\sigma(\Gamma)=\sigma(F(\Gamma))\bigcup\sigma(J(\Gamma)).$$

Hence, to compute the spectrum of Γ, we apply the spectral result of Damanik and Simon for Jacobi matrices of finite rank, based on the Jost solution. The eigenvalues of the finite-dimensional component are the roots of the corresponding characteristic polynomial.

REMARK 1.4. Given a finite graph G, one can attach $p \geqslant 1$ copies of the infinite path $\mathbb{P}_\infty$ to *some* vertex $v\in\mathscr{V}(G)$. Although the graph Γ thus obtained is not exactly the coupling in the sense of Definition 1.1, its adjacency operator acts similarly to one for the coupling. Indeed, it is not hard to see that

$$A(\Gamma)=\begin{bmatrix} A(G) & E_d \\ E_d^* & J_0 \end{bmatrix}\bigoplus\Big(\bigoplus_{i=1}^{p-1} J_0\Big), \quad d:=\sqrt{p}. \tag{1.19}$$

Hence, Theorem 1.2 applies, and the spectral analysis of such graph can be carried out.

Surprisingly enough, the case, when $p\geqslant 1$ infinite rays are attached to *each* vertex of a finite graph G, is easy to work out, and the spectrum of such graph can be found explicitly in terms of the spectrum of G. Denote such graph by $G_\infty(p)$.

THEOREM 1.5. *Given a finite graph G of order n with $\sigma(G)=\{\lambda_j\}_{j=1}^n$, let $\Gamma=G_\infty(p)$, $p\in\mathbb{N}$. Denote by $J(\lambda_j,\sqrt{p})$ the Jacobi matrices of rank* 1

$$J(\lambda_j,\sqrt{p}):=J(\{\lambda_j,0,0,\dots\},\{\sqrt{p},1,1,\dots\}). \tag{1.20}$$

Then the adjacency operator $A(\Gamma)$ is unitarily equivalent to the orthogonal sum

$$A(\Gamma)\simeq\bigoplus_{j=1}^{n} J(\lambda_j,\sqrt{p})\bigoplus\Big(\bigoplus_{i=1}^{(p-1)n} J_0\Big). \tag{1.21}$$

The spectrum of Γ is

$$\sigma(\Gamma)=[-2,2]\bigcup\left(\bigcup_{j=1}^{n}\sigma_d\big(J(\lambda_j,\sqrt{p})\big)\right). \tag{1.22}$$

For the proof see [12, Theorem 1.6].

The spectral theory of infinite graphs with one or several rays attached to certain finite graphs was initiated in [20, 21, 22, 26] wherein several particular examples of unweighted (background) graphs are examined. The spectral analysis of similar graphs appeared earlier in the study of thermodynamical states on complex networks [9, 10].

We argue in the spirit of [3, 4, 31] and supplement to the list of examples. The general canonical form for the adjacency matrices of such graphs and the algorithm of their reducing to this form suggested in the paper apply to a wide class of couplings (not only the graphs with tails), and also to Laplacians on graphs of such type.

The second method, which can be called the "Schur complement method", is based on this well-known notion from the algebra of block matrices. The method is applied to wider classes of infinite graphs, as well as to some other operators (Laplacians) on graphs, see [13].

Let

$$\mathscr{A}=\begin{bmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix} \tag{1.23}$$

be a block operator matrix which acts on the orthogonal sum $\mathscr{H}_1\oplus\mathscr{H}_2$ of two Hilbert spaces. If A_{11} is invertible, the matrix $\mathscr{A}$ can be factorized as

$$\mathscr{A}=\begin{bmatrix} I & 0\\ A_{21}A_{11}^{-1} & I\end{bmatrix}\begin{bmatrix} A_{11} & 0\\ 0 & C_{22}\end{bmatrix}\begin{bmatrix} I & A_{11}^{-1}A_{12}\\ 0 & I\end{bmatrix}, \tag{1.24}$$

I is the unity operator on the corresponding Hilbert space. Similarly, if A_{22} is invertible, one can write

$$\mathscr{A}=\begin{bmatrix} I & A_{12}A_{22}^{-1}\\ 0 & I\end{bmatrix}\begin{bmatrix} C_{11} & 0\\ 0 & A_{22}\end{bmatrix}\begin{bmatrix} I & 0\\ A_{22}^{-1}A_{21} & I\end{bmatrix}. \tag{1.25}$$

Here

$$C_{22}:=A_{22}-A_{21}A_{11}^{-1}A_{12}, \qquad C_{11}:=A_{11}-A_{12}A_{22}^{-1}A_{21} \tag{1.26}$$

are usually referred to as the *Schur complements* [29], [15, Section 0.8.5]. Both equalities can be checked by direct multiplication.

The result below follows immediately from the formulae (1.24) and (1.25).

PROPOSITION 1.6. *Given a block operator matrix $\mathscr{A}$ (1.23), let A_{22} (A_{11}) be invertible. Then $\mathscr{A}$ is invertible if and only if so is C_{11} (C_{22}).*

Note that in the premises of Proposition 1.6 the inverse $\mathscr{A}^{-1}$ takes the form

$$\mathscr{A}^{-1}=\begin{bmatrix} C_{11}^{-1} & -C_{11}^{-1}A_{12}A_{22}^{-1}\\ -A_{22}^{-1}A_{21}C_{11}^{-1} & A_{22}^{-1}+A_{22}^{-1}A_{21}C_{11}^{-1}A_{12}A_{22}^{-1}\end{bmatrix}$$

and, respectively,

$$\mathscr{A}^{-1}=\begin{bmatrix} A_{11}^{-1}+A_{11}^{-1}A_{12}C_{22}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}C_{22}^{-1}\\ -C_{22}^{-1}A_{21}A_{11}^{-1} & C_{22}^{-1}\end{bmatrix}.$$

Denote by $\rho(T)$ the resolvent set of a bounded, linear operator T, i.e., the set of complex numbers λ so that $\lambda I-T$ is boundedly invertible. We apply the latter result to the block matrix

$$\lambda I-\mathscr{A}=\begin{bmatrix} \lambda I-A_{11} & -A_{12}\\ -A_{21} & \lambda I-A_{22}\end{bmatrix}, \qquad \lambda\in\mathbb{C}, \tag{1.27}$$

to obtain

PROPOSITION 1.7. *Given a block operator matrix* $\mathscr{A}$ (1.23), *let* $\lambda \in \rho(A_{22})$ $\big(\lambda \in \rho(A_{11})\big)$. *Then* $\lambda \in \rho(\mathscr{A})$ *if and only if the operator*

$$\begin{aligned} C_{11}(\lambda) &= \lambda I - A_{11} - A_{12}(\lambda I - A_{22})^{-1} A_{21} \\ (C_{22}(\lambda) &= \lambda I - A_{22} - A_{21}(\lambda I - A_{11})^{-1} A_{12}) \end{aligned} \tag{1.28}$$

is invertible.

We proceed as follows. An exposition of the spectral theory for certain classes of Jacobi matrices in Section 2 gives a chance to experts in the graph theory to get acquainted with this fascinating topic, and makes the reasoning self-contained. In the next two sections we collect a number of illuminating examples, the graphs with infinite tails (Section 3) and ladders and chains of cycles (Section 4). We construct explicitly the canonical bases and find the spectra of the corresponding graphs. In Section 5 we discuss the second method based on the Schur complement.

2. Spectral analysis for classes of Jacobi matrices

2.1. Perturbation determinants and Jost functions

A basic object known as the *perturbation determinant* [11] is a key ingredient of perturbation theory.

Given bounded linear operators T_0 and T on the Hilbert space such that $T - T_0$ is a trace class operator, the perturbation determinant is defined as

$$L(\lambda; T, T_0) := \det(I + (T - T_0)R(\lambda, T_0)), \quad R(\lambda, T_0) := (T_0 - \lambda)^{-1} \tag{2.1}$$

is the resolvent of the operator T_0, an analytic operator-function on the resolvent set $\rho(T_0)$.

The perturbation determinant is designed for the spectral analysis of the perturbed operator T, once the spectral analysis for the unperturbed one T_0 is available. In particular, the essential spectra of T and T_0 agree, and the discrete spectrum of T is exactly the zero set of the analytic function L on $\rho(T_0)$, at least if the latter is a domain, i.e., a connected, open set in the complex plane.

In the simplest case, $\operatorname{rank}(T - T_0) < \infty$, the perturbation determinant reduces to the standard finite-dimensional determinant. Indeed, now

$$(T - T_0)h = \sum_{k=1}^{p} \langle h, \varphi_k \rangle\, \psi_k, \quad (T - T_0)R(\lambda, T_0)h = \sum_{k=1}^{p} \langle h, R^*(\lambda, T_0)\varphi_k \rangle\, \psi_k,$$

so L can be computed by the formula (see, e.g., [11, Section IV.1.3])

$$L(\lambda; T, T_0) = \det[\delta_{ij} + \langle R(\lambda, T_0)\psi_i, \varphi_j \rangle]_{i,j=1}^{p}. \tag{2.2}$$

Our particular concern is $T_0 = J_0$, the free Jacobi matrix. The matrix of its resolvent in the standard basis in ℓ^2 is given by (see, e.g., [16])

$$R(\lambda, J_0) = [r_{ij}(z)]_{i,j=1}^{\infty}, \quad r_{ij}(z) = \frac{z^{|i-j|} - z^{i+j}}{z - z^{-1}}, \quad \lambda = z + \frac{1}{z}, \quad z \in \mathbb{D}. \tag{2.3}$$

If $T = J$ is a Jacobi matrix of finite rank p, we end up with computation of the ordinary determinant (2.2) of order p.

It is instructive for the further usage computing two simplest perturbation determinants for $\operatorname{rank}(J - J_0) = 1$ and 2.

EXAMPLE 2.1. Let

$$J = J(\{b_i\}, \{1\}): \quad b_i = 0, \quad i \neq q,$$

so $J - J_0 = \langle \cdot, e_q\rangle b_q e_q$. By (2.3) and (2.2),

$$\widehat{L}(z) := L\Big(z + \frac{1}{z}; J, J_0\Big) = 1 + b_q r_{qq}(z) = 1 - b_q z \frac{z^{2q} - 1}{z^2 - 1}. \tag{2.4}$$

Similarly, let

$$J = J(\{0\}, \{a_i\}): \quad a_i = 1, \quad i \neq q,$$

so $J - J_0 = \langle \cdot, e_q\rangle (a_q - 1) e_{q+1} + \langle \cdot, e_{q+1}\rangle (a_q - 1) e_q$, and again

$$\begin{aligned} \widehat{L}(z) &= \begin{vmatrix} 1 + (a_q - 1) r_{q,q+1}(z) & (a_q - 1) r_{qq}(z) \\ (a_q - 1) r_{q+1,q+1}(z) & 1 + (a_q - 1) r_{q+1,q}(z) \end{vmatrix} \\ &= 1 + (1 - a_q^2) z^2 \frac{z^{2q} - 1}{z^2 - 1}. \end{aligned} \tag{2.5}$$

In the Jacobi matrices setting there is yet another way of computing perturbation determinants based on the so-called Jost solution and Jost function (see, e.g., [33, Section 3.7]).

Consider the basic recurrence relation for the Jacobi matrix J

$$a_{n-1} y_{n-1} + b_n y_n + a_n y_{n+1} = \Big(z + \frac{1}{z}\Big) y_n, \quad z \in \mathbb{D}, \quad n \in \mathbb{N}, \quad a_0 = 1, \tag{2.6}$$

see, e.g., [16, formula (1.26)]. Its solution $y_n = u_n(z)$ is called the *Jost solution* if

$$\lim_{n \to \infty} z^{-n} u_n(z) = 1, \qquad z \in \mathbb{D}. \tag{2.7}$$

In this case the function $u = u_0(z)$ is called the *Jost function*.

The Jost solution certainly exists for finite rank Jacobi matrices. The Jost function is now an algebraic polynomial, called the *Jost polynomial*. Indeed, let

$$b_{q+1} = b_{q+2} = \ldots = 0, \quad a_{q+1} = a_{q+2} = \ldots = 1. \tag{2.8}$$

One can put $u_k(z) = z^k$, $k \geqslant q+1$, and then determine $u_q, u_{q-1}, \ldots, u_0$ consecutively from (2.6). So,

$$\begin{aligned} a_q u_q(z) &= \Big(z + \frac{1}{z}\Big) z^{q+1} - z^{q+2} = z^q, \\ a_{q-1} a_q u_{q-1}(z) &= \Big(z + \frac{1}{z}\Big) z^q - b_q z^q - a_q^2 z^{q+1} \\ &= (1 - a_q^2) z^{q+1} - b_q z^q + z^{q-1}, \end{aligned} \tag{2.9}$$

and so on. The algorithm shows that under conditions (2.8), the Jost polynomial is a real polynomial of degree at most $2q$. In particular, if $\alpha_j := 1-a_j^2$, $j=1,2$, we have for $q=1$

$$a_1\, u(z) = \alpha_1\, z^2 - b_1 z + 1, \tag{2.10}$$

and for $q=2$

$$a_1 a_2\, u(z) = \alpha_2\, z^4 - (b_2 + b_1\alpha_2)\, z^3 + (\alpha_1 + \alpha_2 + b_1 b_2)\, z^2 - (b_1 + b_2)\, z + 1. \tag{2.11}$$

The relation between the perturbation determinant and the Jost function is given by

$$u(z) = \prod_{i=1}^{\infty} a_i^{-1} \cdot \widehat{L}(z), \tag{2.12}$$

see [16], and such recursive way of computing perturbation determinants is sometimes far easier than computing ordinary determinants (2.2), especially for large enough ranks of perturbation. On the other hand, for small ranks of perturbation (as in Example 2.1) with large q it is much easier applying formula (2.2).

EXAMPLE 2.2. Let $J = J(\{b_i\},\{a_i\})$ be a Jacobi matrix such that

$$b_i = 0, \quad i \neq 1; \qquad a_i = 1, \quad i \neq q.$$

We have $u_{q+j}(z) = z^{q+j}$, $j = 1,2\ldots$,

$$a_q u_q(z) = z^q, \quad a_q u_{q-1}(z) = \alpha_q\, z^{q+1} + z^{q-1},$$
$$a_q u_{q-2} = \alpha_q\, (z^{q+2} + z^q) + z^{q-2},$$

and, by induction,

$$a_q u_{q-k}(z) = \alpha_q\, z^{q-k+2}\, \frac{z^{2k}-1}{z^2-1}, \qquad k = 1,2,\ldots,q-1. \tag{2.13}$$

Next, for $q=1$ we have exactly (2.10), so let $q \geqslant 2$. The recurrence relation (2.6) with $n=1$ gives

$$a_q u(z) + b_1 a_q u_1(z) + a_q u_2(z) = \left(z + \frac{1}{z}\right) a_q u_1(z),$$

and so we come to the following expression for the Jost polynomial

$$a_q u(z)(z^2-1) = \alpha_q\, (z - b_1)\, z^{2q+1} - b_1 a_q^2 z^3 + a_q^2 z^2 + b_1 z - 1. \tag{2.14}$$

Similarly, for the Jacobi matrix $J = J(\{b_i\},\{a_i\})$ with

$$b_i = 0, \ \ i \neq q; \quad a_i = 1, \quad i \neq 1$$

one has

$$a_1 u(z) = -b_q\, \frac{z^{2q+1} + \alpha_1 z^{2q-1} - \alpha_1 z^3 - z}{z^2-1} + \alpha_1 z^2 + 1. \tag{2.15}$$

For the Jacobi matrix $J = J(\{0\},\{a_i\})$ with $a_i = 1$, $i \neq 1$, q, the Jost polynomial is given by

$$a_1 a_q u(z) = \alpha_q \frac{z^{2q+2} + \alpha_1 z^{2q} - \alpha_1 z^4 - z^2}{z^2 - 1} + \alpha_1 z^2 + 1. \tag{2.16}$$

For the Jacobi matrices of finite rank a complete spectral analysis is available at the moment, see [8, 17]. The spectral theorem of Damanik and Simon [8] provides a complete description of the spectral measure for such matrices.

THEOREM. (Damanik–Simon). *Let $J = J(\{b_i\},\{a_i\})_{i\in\mathbb{N}}$ be a Jacobi matrix of finite rank*

$$a_{q+1} = a_{q+2} = \ldots = 1, \qquad b_{q+1} = b_{q+2} = \ldots = 0,$$

and $u = u_0(J)$ be its Jost polynomial. Then

- *u is a real polynomial of degree* $\deg u \leqslant 2q$, $\deg u = 2q$ *if and only if $a_q \neq 1$.*
- *All roots of u in the unit disk $\mathbb{D}$ are real and simple, $u(0) \neq 0$. A number λ_j is an eigenvalue of J if and only if*

$$\lambda_j = z_j + \frac{1}{z_j}, \qquad z_j \in (-1,1), \quad u(z_j) = 0. \tag{2.17}$$

- *The spectral measure $\sigma(J)$ is of the form*

$$\sigma(J,dx) = \sigma_{ac}(J,dx) + \sigma_d(J,dx) = w(x)\,dx + \sum_{j=1}^{N} \sigma_j \delta(\lambda_j), \tag{2.18}$$

where

$$w(x) := \frac{\sqrt{4-x^2}}{2\pi|u(e^{it})|^2}, \quad x = 2\cos t, \quad \sigma_j = \frac{z_j(1-z_j^{-2})^2}{u'(z_j)u(1/z_j)}.$$

Note that $|u(e^{it})|^2 = Q(x)$, $x = 2\cos t$, Q is a real polynomial of the same degree as the Jost polynomial u.

The algebraic equations which we encounter later on cannot in general be solved explicitly. By means of the following well-known result [27, p. 41], we can determine how many roots (if any) they have in $(-1,1)$.

THEOREM. (Descarte's rule). *Let $a(x) = a_0 x^n + \ldots + a_n$ be a real polynomial. Denote by $\mu(a)$ the number of its positive roots, and $\nu(a)$ the number of the sign changes in the sequence $\{a_0,\ldots,a_n\}$ of its coefficients (the zero coefficients are not taken into account). Then $\nu(a) - \mu(a)$ is a nonnegative even number.*

2.2. Weyl function and transfer matrix

Let us go back to the basic recurrence relations (2.6)

$$a_{n-1}y_{n-1}(\lambda)+b_ny_n(\lambda)+a_ny_{n+1}(\lambda)=\lambda\, y_n, \quad n\in\mathbb{N}, \quad a_0=1, \tag{2.19}$$

and consider its two solutions

$$y_n(\lambda)=p_n(\lambda): \;\; p_0=0,\; p_1=1; \qquad y_n(\lambda)=q_n(\lambda): \;\; q_0=-1,\; q_1=0.$$

The polynomials p_n (q_n) are called the first (second) kind polynomials for the Jacobi matrix J. We have

$$p_2(\lambda)=\frac{\lambda-b_1}{a_1}, \quad p_3(\lambda)=\frac{(\lambda-b_2)(\lambda-b_1)}{a_1a_2}-\frac{a_1}{a_2},\dots \tag{2.20}$$

so $\deg p_k=k-1$.

Recall that 1-stripped matrix J_1 for J is given by $J_1=J(\{b_{i+1}\},\{a_{i+1}\})$. The stripping formula [33, formula (3.2.16)] relates the second kind polynomials q_n for J and the first kind ones $p_n^{(1)}$ for J_1

$$q_n(\lambda)=\frac{1}{a_1}\,p_{n-1}^{(1)}(\lambda), \qquad n\in\mathbb{N}, \tag{2.21}$$

so $\deg q_k=k-2$.

EXAMPLE 2.3. “Chebyshev polynomials”.

We compute the 1st (2nd) kind polynomials for two particular Jacobi matrices. Recall the notion of Chebyshev polynomials of the 1st and 2nd kind

$$T_n(\cos\theta)=\cos n\theta, \quad U_n(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta}, \qquad n=0,1,\dots \tag{2.22}$$

so $T_0=U_0=1$,

$$\begin{aligned} &T_1(\lambda)=\lambda, \quad T_2(\lambda)=2\lambda^2-1,\; T_3(\lambda)=4\lambda^3-3\lambda,\; T_4(\lambda)=8\lambda^4-8\lambda^2+1;\\ &U_1(\lambda)=2\lambda,\; U_2(\lambda)=4\lambda^2-1,\; U_3(\lambda)=8\lambda^3-4\lambda \;\; U_4(\lambda)=16\lambda^4-12\lambda^2+1. \end{aligned}$$

The general expressions as the products are

$$T_n(\lambda)=2^{n-1}\prod_{k=1}^{n}\Bigl(\lambda-\cos\frac{(2k-1)\pi}{2n}\Bigr), \quad U_n(\lambda)=2^n\prod_{k=1}^{n}\Bigl(\lambda-\cos\frac{k\pi}{n+1}\Bigr),$$

The standard equalities

$$\begin{aligned} \cos(n-1)\theta+\cos(n+1)\theta&=2\cos\theta\,\cos n\theta,\\ \sin(n-1)\theta+\sin(n+1)\theta&=2\cos\theta\,\sin n\theta \end{aligned}$$

lead to

$$T_{n-1}(\lambda)+T_{n+1}(\lambda)=2\lambda\, T_n(\lambda),$$
$$U_{n-1}(\lambda)+U_{n+1}(\lambda)=2\lambda\, U_n(\lambda), \quad n\in\mathbb{N}.$$

It is clear now that the 1st kind polynomials for $J=J_0$ are

$$p_0(J_0)=0, \quad p_n(\lambda,J_0)=U_{n-1}\Big(\frac{\lambda}{2}\Big), \quad n=1,2,\dots,$$

and 1st kind polynomials for $J=J_0'=J(\{0\},\{\sqrt{2},1,1,\dots\})$ are

$$p_0(J_0')=0, \;\; p_1(\lambda,J_0')=1, \;\; p_n(\lambda,J_0')=\sqrt{2}\,T_{n-1}\Big(\frac{\lambda}{2}\Big), \quad n=2,3,\dots.$$

More generally, if $b_1=b_2=\ldots=0$, $a_1=a_2=\ldots=a_k=1$, then

$$p_n(\lambda,J)=U_{n-1}\Big(\frac{\lambda}{2}\Big), \quad n=1,2,\dots,k+1. \tag{2.23}$$

If $b_1=b_2=\ldots=0$, $a_1=\sqrt{2}$, $a_2=\ldots=a_k=1$, then

$$p_n(\lambda,J)=\sqrt{2}\,T_{n-1}\Big(\frac{\lambda}{2}\Big), \quad n=2,3,\dots,k+1. \tag{2.24}$$

The matrix form of (2.19) is

$$\begin{bmatrix} y_{n+1} \\ a_n y_n \end{bmatrix}=A(\lambda;a_n,b_n)\begin{bmatrix} y_n \\ a_{n-1}y_{n-1} \end{bmatrix}, \quad A(\lambda;a_n,b_n)=\begin{bmatrix} \frac{\lambda-b_n}{a_n} & -\frac{1}{a_n} \\ a_n & 0 \end{bmatrix}. \tag{2.25}$$

The product of the matrices in (2.25) produces the *transfer matrix*

$$\mathscr{T}_n(\lambda):=A(\lambda;a_n,b_n)A(\lambda;a_{n-1},b_{n-1})\dots A(\lambda;a_1,b_1), \quad n\in\mathbb{N}. \tag{2.26}$$

Precisely,

$$\begin{gathered}\begin{bmatrix} p_{n+1} \\ a_n p_n \end{bmatrix}=\mathscr{T}_n(\lambda)\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} q_{n+1} \\ a_n q_n \end{bmatrix}=\mathscr{T}_n(\lambda)\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\ \mathscr{T}_n(\lambda)=\begin{bmatrix} p_{n+1}(\lambda) & -q_{n+1}(\lambda) \\ a_n p_n(\lambda) & -a_n q_n(\lambda) \end{bmatrix}, \;\; n\in\mathbb{N}. \end{gathered} \tag{2.27}$$

Since $\det A(\lambda;a_n,b_n)=\det\mathscr{T}_n(\lambda)=1$, we see that for each $n\in\mathbb{N}$ and complex λ

$$a_n\big(p_n(\lambda)q_{n+1}(\lambda)-p_{n+1}(\lambda)q_n(\lambda)\big)=1. \tag{2.28}$$

Let μ be the spectral measure of the Jacobi operator J. The *Weyl function* is defined by

$$m(\lambda)=m(\lambda,J):=\int_{\mathbb{R}}\frac{\mu(dt)}{t-\lambda}.$$

For the initial data

$$v_0(\lambda) = \begin{bmatrix} m(\lambda) \\ -1 \end{bmatrix}$$

the solution of (2.25) is

$$v_n(\lambda) = \mathscr{T}_n(\lambda)v_0(\lambda) = \begin{bmatrix} m(\lambda)p_{n+1}(\lambda) + q_{n+1}(\lambda) \\ a_n\big(m(\lambda)p_{n+1}(\lambda) + q_{n+1}(\lambda)\big) \end{bmatrix}.$$

As is known, this solution is square summable for each $\lambda \in \rho(J)$. Denote $w_n := mp_{n+1} + q_{n+1}$. The Weyl function for the k-stripped Jacobi matrix J_k can be expressed in terms of w_k as

$$m(\lambda, J_k) = -\frac{w_k(\lambda)}{a_k w_{k-1}(\lambda)} = -\frac{m(\lambda)p_{k+1}(\lambda) + q_{k+1}(\lambda)}{a_k\big(m(\lambda)p_k(\lambda) + q_k(\lambda)\big)}. \tag{2.29}$$

In particular, for $k = 1$ we have [33, Theorem 3.2.4]

$$m(\lambda, J) = \frac{1}{b_1 - \lambda - a_1^2 m(\lambda, J_1)}.$$

2.3. Periodic Jacobi matrices

See [35, Chapter 7], [33, Chapter 5] for an extensive exposition of the theory.

A Jacobi matrix J is called *N-periodic*, $N \in \mathbb{N}$, if

$$a_{n+N} = a_n, \quad b_{n+N} = b_n, \qquad n = 1, 2, \dots.$$

In other word, J is N-periodic if and only if the N stripped matrix $J_N = J$. Equality (2.29) shows that the Weyl function satisfies the following quadratic equation

$$a_N p_N(\lambda) m^2(\lambda) + \big(p_{N+1}(\lambda) + a_N q_N(\lambda)\big) m(\lambda) + q_{N+1}(\lambda) = 0. \tag{2.30}$$

In view of (2.28), the discriminant of this equation equals

$$\begin{aligned} D(\lambda) &= \big(p_{N+1}(\lambda) + a_N q_N(\lambda)\big)^2 - 4a_N p_N(\lambda) q_{N+1}(\lambda) \\ &= \big(p_{N+1}(\lambda) - a_N q_N(\lambda)\big)^2 - 4 = \mathscr{D}^2(\lambda) - 4, \end{aligned} \tag{2.31}$$

where the polynomial

$$\mathscr{D}(\lambda) := p_{N+1}(\lambda) - a_N q_N(\lambda) = p_{N+1}(\lambda) - \frac{a_N}{a_1} p_{N-1}^{(1)}(\lambda) \tag{2.32}$$

is the well-known *discriminant* of the Jacobi matrix J, which plays a key role in the spectral theory of periodic Jacobi matrices.

The "right" root of (2.30) for the Weyl function can be singled out from the condition $m(z) = O(z^{-1})$, $z \to \infty$,

$$m(\lambda) = \frac{-\gamma_N(\lambda) + \sqrt{\mathscr{D}^2(\lambda) - 4}}{2a_N p_N(\lambda)}, \quad \gamma_N(\lambda) := p_{N+1}(\lambda) + \frac{a_N}{a_1} p_{N-1}^{(1)}(\lambda). \tag{2.33}$$

Let us now turn to the structure of the spectrum for a periodic Jacobi matrix. It is well known, that the essential spectrum of a periodic Jacobi matrix has a banded structure, i.e., it is a union of nondegenerate closed intervals, some of them may touch each other. Precisely, let J be an N-periodic Jacobi matrix. Then [33, Section 5.4]

$$\begin{gathered}\sigma_{ess}(J)=\{x\in\mathbb{R}:\ -2\leqslant\mathscr{D}(x)\leqslant 2\}=\bigcup_{j=1}^{N}[\alpha_j,\beta_j],\\ \alpha_1<\beta_1\leqslant\alpha_2<\beta_2\leqslant\ldots\leqslant\alpha_N<\beta_N.\end{gathered}\tag{2.34}$$

The open intervals $G_{N-j}:=(\beta_j,\alpha_{j+1})$, $j=1,2,\ldots,N-1$, are called the *spectral gaps*. The gap (β_j,α_{j+1}) is open as long as $\beta_j<\alpha_{j+1}$, and it is closed otherwise (it is convenient counting closed gaps as the "genuine ones"). Note that enumeration of the gaps (including the closed ones) goes from the right to the left, so $G_1=(\beta_{N-1},\alpha_N)$ is the rightmost gap.

We have

$$\begin{aligned}\mathscr{D}(\lambda)=2\ &\Leftrightarrow\quad \lambda=\beta_N,\alpha_{N-1},\beta_{N-2},\ldots\\ \mathscr{D}(\lambda)=-2\ &\Leftrightarrow\ \lambda=\alpha_N,\beta_{N-1},\alpha_{N-2},\ldots.\end{aligned}$$

The rest of the spectrum of J is the discrete spectrum, which consists of a finite number of eigenvalues. The eigenvalues of J agree with the poles of the Weyl function (2.33). So, the discrete spectrum $\sigma_d(J)$ is a part of the zero set of the polynomial p_N (the denominator of (2.33)). It is known, that the closure of each spectral gap (including the closed ones) contains exactly one root of p_N. To specify those, which produce the eigenvalues, we should first choose the roots *inside* the open gaps. If $p_N(\lambda_0)=0$, and λ_0 lies inside the gap, we need the numerator in (2.33) be *nonzero*, so not to cancel the root of the denominator. By (2.31),

$$\mathscr{D}^2(\lambda_0)-4=\gamma_N^2(\lambda_0)\ \Rightarrow\ \sqrt{\mathscr{D}^2(\lambda_0)-4}=|\gamma_N(\lambda_0)|,$$

so $\lambda_0\in G_q$ is the eigenvalue of J if and only if

$$\operatorname{sgn}\gamma_N(\lambda_0)=(-1)^{q+1}.\tag{2.35}$$

So, to find the spectrum $\sigma(J)=\sigma_{ess}(J)\cup\sigma_d(J)$, we proceed in three steps.

Step 1. Find the 1st kind polynomials p_{N+1} for J and $p_{N-1}^{(1)}$ for 1-stripped matrix J_1. Compute the discriminant $\mathscr{D}(J)$ in (2.32) and the polynomial γ_N in (2.33).

Step 2. Find the essential spectrum by (2.34).

Step 3. Find all roots of the polynomial p_N and choose those inside the gaps, for which (2.35) holds.

2.4. Right limits and eigenvalues

Our first topic here concerns the notion of a right limit, see [33, Chapter 7].

Let $f = \{f_i\}_{i\in\mathbb{N}} \in \ell^\infty(\mathbb{N})$. A two-sided sequence $\varphi = \{\varphi_i\}_{i\in\mathbb{Z}}$ is said to be the *right limit* for f, $\varphi \in RL(f)$, if there is a sequence of indices $\mathscr{M} = \{m_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ so that

$$\varphi_i = \lim_{j\to\infty} f_{i+m_j} \quad \forall i \in \mathbb{Z}. \tag{2.36}$$

Sometimes we write (2.36) as $\varphi_i = \lim_{m\in\mathscr{M}} f_{i+m}$. Note that, although the individual value f_{i+m_j} may be senseless for "large enough" negative i, the φ_i in (2.36) is well-defined. We say that $\mathscr{M}$ generates the right limit φ.

A simple compactness argument implies the following result.

PROPOSITION 2.4. *For an arbitrary $f \in \ell^\infty(\mathbb{N})$, the set $RL(f)$ is nonempty. Moreover, for each $\varphi \in RL(f)$ and each sequence of indices $\Lambda \subset \mathbb{N}$ there is a subsequence $\mathscr{M} \subset \Lambda$ generating φ.*

It is clear that the set $RL(f)$ is closed under the shift S

$$\varphi \in RL(f) \;\Leftrightarrow\; S^k\varphi = \{\varphi_{i+k}\}_{i\in\mathbb{Z}} \in RL(f) \quad \forall k \in \mathbb{Z}. \tag{2.37}$$

It follows directly from the definition, that

$$\lim_{i\to\infty} f_i = g \;\Rightarrow\; RL(f) = \{\dots, g, g, g, \dots\},$$

i.e., $RL(f)$ consists of a single, constant sequence. More generally,

$$\lim_{i\to\infty} (\tilde{f}_i - f_i) = 0 \;\Rightarrow\; RL(\tilde{f}) = RL(f).$$

In particular, if f and $\tilde{f}$ agree from some point on, the sets of right limits are the same, $RL(f) = RL(\tilde{f})$.

Denote by $L(f) \subset \mathbb{C}$ the set of all limit points of f. It is clear that $\varphi = \{\varphi_i\}_{i\in\mathbb{Z}} \in RL(f)$ implies $\varphi_i \in L(f)$ for each $i \in \mathbb{Z}$.

If f does not converge, the cardinality of $L(f)$ is at least 2. It is easy to see from the second statement of Proposition 2.4 that the following holds.

PROPOSITION 2.5. *Given $g_1, g_2 \in L(f)$, $g_1 \neq g_2$, for each $k \in \mathbb{Z}$ there are $\varphi(j) = \{\varphi_i(j)\}_{i\in\mathbb{Z}} \in RL(f)$, $j = 1, 2$, which depend on k, so that*

$$\varphi_k(1) = g_1, \qquad \varphi_k(2) = g_2.$$

In particular, uniqueness of the right limit yields the convergence of f.

EXAMPLE 2.6. Let f be an N-periodic sequence, $f_{i+N} = f_i$, $i \in \mathbb{N}$. We extend it to N-periodic, two-sided sequence $\varphi = \{\varphi_i\}_{i\in\mathbb{Z}}$

$$\varphi_{i+N} = \varphi_i, \quad i \in \mathbb{Z}, \qquad \varphi_i = f_i, \quad i \in \mathbb{N},$$

so $\varphi = \{\dots, f_1, f_2, \dots, f_N, f_1, f_2, \dots\}$. It is not hard to show that

$$RL(f) = \{S^q\varphi\}_{q=0}^{N-1},$$

so the right limits are exhausted by the shifts of φ.

There is another, opposite in a sense, situation when the set of right limits is available.

EXAMPLE 2.7. An increasing sequence of positive numbers $\Lambda = \{\lambda_i\}_{i\in\mathbb{N}}$ is called *sparse*, if

$$\lim_{i\to\infty}(\lambda_{i+1} - \lambda_i) = +\infty.$$

Given a sparse sequence of indices Λ, put

$$b = \{b_i\}_{i\in\mathbb{N}}, \quad b_i = \begin{cases} \beta_1, \ i \in \Lambda; \\ \beta_0, \ i \notin \Lambda. \end{cases}, \quad \beta_0, \beta_1 \in \mathbb{C}.$$

Our goal is to describe the set $RL(b)$.

We show first that the constant sequence $b(0) = \{\dots, \beta_0, \beta_0, \beta_0, \dots\}$ is in $RL(b)$. By the sparseness, there is j_1 so that $\lambda_{j+1} - \lambda_j \geqslant 3$ for $j \geqslant j_1$. Define the sequence of indices $\mathscr{M}_0 = \{m_j\}$

$$m_j := \lambda_{j_1+j} + (-1)^j, \qquad j \in \mathbb{N}.$$

Then for each $i \in \mathbb{Z}$ we have $i + m_j \notin \Lambda$ for all large enough j. Indeed, assume on the contrary, that $i + m_j = \lambda_{s(j)}$ for an infinite number of j's, that is,

$$i + (-1)^j = \lambda_{s(j)} - \lambda_{j_1+j}$$

for such values of j. But the latter contradicts the sparseness of Λ. So, $\mathscr{M}_0$ generates $b(0)$.

Next, note that $\beta_1 \in L(b)$, i.e., β_1 is the limit point of b. By Proposition 2.5, for each $k \in \mathbb{Z}$ there is a right limit $\varphi(k) \in RL(b)$ so that $\varphi_k(k) = \beta_1$. The latter means that there is a sequence $\{\nu_j\}_{j\in\mathbb{N}}$ (generating $\varphi(k)$), for which

$$b_{k+\nu_j} = \beta_1, \ \sim \ k + nu_j \in \Lambda,$$

for all large enough j.

We show that for any $k' \neq k$ the relation $k' + \nu_j \notin \Lambda$ for all large enough j. Indeed, assume on the contrary, that

$$k' + \nu_j \in \Lambda, \quad k' + \nu_j = \lambda_{r(j)}$$

for an infinite number of j's. But

$$k' + \nu_j = k' - k + k + \nu_j = k' - k + \lambda_{t(j)}$$

and so

$$k' - k = \lambda_{r(j)} - \lambda_{t(j)}$$

for such values of j, that contradicts the sparseness of Λ. Hence, $\varphi_{k'}(k)=\beta_0$ for $k'\neq k$, which means that

$$\varphi(k)=b(k,1):=\{\dots,\beta_0,\beta_0,\beta_1,\beta_0,\beta_0,\dots\},\quad k\in\mathbb{Z},$$

β_1 occurs at k-th place. Finally,

$$RL(b)=\{b(0);b(k,1),\ k\in\mathbb{Z}\}. \tag{2.38}$$

In exactly the same way we can examine a union of two sparse sequences. Precisely, let $R=\{r_i\}_{i\in\mathbb{N}}$, be a sparse sequence of positive integers, and assume for simplicity, that $r_{i+1}-r_i$ is strictly increasing, and $r_{i+1}-r_i\geqslant 2$. Consider the union $\tilde{R}:=\{r_1,r_1+1,r_2,r_2+1,\dots\}$. Let

$$a=\{a_i\}_{i\in\mathbb{N}},\quad a_i=\begin{cases}\alpha_1,\ i\in\tilde{R};\\ \alpha_0,\ i\notin\tilde{R}.\end{cases},$$

The similar reasoning leads to the following conclusion

$$\begin{aligned}RL(a)&=\{a(0);a(k,2),k\in\mathbb{Z}\},\quad a(0)=\{\dots,\alpha_0,\alpha_0,\alpha_0,\dots\},\\ a(k,2):&=\{\dots,\alpha_0,\alpha_0,\alpha_0,\alpha_1,\alpha_1,\alpha_0,\alpha_0,\alpha_0,\dots\},\end{aligned} \tag{2.39}$$

α_1 occurs at the places k, $k+1$.

Going back to Jacobi matrices $J=J(\{b_i\},\{a_i\})_{i\in\mathbb{N}}$, we say that a two-sided Jacobi matrix

$$J_{right}=J\Big(\{b_i^{(r)}\},\{a_i^{(r)}\}\Big)_{i\in\mathbb{Z}}$$

is a *right limit* of J if for some sequence of indices $\{m_j\}_{j\in\mathbb{N}}$

$$\lim_{j\to\infty}a_{i+m_j}=a_i^{(r)},\quad \lim_{j\to\infty}b_{i+m_j}=b_i^{(r)},\quad \forall i\in\mathbb{Z}.$$

Dealing with certain sparse graphs, we will encounter the following Jacobi matrices

$$J^{\pm}=J(\{b_i^{\pm}\},\{1\})_{i\in\mathbb{N}},\quad b_i^{\pm}=\begin{cases}\pm1,\ i\in\Lambda;\\ 0,\quad i\notin\Lambda.\end{cases} \tag{2.40}$$

and

$$\tilde{J}=J(\{0\},\{a_i\})_{i\in\mathbb{N}},\quad a_i=\begin{cases}\sqrt{2},\ i\in\tilde{R};\\ 1,\quad i\notin\tilde{R}.\end{cases}, \tag{2.41}$$

Λ and $\tilde{R}$ being sparse sequences of indices above. The set of right limits in the first case is given by

$$RL(J^{\pm})=\big\{J_0(\mathbb{Z});J(\{\dots,0,0,\pm1,0,0,\dots\},\{1\})\big\}, \tag{2.42}$$

±1 occurs at k-th place, $k\in\mathbb{Z}$. In the second case

$$RL(J)=\big\{J_0(\mathbb{Z});J(\{0\},\{\dots,1,1,1,\sqrt{2},\sqrt{2},1,1,1,\dots\})\big\}, \tag{2.43}$$

$\sqrt{2}$ occurs at the places k, $k+1$; $k\in\mathbb{Z}$.

The following result of Last–Simon [19], [33, Theorem 7.2.1], plays a key role for computations in Section 4.

THEOREM. (Last–Simon). *Let* $J = J(\{b_i\},\{a_i\})_{i\in\mathbb{N}}$ *be a Jacobi matrix with bounded entries*

$$a := \sup_i(|a_i|+|b_i|) < \infty. \tag{2.44}$$

Then

$$\sigma_{ess}(J) = \bigcup_{J_{right}\in RL(J)} \sigma(J_{right}).$$

Our second topic here concerns the eigenvalues of Jacobi matrices, in particular, a result of Simon–Stolz [34] which provides a condition for a real λ not to be an eigenvalue of J ($\lambda \notin \sigma_p(J)$). The condition is given in terms of the asymptotic behavior for the norms of the transfer matrices $\mathscr{T}_n$ (2.26).

Let $J = J(\{b_i\},\{a_i\})_{i\in\mathbb{N}}$ be a Jacobi matrix with bounded entries (2.44). Let $\{p_n\}_{n\geqslant 1}$ be the 1st kind polynomials for J. By definition, $\lambda \in \sigma_p(J)$ is equivalent to $\{p_n\}_{n\geqslant 1} \in \ell^2$.

THEOREM. (Simon–Stolz). *A real number* $\lambda \notin \sigma_p(J)$ *as long as*

$$\sum_{n=1}^{\infty} \|\mathscr{T}_n(\lambda)\|^{-2} = +\infty. \tag{2.45}$$

Here and in the rest of the paper under the norm $\|A\|$ *of a matrix* A *we always mean its operator norm.*

The argument is simple enough. Note first, that for an invertible 2×2 matrix C the following holds

$$\|C^{-1}\| = \frac{\|C\|}{|\det C|}, \tag{2.46}$$

see, e.g., [32, Lemma 10.5.1]. Next, by (2.27) and (2.46) with $\det \mathscr{T}_n = 1$,

$$1 \leqslant \|\mathscr{T}_n^{-1}(\lambda)\|^2\,(p_{n+1}^2(\lambda)+a_n^2p_n^2(\lambda)) \leqslant \|\mathscr{T}_n(\lambda)\|^2\,(1+a^2)(p_{n+1}^2(\lambda)+p_n^2(\lambda)).$$

Hence,

$$\frac{1}{1+a^2}\sum_{n=1}^{k} \|\mathscr{T}_n(\lambda)\|^{-2} \leqslant 2\sum_{n=1}^{k+1} p_n^2(\lambda),$$

and we are done.

In some examples below the condition (2.45) can be employed.

3. Spectra of graphs with infinite tails

To obtain the canonical form for particular graphs one should apply the reversed Gram–Schmidt algorithm by hand. Its efficiency strongly depends on complexity of the graph in question.

3.1. Trees with tails

The examples below are taken partially from [12, Section 4]. Some of them are new.

EXAMPLE 3.1. "A weighted star".

Let $S_n(w)$ be a simple weighted star graph of order $n+1$, $n \geqslant 2$, with vertices $1,\dots,n$ of degree 1, and the vertex $n+1$ of degree n being a root. The weight of the edge $(i,n+1)$ equals w_i, $1 \leqslant i \leqslant n$. We consider the coupling $\Gamma = S_n(w) + \mathbb{P}_\infty$, where the infinite ray is attached to the root.

The canonical basis $\{h_j\}_{j\in\mathbb{N}}$ looks as follows. We put

$$h_j := e_j, \quad j \geqslant n+1 \implies A(\Gamma)\,h_j = h_{j-1} + h_{j+1}, \quad j \geqslant n+2.$$

Next, let $w := (w_1, w_2, \dots, w_n)$, $\|w\| = \sqrt{w_1^2 + \ldots + w_n^2}$, and let

$$h_n := \frac{1}{\|w\|} \sum_{j=1}^{n} w_k\, e_j\,.$$

Then

$$A(\Gamma)\,h_{n+1} = \|w\|\,h_n + h_{n+2}, \qquad A(\Gamma)\,h_n = \|w\|\,h_{n+1}.$$

So the Jacobi subspace and Jacobi component of Γ are

$$\mathscr{J}(\Gamma) = \operatorname{span}\{h_j\}_{j\geqslant n}, \quad J(\Gamma) = J\big(\{0\}, \{\|w\|, 1, 1, \dots\}\big). \tag{3.1}$$

To find the finite-dimensional component, let $\xi = [\xi_{kj}]_{k,j=1}^n$ be a unitary matrix with the specified last column

$$\xi_{kn} = \frac{w_k}{\|w\|}, \quad k = 1, \dots, n. \tag{3.2}$$

We construct an orthonormal basis in $\mathbb{C}^n$

$$f_j := \sum_{k=1}^{n} \xi_{kj} e_k(n), \quad j = 1, \dots, n, \tag{3.3}$$

where $\{e_k(n)\}_{k=1}^n$ is the standard basis in $\mathbb{C}^n$. Put

$$h_j := \{f_j, 0, 0, \dots\}, \quad j = 1, \dots, n. \tag{3.4}$$

The orthogonality relations $\langle h_k, h_n\rangle = 0$, $1 \leqslant k \leqslant n-1$, give

$$A(\Gamma)\,h_k = \sum_{j=1}^{n} \xi_{kj}\xi_{kn} \cdot h_{n+1} = 0. \tag{3.5}$$

Hence the finite-dimensional component $F(\Gamma) = \mathbb{O}_{n-1}$ on the subspace $\mathscr{F}(\Gamma) = \operatorname{span}\{h_j\}_{j=1}^{n-1}$ of the dimension $n-1$ (note that the size of the star is now $n+1$). So the canonical form is

$$A(\Gamma) \simeq \mathbb{O}_{n-1} \bigoplus J\big(\{0\}, \{\|w\|, 1, 1, \dots\}\big). \tag{3.6}$$

The Jost polynomial is now given by (2.10)

$$\|w\|\,u(x) = (1-\|w\|^2)x^2 + 1.$$

Clearly, $u > 0$ for $\|w\| \leqslant 1$, and it has zeros inside $(-1,1)$ if and only if $\|w\| > \sqrt{2}$. In this case the spectrum is $\sigma(\Gamma) = [-2,2] \cup \sigma_d(\Gamma) \cup \sigma_h(\Gamma)$ with the discrete spectrum being a pair of eigenvalues off $[-2,2]$,

$$\sigma_d(S_n(w)+\mathbb{P}_\infty) = \left\{\pm\left(\sqrt{\|w\|^2-1} + \frac{1}{\sqrt{\|w\|^2-1}}\right)\right\}, \tag{3.7}$$

and the hidden spectrum $\sigma_h(\Gamma) = \{0_{n-1}\}$, the zero eigenvalue of multiplicity $n-1$.

For the unweighted star S_n we have

$$\sigma_d(S_n+\mathbb{P}_\infty) = \left\{\pm\left(\sqrt{n-1} + \frac{1}{\sqrt{n-1}}\right)\right\}, \quad \sigma_h(S_n+\mathbb{P}_\infty) = 0_{n-1}, \quad n \geqslant 3,$$
$$\sigma_d(S_2+\mathbb{P}_\infty) = \emptyset, \quad \sigma_h(S_2+\mathbb{P}_\infty) = 0_1.$$

The spectrum $\sigma(S_n+\mathbb{P}_\infty)$ was found in [22].

Note that the unweighted star graph S_n is a complete bipartite graph, $S_n = K_{1,n}$. For the general complete bipartite graph $K_{p,n+1-p}$ see [12, Example 5.6].

Although an explicit form of the matrix $\xi = [\xi_{kj}]_{k,j=1}^n$ in (3.2) is immaterial, it is worth noting that in the unweighted case $\xi_{kn} = n^{-1/2}$, $1 \leqslant k \leqslant n$, and one can take

$$\xi = \Phi_n := \frac{1}{\sqrt{n}}\left[e^{\frac{2\pi ikj}{n}}\right]_{k,j=1}^n, \tag{3.8}$$

which is known as the *Fourier matrix*. Clearly, there is a number of other options for ξ to be a real orthogonal matrix (rotation in $\mathbb{R}^n$ with appropriate Euler's angles, orthogonal polynomials etc.).

EXAMPLE 3.2. "A multiple star".

Consider the unweighted star-like graph $S_{n,p}$ with n rays, $n \geqslant 2$, each of which contains $p+1$ vertices, $p \geqslant 2$. The vertices along each ray (without the root) are numbered as

$$\{1, n+1, \dots, (p-1)n+1\}, \ \{2, n+2, \dots, (p-1)n+2\}, \ \dots \ \{n, 2n, \dots, pn\},$$

and the root is $pn+1$, so $S_{n,1} = S_n$. Let $\Gamma = S_{n,p} + \mathbb{P}_\infty$, with the path attached to the root. As above, we put $h_j := e_j$, $j = pn+1, \dots$, and

$$h_{p(n-1)+i} := \frac{1}{\sqrt{n}}\sum_{q=1}^n e_{(i-1)n+q}, \qquad i = 1,2,\dots,p. \tag{3.9}$$

Then $A(\Gamma)h_j = h_{j-1} + h_{j+1}$, $j = pn+2\dots$, and

$$A(\Gamma)h_{pn+1} = \sqrt{n}h_{pn} + h_{pn+2}, \ A(\Gamma)h_{pn} = h_{pn-1} + \sqrt{n}\,h_{pn+1},$$
$$A(\Gamma)h_{pn} = h_{pn-1} + \sqrt{n}\,h_{pn+1},$$
$$A(\Gamma)h_{p(n-1)+i} = h_{p(n-1)+i-1} + h_{p(n-1)+i+1}, \quad i = 2,\dots,p-1,$$
$$A(\Gamma)h_{p(n-1)+1} = h_{p(n-1)+2},$$

so the Jacobi subspace and Jacobi component of Γ are

$$\begin{aligned} \mathscr{J}(\Gamma) &= \operatorname{span}\{h_j\}_{j\geqslant p(n-1)+1}, \\ J(\Gamma) &= J(\{0\},\{a_j\}), \quad a_j = \begin{cases} \sqrt{n}, & j=p; \\ 1, & j\neq p. \end{cases} \end{aligned} \tag{3.10}$$

To find the finite-dimensional component note that, by the construction, $h_{p(n-1)+i} \in \operatorname{span}\{e_{(i-1)n+1},\dots,e_{in}\}$. As in the above example, we supplement each $h_{p(n-1)+i}$ to the basis in this subspace by means of the Fourier matrix (3.8)

$$f_j^{(k)} := \sum_{q=1}^{n} \xi_{qj} e_{(k-1)n+q}, \quad f_n^{(k)} := \sum_{q=1}^{n} \xi_{qn} e_{(k-1)n+q} = h_{p(n-1)+k}$$

for $1 \leqslant j \leqslant n-1$, $1 \leqslant k \leqslant p$. As in (3.5), we have

$$\begin{aligned} A(\Gamma) f_j^{(1)} &= f_j^{(2)}, \quad A(\Gamma) f_j^{(2)} = f_j^{(1)} + f_j^{(3)}, \dots, \\ A(\Gamma) f_j^{(p-1)} &= f_j^{(p-2)} + f_j^{(p)}, \quad A(\Gamma) f_j^{(p)} = f_j^{(p-1)}. \end{aligned} \tag{3.11}$$

Relations (3.11) mean that the subspace $\mathscr{H}_j := \operatorname{span}\{f_j^{(1)},\dots,f_j^{(p)}\}$ is $A(\Gamma)$-invariant, and $A(\Gamma)|\mathscr{H}_j = J_{0,p}$. There are exactly $n-1$ such subspaces for $j=1,\dots,n-1$. Finally, we come to the following canonical form for the adjacency matrix

$$A(\Gamma) \simeq F(\Gamma) \bigoplus J(\Gamma), \qquad F(\Gamma) = \bigoplus_{j=1}^{n-1} J_{0,p}. \tag{3.12}$$

The Jost polynomial is computed in (2.5)

$$-\sqrt{n}u(x) = (n-1)x^2 \frac{x^{2p}-1}{x^2-1} - 1 = (n-1)(x^{2p}+x^{2p-2}+\ldots+x^2) - 1.$$

It is easy to see that u has exactly a pair of symmetric roots $\pm x_0(p,n)$ in $(-1,1)$, which have the spectral meaning. Hence

$$\sigma(\Gamma) = [-2,2] \cup \sigma_d(\Gamma) \cup \sigma_h(\Gamma)$$

with

$$\sigma_d(\Gamma) = \left\{\pm\left(x_0(p,n) + \frac{1}{x_0(p,n)}\right)\right\}, \quad \sigma_h(\Gamma) = \left\{2\cos\frac{\pi j}{p+1}\right\}_{j=1}^{p}, \tag{3.13}$$

the hidden spectrum comes from the finite component $F(\Gamma)$ (3.12), see (1.10), and each hidden eigenvalue has multiplicity $n-1$.

The problem becomes harder (in the sense of computation) if the original finite star-like graph is nonsymmetric (the rays are different).

EXAMPLE 3.3. "$T_{n,n-1,\infty}$".

Consider a finite path of order $2n$ with the vertices labeled

$$\{1,2,4,\dots,2n-2,2n,2n-1,\dots,5,3\},$$

with the tail attached to the root $2n$. So there are two finite rays of different length, n and $n-1$, respectively. The graph Γ thus obtained is known as $T_{n,n-1,\infty}$-graph.

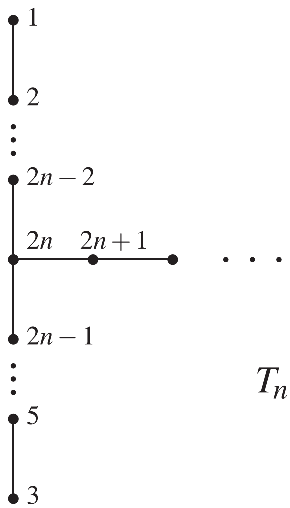
The canonical basis $\{h_j\}_{j\geqslant 1}$ looks as follows: $h_j=e_j$ for $j\geqslant 2n$,

$$h_{n+i}=\frac{e_{2i}+e_{2i+1}}{\sqrt{2}}, \quad h_{n-i}=\frac{e_{2i}-e_{2i+1}}{\sqrt{2}}, \qquad i=1,\dots,n-1,$$

and $h_n=e_1$.

The adjacency operator $A(\Gamma)$ acts on the basis vectors in a simple way:

$$\begin{aligned} A(\Gamma)\,h_{2n+j}&=h_{2n+j+1}+h_{2n+j-1}, \quad j=1,2,\dots,\\ A(\Gamma)\,h_{2n}&=h_{2n+1}+\sqrt{2}\,h_{2n-1}, \qquad A(\Gamma)\,h_{2n-1}=\sqrt{2}\,h_{2n}+h_{2n-2},\\ A(\Gamma)\,h_{2n-k}&=h_{2n-k+1}+h_{2n-k-1}, \quad k=2,\dots,n-2. \end{aligned}$$



$T_{n,n-1,\infty}$

Furthermore,

$$\begin{aligned} A(\Gamma)\,h_{n+1}&=\frac{e_1+e_4+e_5}{\sqrt{2}}=h_{n+2}+\frac{1}{\sqrt{2}}h_n\\ A(\Gamma)\,h_n&=e_2=\frac{h_{n+1}+h_{n-1}}{\sqrt{2}}, \qquad A(\Gamma)\,h_{n-j}=h_{n-j+1}+h_{n-j-1}, \quad j=2,\dots,n-2,\\ A(\Gamma)\,h_{n-1}&=\frac{e_1+e_4-e_5}{\sqrt{2}}=\frac{1}{\sqrt{2}}h_n+h_{n-2}, \quad A(\Gamma)\,h_1=h_2. \end{aligned}$$

So, the finite-dimensional component is missing, and $A(\Gamma)\simeq J(\{0\},\{a_j\})$,

$$a_j=1, \;\; j\neq n-1,n,2n-1, \quad a_{n-1}=a_n=\frac{1}{\sqrt{2}}, \quad a_{2n-1}=\sqrt{2}.$$

The Jost polynomial can be found from the definition (2.6). For instance, for $n=3$ we have

$$\frac{1}{\sqrt{2}}u(x) = -ax^{10} - 2ax^8 - \Big(\frac{5}{2}a - 1\Big)x^6 - \Big(2a + \frac{3}{2}\Big)x^4 - ax^2 + 1, \quad a = \sqrt{2} - 1.$$

Then u has a pair of symmetric roots $\pm x_1$ in $(-1,1)$, and so

$$\sigma(T_{3,2,\infty}) = [-2,2] \cup \sigma_d(T_{3,2,\infty}), \qquad \sigma_d(T_{3,2,\infty}) = \pm\Big(x_1 + \frac{1}{x_1}\Big). \tag{3.14}$$

3.2. Graphs with cycles and tails

EXAMPLE 3.4. "The complete graph with tail".

Let K_n be a complete graph of order $n \geqslant 3$, $\Gamma = K_n + \mathbb{P}_\infty$, the ray $\{n, n+1, \dots\}$ is attached to the vertex n. Put

$$h_j = e_j, \quad j = n, n+1 \dots, \quad h_{n-1} = \frac{1}{\sqrt{n-1}} \sum_{k=1}^{n-1} e_k.$$

Since

$$A(\Gamma)e_k = \sum_{j \neq k} e_j = S - e_k, \qquad S := \sum_{j=1}^{n} e_j, \quad k = 1, 2, \dots, n-1,$$

we see that

$$\begin{aligned} A(\Gamma)h_{n-1} &= \frac{1}{\sqrt{n-1}} \sum_{k=1}^{n-1}(S - e_k) = \sqrt{n-1}\,S - \widehat{e}_{n-1} \\ &= \sqrt{n-1}\big(\sqrt{n-1}\,h_{n-1} + h_n\big) - h_{n-1} = \sqrt{n-1}\,h_n + (n-2)\,h_{n-1}. \end{aligned}$$

Hence, the Jacobi component is

$$\mathscr{J}(\Gamma) = \operatorname{span}\{h_j\}_{j \geqslant n-1}, \quad J(\Gamma) = J(\{n-2, 0, 0, \dots\}, \{\sqrt{n-1}, 1, 1, \dots\}).$$

Next, put

$$h_k = \sum_{j=1}^{n-2} \xi_{kj} e_j, \quad \xi_{k,n-2} = \frac{1}{\sqrt{n-2}}, \quad k = 1, 2, \dots, n-2,$$

$[\xi_{ij}]$ is a unitary matrix of order $n-2$. Then,

$$A(\Gamma)h_k = \sum_{j=1}^{n-2} \xi_{kj}(S - e_j) = S \sum_{j=1}^{n-2} \xi_{kj} - h_k = -h_k, \quad k = 1, \dots, n-2.$$

So, the finite dimensional component is

$$\mathscr{F}(\Gamma) = \operatorname{span}\{h_j\}_{j=1}^{n-2}, \quad F(\Gamma) = -I_{n-2}.$$

The Jost polynomial, computed in (2.10)

$$\sqrt{n-1}\,u(x) = -(n-2)x^2-(n-2)x+1,$$

has one root x_6 in $(-1,1)$,

$$x_6 = \frac{1}{2}\Big(\sqrt{\frac{n+2}{n-2}}-1\Big).$$

Hence,

$$\sigma(\Gamma) = [-2,2]\cup\Big\{x_6+\frac{1}{x_6}\Big\}\cup\{(-1)_{n-2}\}.$$

EXAMPLE 3.5. "A cycle with two tails".

Let $\mathbb{C}_m$ be a cycle of order m. The spectral analysis of the coupling $\mathbb{C}_m+\mathbb{P}_\infty$ was carried out in [12, Proposition 1.4]. We consider here the graph $\Gamma = \mathbb{C}_{2n+1}+\mathbb{P}_\infty+\mathbb{P}_\infty$, $n\geqslant 2$, with two tails,

$$\{2n,2n+2\dots\},\quad \{2n+1,2n+3,\dots\},$$

attached to the adjacent vertices $2n$ and $2n+1$, respectively. Although Γ is not a coupling in the sense of Definition 1.1, the method works in this situation as well.

Define a system of vectors $h_0=e_n$,

$$\begin{aligned} h_k^{\pm} &= \frac{e_{n+k}\pm e_{n-k}}{\sqrt{2}}, \qquad k=1,2,\dots,n-1,\\ h_{n+i}^{\pm} &= \frac{e_{2n+2i}\pm e_{2n+2i+1}}{\sqrt{2}}, \qquad i=0,1,\dots \end{aligned}$$

The system $\{h_0,h_j^{\pm}\}_{j\geqslant 1}$ is the canonical orthonormal basis. The adjacency operator $A(\Gamma)$ acts as

$$\begin{aligned} &A(\Gamma)\,h_0 = \sqrt{2}h_1^+, \quad A(\Gamma)\,h_1^+ = h_2^+ + \sqrt{2}h_0,\\ &A(\Gamma)\,h_k^+ = h_{k+1}^+ + h_{k-1}^+, \quad k=2,\dots,n-1,\\ &A(\Gamma)\,h_n^+ = h_{n+1}^+ + h_n^+ + h_{n-1}^+,\\ &A(\Gamma)\,h_k^+ = h_{k+1}^+ + h_{k-1}^+, \quad k\geqslant n+1. \end{aligned}$$

So, the first Jacobi component is

$$\mathscr{J}^+(\Gamma) = \mathrm{span}\{h_0,\{h_j^+\}\}_{j\geqslant 1}, \quad J^+(\Gamma) = J(\{b_j^+\},\{a_j^+\})$$

with

$$b_j^+ = \begin{cases} 0, & j\neq n+1;\\ 1, & j=n+1, \end{cases} \qquad a_j^+ = \begin{cases} 1, & j\neq 1;\\ \sqrt{2}, & j=1. \end{cases}$$

Next,

$$\begin{aligned} &A(\Gamma)\,h_1^- = h_2^-, \quad A(\Gamma)\,h_k^- = h_{k+1}^- + h_{k-1}^-, \quad k=2,\dots,n-1,\\ &A(\Gamma)\,h_n^- = h_{n+1}^- - h_n^- + h_{n-1}^-,\\ &A(\Gamma)\,h_k^- = h_{k+1}^- + h_{k-1}^-, \quad k\geqslant n+1. \end{aligned}$$

So, the second Jacobi component is

$$\mathscr{J}^-(\Gamma) = \operatorname{span}\{h_j^-\}_{j\geqslant 1}, \quad J^-(\Gamma) = J(\{b_j^-\},\{a_j^-\})$$

with

$$b_j^- = \begin{cases} 0, & j \neq n; \\ -1, & j = n, \end{cases} \qquad a_j^- \equiv 1, \qquad j = 1,2,\dots$$

The canonical form of the adjacency operator is the orthogonal sum of two Jacobi operators

$$A(\Gamma) \simeq J^+(\Gamma) \bigoplus J^-(\Gamma).$$

The Jost polynomial for $J^+(\Gamma)$, given in (2.15) with $q = n+1$,

$$\sqrt{2}u_+(x) = -x^{2n+1} - x^2 - x + 1,$$

has one root $x_8 > 0$ in $(-1,1)$. Similarly, the Jost polynomial for $J^-(\Gamma)$, given in (2.4),

$$-u(x) = 1 + x\frac{x^{2n}-1}{x^2-1},$$

has one root $x_9 < 0$ in $(-1,1)$. Hence,

$$\sigma(\Gamma) = [-2,2] \cup \sigma_d(\Gamma), \quad \sigma_d(\Gamma) = \left\{x_8 + \frac{1}{x_8}, x_9 + \frac{1}{x_9}\right\}.$$

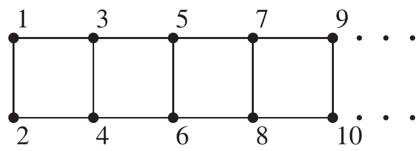
The case of the cycle $\mathbb{C}_{2n+1}$ with two tails attached to the *same* vertex is studied in [12, Example 5.1].

4. Ladders and chains of cycles

4.1. Canonical form and periodic structure

For certain infinite graphs (not necessarily finite graphs with tails) the canonical basis arises in a natural way. The canonical form, distinct from (1.15), enables one to describe explicitly the spectrum of the graph in question.

EXAMPLE 4.1. “A complete ladder”.



We define the canonical basis $\{h_n^+, h_n^-\}_{n\geqslant 1}$ by the relations

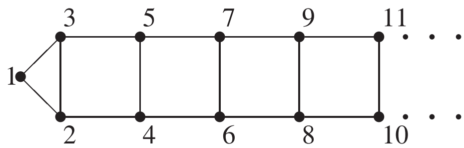
$$h_n^\pm := \frac{e_{2n-1} \pm e_{2n}}{\sqrt{2}}, \qquad n = 1,2,\dots.$$

The subspaces $\mathscr{J}^{\pm} = \mathrm{span}\{h_n^{\pm}\}_{n\geqslant 1}$ are invariant for $A(\Gamma)$, $\ell^2 = \mathscr{J}^{+} \oplus \mathscr{J}^{-}$, and the restrictions on these subspaces $J^{\pm}(\Gamma) = \pm I + J_0$. So, the canonical form of Γ is

$$A(\Gamma) \simeq \begin{bmatrix} I+J_0 & \\ & -I+J_0 \end{bmatrix}.$$

The spectrum is $\sigma(\Gamma) = [-3,3]$, with the interval $[-1,1]$ of multiplicity 2.

Similarly, for the ladder below



we define

$$h_n^{\pm} := \frac{e_{2n} \pm e_{2n+1}}{\sqrt{2}}, \quad n = 1,2,\dots, \quad h_0^{+} = e_1.$$

The adjacency operator $A(\Gamma)$ acts as

$$A(\Gamma)\,h_0^{+} = \sqrt{2}h_1^{+}, \quad A(\Gamma)\,h_1^{+} = \sqrt{2}h_0^{+} + h_1^{+} + h_2^{+},$$
$$A(\Gamma)\,h_n^{+} = h_{n-1}^{+} + h_n^{+} + h_{n+1}^{+}, \quad n = 2,3,\dots,$$

$$A(\Gamma)\,h_1^{-} = -h_1^{-} + h_2^{-},$$
$$A(\Gamma)\,h_n^{-} = h_{n-1}^{-} - h_n^{-} + h_{n+1}^{-}, \quad n = 2,3,\dots.$$

Again, the subspaces $\mathscr{J}^{+} = \mathrm{span}\{h_n^{+}\}_{n\geqslant 0}$ and $\mathscr{J}^{-} = \mathrm{span}\{h_n^{-}\}_{n\geqslant 1}$ are invariant for $A(\Gamma)$, $\ell^2 = \mathscr{J}^{+} \oplus \mathscr{J}^{-}$, and the canonical form of Γ is

$$A(\Gamma) \simeq \begin{bmatrix} I+J^{+} & \\ & -I+J^{-} \end{bmatrix}, \tag{4.1}$$

where

$$J^{-} = J_0, \qquad J^{+} = J(\{-1,0,0,\dots\},\{\sqrt{2},1,1,\dots\}).$$

The Jost polynomial for J^{+} is computed in (2.10)

$$\sqrt{2}\,u(x) = -x^2 + x + 1,$$

with the roots

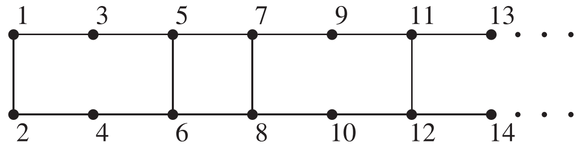
$$x_{10,11} = \frac{1 \pm \sqrt{5}}{2},$$

so $x_{11} \in (-1,1)$. Finally, $\sigma(I+J^{+}) = [-1,3] \cup \{1-\sqrt{5}\}$, and

$$\sigma(\Gamma) = [-3,3] \cup \sigma_h(\Gamma), \quad \sigma_h(\Gamma) = \{1-\sqrt{5}\},$$

with the eigenvalue $1-\sqrt{5} \in [-2,-1]$ lying on the simple spectrum.

EXAMPLE 4.2. "Simon's ladder" [31].



Consider the complete ladder with some rungs missing. Precisely, let $\Lambda = \{\lambda_j\}_{j\geqslant 1} \subset \mathbb{N}$ be a sequence of positive integers, $\lambda_1 = 1$, and assume that the rungs $\{2n-1, 2n\}_{n\in\Lambda}$ are present, i.e., the vertices $2n-1$ and $2n$ are incident. Put

$$\chi_k := \begin{cases} 1, & k \in \Lambda; \\ 0, & k \notin \Lambda. \end{cases} \qquad (\chi_1 = 1).$$

We define the canonical basis as in the case of the complete ladder. The subspaces $\mathscr{J}^{\pm} = \mathrm{span}\{h_n^{\pm}\}_{n\geqslant 1}$ are invariant for $A(\Gamma)$, $\ell^2 = \mathscr{J}^{+} \oplus \mathscr{J}^{-}$, and the adjacency operator $A(\Gamma)$ acts as

$$\begin{aligned} A(\Gamma)h_1^{+} &= h_1^{+} + h_2^{+}, \qquad & A(\Gamma)h_n^{+} &= h_{n-1}^{+} + \chi_n h_n^{+} + h_{n+1}^{+}, \quad n = 2,3,\ldots, \\ A(\Gamma)h_1^{-} &= -h_1^{-} + h_2^{-}, \qquad & A(\Gamma)h_n^{-} &= h_{n-1}^{-} - \chi_n h_n^{-} + h_{n+1}^{-}, \quad n = 2,3,\ldots. \end{aligned}$$

So, the canonical form of Γ is

$$A(\Gamma) = J^{+} \bigoplus J^{-}, \quad J^{\pm} = J(\{\pm\chi_1, \pm\chi_2, \ldots\}, \{1\}). \tag{4.2}$$

There is no hope computing the spectra of the above matrices $J^{\pm}$ explicitly. So we will focus here on two particular cases with periodic structure, and on the sparse case in the next section.

A periodic structure appears when Λ is an arithmetic progression. We consider two simplest examples of such kind.

Let $\Lambda = \{1,3,5,\ldots\}$ (the infinite linear hexagon L_∞). The terms in (4.2) are

$$J^{\pm} = J(\{\pm 1, 0, \pm 1, 0, \ldots\}, \{1\}),$$

so we have two 2-periodic Jacobi matrices.

A detailed computation is carried out for J^{+}. The discriminant (2.32) and the polynomial γ_2 in (2.33) are

$$\mathscr{D}^{+}(\lambda) = \lambda(\lambda-1) - 2, \qquad \gamma_2(\lambda) = \lambda(\lambda-1).$$

So,

$$\mathscr{D}^{+}(\lambda) = 2 \sim \lambda_{1,2}^{+} = \frac{1 \pm \sqrt{17}}{2}, \quad \mathscr{D}^{+}(\lambda) = -2 \sim \lambda_{3,4}^{+} = 0, 1.$$

The essential spectrum is

$$\sigma_{ess}(J^{+}) = \Big[\frac{1-\sqrt{17}}{2}, 0\Big] \bigcup \Big[1, \frac{1+\sqrt{17}}{2}\Big]. \tag{4.3}$$

The only root of the polynomial p_2 is 1, which is at the edge of the gap. So, there are no eigenvalues, and $\sigma(J^+) = \sigma_{ess}(J^+)$.

The computation for J^- is identical. Now the essential spectrum is

$$\sigma_{ess}(J^-) = \Big[\frac{-1-\sqrt{17}}{2}, -1\Big] \bigcup \Big[0, \frac{-1+\sqrt{17}}{2}\Big], \tag{4.4}$$

and again, there are no eigenvalues of J^-. Finally

$$\sigma(\Gamma) = \sigma_{ess}(\Gamma) = \Big[\frac{-1-\sqrt{17}}{2}, \frac{1+\sqrt{17}}{2}\Big]. \tag{4.5}$$

Let now $\Lambda = \{1,4,7,10,\dots\}$ (the infinite linear octagon), so the Jacobi components are

$$J^{\pm} = J(\{\pm 1,0,0,\pm 1,0,0,\dots\},\{1\}),$$

and we have two 3-periodic Jacobi matrices. Again, a direct computation provides

$$\mathscr{D}^+(\lambda) = \lambda^3 - \lambda^2 - 3\lambda + 1, \quad \gamma_3(\lambda) = (\lambda-1)^2(\lambda+1).$$

The endpoints of the gaps are

$$\mathscr{D}^+ = 2 \sim \lambda_1^+ = -1, \ \lambda_{2,3}^+ = 1 \pm \sqrt{2};$$
$$\mathscr{D}^+ = -2 \sim \lambda_4^+ = 1, \ \lambda_{5,6}^+ = \pm\sqrt{3},$$

so the essential spectrum is

$$\sigma_{ess}(J^+) = [-\sqrt{3},-1] \cup [1-\sqrt{2},1] \cup [\sqrt{3},1+\sqrt{2}].$$

Now the both roots of the polynomial $p_3(\lambda) = \lambda^2 - \lambda - 1$ lie inside the gaps

$$\lambda_7^+ = \frac{1+\sqrt{5}}{2} \in (1,\sqrt{3}), \quad \lambda_8^+ = \frac{1-\sqrt{5}}{2} \in (-1, 1-\sqrt{2}),$$

and $\gamma(\lambda_{7,8}^+) > 0$. According to the "rule of signs" (2.35), the point λ_7^+ is the eigenvalue of J^+, and λ_8^+ is not.

The computation for J^- is identical. We have

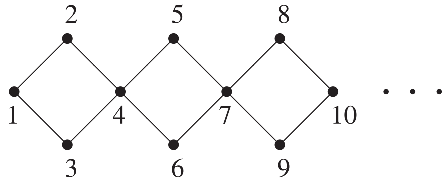
$$\sigma_{ess}(J^-) = [-1-\sqrt{2},-\sqrt{3}] \cup [-1,-1+\sqrt{2}] \cup [1,\sqrt{3}],$$

and the only eigenvalue is $(-1-\sqrt{5})/2$. Finally,

$$\sigma(\Gamma) = [-1-\sqrt{2}, 1+\sqrt{2}] \bigcup \Big\{\pm\frac{1+\sqrt{5}}{2}\Big\}, \tag{4.6}$$

and the spectrum has multiplicity 2 on $[1-\sqrt{2},\sqrt{2}-1]$.

EXAMPLE 4.3. "Squares and cubes".
Consider the simplest chain of squares



We define the canonical basis as

$$h_{3k-2} = e_{3k-2}, \quad h_{3k-1} = \frac{e_{3k-1} + e_{3k}}{\sqrt{2}}, \quad h_{3k} = \frac{e_{3k-1} - e_{3k}}{\sqrt{2}}, \quad k = 1, 2, \dots.$$

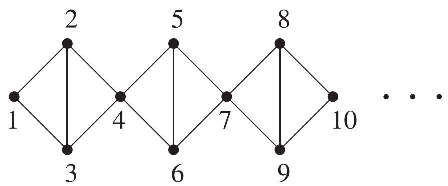
The Hilbert space is decomposed as

$$\ell^2 = \mathscr{H}_0 \oplus \mathscr{H}_1, \qquad \mathscr{H}_0 := \text{span}\{h_{3k}\}_{k \geqslant 1}, \quad \mathscr{H}_1 := \text{span}\{h_{3k-2}, h_{3k-1}\}_{k \geqslant 1},$$

both $\mathscr{H}_0$, $\mathscr{H}_1$ are $A(\Gamma)$ -invariant, and

$$A(\Gamma) = \mathbb{O}_\infty \bigoplus \sqrt{2} J_0.$$

So the spectrum is $\sigma(\Gamma) = \{0_\infty\} \cup [-2\sqrt{2}, 2\sqrt{2}]$.
A similar graph below is more delicate.



With the same canonical basis and decomposition of ℓ^2 we have now

$$A(\Gamma) = -\mathbb{I}_\infty \bigoplus J(\{0, 1, 0, 1, \dots\}, \{\sqrt{2}\}). \tag{4.7}$$

The second term is a 2-periodic Jacobi matrix. Its discriminant $\mathscr{D}$ in (2.32) and the polynomial γ_2 in (2.33) are

$$\mathscr{D}(\lambda) = p_3(\lambda) - p_1^{(1)}(\lambda) = \frac{\lambda(\lambda-1)}{2} - 2, \quad \gamma_2(\lambda) = p_3(\lambda) + p_1^{(1)}(\lambda) = \frac{\lambda(\lambda-1)}{2}.$$

The essential spectrum is a union of two intervals (with the eigenvalue -1 of infinite multiplicity on one of them)

$$\sigma_{ess}(\Gamma) = \Big[\frac{1-\sqrt{33}}{2}, 0\Big] \bigcup \Big[1, \frac{1+\sqrt{33}}{2}\Big]. \tag{4.8}$$

The polynomial $p_2=\lambda/\sqrt{2}$ has the only root $\lambda_0=0$ which lies at the edge of the gap, so there are no eigenvalues. The whole spectrum now is (4.8) along with the eigenvalue -1 of infinite multiplicity.

We might equally well have considered cubes in $\mathbb{R}^3$ in place of squares, with one common vertex for each pair of adjacent cubes.

Define an orthonormal sequence $\{h_n\}_{n\geqslant 1}$ by the recipe

$$\begin{aligned} h_{3k+1} &= e_{3k+1}, \\ h_{3k+2} &= \frac{1}{\sqrt{3}}\big(e_{7k+2}+e_{7k+3}+e_{7k+4}\big), \\ h_{3k+3} &= \frac{1}{\sqrt{3}}\big(e_{7k+5}+e_{7k+6}+e_{7k+7}\big), \quad k=0,1,\dots. \end{aligned}$$

It is easy to see that the subspace $\mathscr{H}:=\operatorname{span}\{h_k\}_{k\geqslant 0}$ is $A(\Gamma)$ -invariant, and the matrix of $A(\Gamma)\,|\mathscr{H}$ is a 3-periodic Jacobi matrix

$$A(\Gamma)\,|\mathscr{H}\sim J(\{0\},\{\sqrt{3},2,\sqrt{3},\dots\}).$$

To complete the above system to an orthonormal basis, we put

$$\begin{aligned} h^{(1)}_{3k+2} &= \frac{1}{\sqrt{6}}\big(e_{7k+2}+e_{7k+3}-2e_{7k+4}\big), \quad h^{(2)}_{3k+2}=\frac{1}{\sqrt{2}}\big(e_{7k+2}-e_{7k+3}\big), \\ h^{(1)}_{3k+3} &= \frac{1}{\sqrt{6}}\big(-e_{7k+5}+2e_{7k+6}-e_{7k+7}\big), \quad h^{(2)}_{3k+2}=\frac{1}{\sqrt{2}}\big(e_{7k+5}-e_{7k+7}\big), \end{aligned}$$

$k=0,1,\dots$. The system thus obtained is complete, and

$$A(\Gamma)\,h^{(q)}_{3k+2}=h^{(q)}_{3k+3}, \quad A(\Gamma)\,h^{(q)}_{3k+3}=h^{(q)}_{3k+2}, \quad q=1,2.$$

So, $\mathscr{H}^{(q)}_k:=\operatorname{span}\{h^{(q)}_{3k+2},h^{(q)}_{3k+3}\}$ are invariant subspaces of dimension 2, and

$$A(\Gamma)\,|\mathscr{H}^{(q)}_k\sim\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Finally,

$$A(\Gamma)\sim J(\{0\},\{\sqrt{3},2,\sqrt{3},\dots\})\bigoplus\Big(\bigoplus_{n=1}^{\infty}\Big)\begin{bmatrix}0&1\\1&0\end{bmatrix}\Big). \tag{4.9}$$

For the Jacobi component we find

$$\begin{aligned} p_4(\lambda) &= \frac{\lambda^3-7\lambda}{6}, \quad p^{(1)}_2(\lambda)=\frac{\lambda}{2}, \\ \mathscr{D}_4(\lambda) &= \frac{\lambda^3-10\lambda}{6}, \quad \gamma_4(\lambda)=\frac{\lambda^3-4\lambda}{6}, \end{aligned}$$

and

$$\sigma_{ess}(\Gamma)=[-1-\sqrt{7},-2]\cup[1-\sqrt{7},\sqrt{7}-1]\cup[2,1+\sqrt{7}]. \tag{4.10}$$

The denominator in (2.33)

$$p_3(\lambda) = \frac{\lambda^2 - 3}{2\sqrt{3}}, \qquad \lambda_{1,2} = \pm\sqrt{3}$$

lie inside the gaps, and the rule of signs (2.35) says that there are no eigenvalues for Γ. So, $\sigma(\Gamma) = \sigma_{ess}(\Gamma)$ (4.10) along with the eigenvalues ± 1 of infinite multiplicity.

EXAMPLE 4.4. "A chain of cycles".

Let $\{n_J\}_{j\geqslant 1}$ be a sequence of positive integers, $n_j \geqslant 2$. Consider a sequence of cycles $\{\mathbb{C}_{2n_j}\}_{j\geqslant 1}$ of even orders $|\mathbb{C}_{2n_j}| = 2n_j$. We connect them in a chain in such a way that two adjacent cycles have one vertex in common.

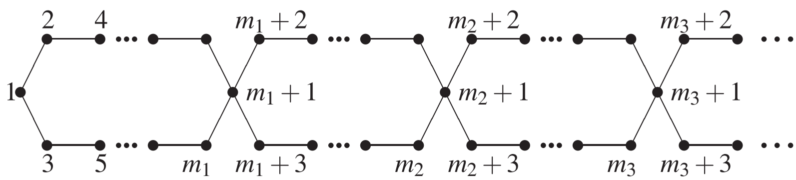
Put

$$m_j := 2(n_1 + \ldots + n_j) - j, \qquad m_0 := 0.$$

The vertices of k-th cycle $\mathbb{C}_{2n_k}$ are enumerated with the numbers

$$\{m_{k-1}+1, \ldots, m_k+1\},$$

so $\{m_{j-1}+1\}_{j\geqslant 1}$ are the vertices of valency 4.



The canonical basis $\{h_n\}_{n\geqslant 1}$ is defined by $h_{m_k+1} = e_{m_k+1}$ for $k = 0, 1, \ldots,$

$$h_{m_k+2j} = \frac{1}{\sqrt{2}}\left(e_{m_k+2j} + e_{m_k+2j+1}\right),$$

$$h_{m_k+2j+1} = \frac{1}{\sqrt{2}}\left(e_{m_k+2j} - e_{m_k+2j+1}\right), \quad j - 1, 2, \ldots, n_{k+1} - 1.$$

Next, let

$$\mathscr{H}^+ = \operatorname{span}\{h_{m_k+1}; h_{m_k+2j},\ j = 1, \ldots, n_{k+1} - 1\}_{k=0}^{\infty},$$

$$\mathscr{H}_k^- = \operatorname{span}\{h_{m_k+2j+1},\ j = 1, \ldots, n_{k+1} - 1\}, \qquad \mathscr{H}^- = \bigoplus_{k=1}^{\infty} \mathscr{H}_{k-1}^-.$$

We come to decomposition $\ell^2 = \mathscr{H}^+ \oplus \mathscr{H}^-$ on two $A(\Gamma)$-invariant subspaces.

The action of the adjacency operator $A(\Gamma)$ can be easily traced. First, put $r_j := n_1 + \ldots + n_j$, $j \in \mathbb{N}$. The restriction of $A(\Gamma)$ on $\mathscr{H}^+$ is of the form

$$A(\Gamma)\,|\mathscr{H}^+ = J(\{0\}, \{a_j\}_{j\geqslant 1}), \quad a_i = \begin{cases} \sqrt{2}, & i = 1, r_1, r_1+1, r_2, r_2+1, \ldots; \\ 1, & \text{otewise.} \end{cases}$$

Secondly, the restriction $A(\Gamma)\,|\mathscr{H}_{k-1}^{-} = J_{0,n_k-1}$, the discrete Laplacian of order n_k-1. So,

$$A(\Gamma) \simeq J(\{0\},\{a_n\}) \bigoplus \Big(\bigoplus_{k=1}^{\infty} J_{0,n_k-1} \Big). \tag{4.11}$$

There is no hope computing the spectrum of the above Jacobi matrix J explicitly. So we will focus here on some particular cases with periodic structure, and on the sparse case in the next section.

A periodic structure appears when, e.g., all cycles are identical: $n_j = N$, $j = 1,2,\ldots$. For $N=2$ we have a simple chain of squares, so assume that $N \geqslant 3$. The above Jacobi matrix is N-periodic with

$$\{a_1,a_2,\ldots,a_{N-1},a_N\} = \{\sqrt{2},\underbrace{1,1\ldots,1}_{N-2},\sqrt{2}\}.$$

The argument in Example 2.3 (see equations (2.23) and (2.24)) provides an explicit expression for the discriminant

$$\mathscr{D}_N(\lambda) = T_N\Big(\frac{\lambda}{2}\Big) - U_{N-2}\Big(\frac{\lambda}{2}\Big).$$

Since we are unable to solve the equations $\mathscr{D}_N = \pm 2$ for an arbitrary N, we restrict ourselves with the cases $N = 3,4$.

Let $N=4$ (a chain of octagons). Then J is 4-periodic Jacobi matrix, and the discriminant and γ_4 in (2.33) are

$$\mathscr{D}_4(\lambda) = \frac{\lambda^4 - 6\lambda^2 + 4}{2}, \qquad \gamma_4(\lambda) = \frac{\lambda^4 - 2\lambda^2}{2}.$$

It is easy to solve $\mathscr{D}_4 = \pm 2$ and find the spectral bands and the essential spectrum

$$\sigma_{ess}(J) = [-\sqrt{6},-2] \cup [-\sqrt{2},0] \cup [0,\sqrt{2}] \cup [2,\sqrt{6}].$$

Note that there is a closed gap at the origin.

The denominator p_4 in (2.33) is now

$$p_4(\lambda) = \frac{\lambda(\lambda^2-3)}{\sqrt{2}},$$

with the roots $\lambda_0 = 0$, $\lambda_{1,2} = \pm\sqrt{3}$. The last two lie inside the gaps, and the rule of signs (2.35) shows that both of them are the eigenvalues of J. So,

$$\sigma(J) = [-\sqrt{6},-2] \cup [-\sqrt{2},\sqrt{2}] \cup [2,\sqrt{6}] \cup \{\pm\sqrt{3}\}. \tag{4.12}$$

There is another part of the spectrum which comes from Laplacians of order $n_k - 1 = 3$ in (4.11). By (1.10),

$$\sigma(J_{0,3}) = \{0, \pm\sqrt{2}\}, \tag{4.13}$$

each eigenvalue has infinite multiplicity.

The same computation can be carried out for $N=3$ (a chain of hexagons). Now

$$\begin{aligned}\sigma(\Gamma) &= [-\omega_+,-\omega_-]\cup[-1,1]\cup[\omega_-,\omega_+]\cup\{\pm\sqrt{2}\},\\ \omega_\pm &:= \frac{\pm 1+\sqrt{17}}{2},\end{aligned} \tag{4.14}$$

along with two eigenvalues ± 1 of infinite multiplicity.

4.2. Sparse ladders and chains of cycles

One can gather some information about the spectrum of the graphs in Examples 4.3 and 4.5 in the situation opposite in a sense to one considered above. We assume that the graph in question is sparse.

EXAMPLE 4.2. (cont.)
Simon's ladder Γ is said to be sparse if the set Λ is sparse in the sense of Example 2.7

$$\lim_{i\to\infty}(\lambda_{i+1}-\lambda_i)=+\infty.$$

The adjacency operator $A(\Gamma)$ (4.2) is now the orthogonal sum of two sparse Jacobi matrices $J^\pm$ (4.2). The sets of all right limits $RL(J^\pm)$ is available, see (2.42), so the Last–Simon theorem applies for the description of their essential spectrum. The nonzero right limits for J^+ have the same spectrum. Take $k=0$ and compute the spectrum of J^+_{right} (2.42) by using the perturbation determinant (2.2) with

$$T_0=J_0(\mathbb{Z})=J(\{0\},\{1\})_{n\in\mathbb{Z}},\quad T=J^+_{right},\quad T-T_0=(\cdot,e_0)e_0.$$

Note that the resolvent matrix is known [16]

$$(T_0-\lambda)^{-1}=\|r_{ij}(z)\|_{i,j\in\mathbb{Z}},\quad r_{ij}(z)=\frac{z^{|i-j|}}{z-z^{-1}},\quad \lambda=z+\frac{1}{z}.$$

So,

$$L\Big(z+\frac{1}{z},J^+_{right}\Big)=1+r_{00}(z)=\frac{z^2+z-1}{z^2-1},\quad z_\pm=\frac{-1\pm\sqrt{5}}{2}$$

are the roots of the perturbation determinant, and one of them, z_+, is in $(0,1)$. The spectrum

$$\sigma\big(J^+_{right}\big)=[-2,2]\cup\{\sqrt{5}\}.$$

Similarly,

$$\sigma\big(J^-_{right}\big)=[-2,2]\cup\{-\sqrt{5}\},$$

and so

$$\sigma_{ess}(\Gamma)=[-2,2]\cup\{\pm\sqrt{5}\}. \tag{4.15}$$

We observe here two isolated points $\pm\sqrt{5}$ of the essential spectrum. As a Jacobi matrix can not have multiple eigenvalues, those two are accumulation points for the eigenvalues. The endpoints ± 2 can also attract some eigenvalues of Γ.

The structure of the spectrum on $[-2,2]$ is subtle. Note first, the all the Jacobi matrices in our consideration are finite valued, that is, the diagonals $\{b_n\}$, $\{a_n\}$ take a finite number of values (3 in the latter example). A result of Remling [28], [33, Theorem 7.4.6] states that finite valued Jacobi matrices with nontrivial absolutely continuous spectrum are eventually periodic

$$a_{n+N} = a_n, \quad b_{n+N} = b_n, \quad n \geqslant n_0.$$

Clearly, finite valued and sparse Jacobi matrices can not be eventually periodic, so their spectra are *purely singular*. Hence, the spectra of the sparse Simon's ladders on $[-2,2]$ are purely singular.

Moreover, assume that a Simon's ladder is strongly sparse:

$$\limsup_{j\to\infty} \frac{\log(\lambda_{j+1}-\lambda_j)}{j} = +\infty. \tag{4.16}$$

Now we can bound the norms of the transfer matrices $\mathscr{T}_n^{\pm}$ (2.26), so the Simon–Stolz theorem applies. Indeed, the main diagonal of $J^{\pm}$ looks

$$\{\pm 1, \underbrace{0,0\ldots,0}_{\lambda_2-\lambda_1-1}, \pm 1, \underbrace{0,0\ldots,0}_{\lambda_3-\lambda_3-1}, \pm 1, \ldots\}.$$

Then

$$\begin{aligned} V_{\pm}(\lambda) &:= A^{\pm}(\lambda; a_n, b_n) = \begin{bmatrix} \lambda \mp 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad n \in \Lambda, \\ V(\lambda) &:= A^{\pm}(\lambda; a_n, b_n) = \begin{bmatrix} \lambda & -1 \\ 1 & 0 \end{bmatrix}, \quad n \notin \Lambda, \end{aligned}$$

and so for $n = \lambda_q$

$$\begin{aligned} \mathscr{T}_n^{\pm}(\lambda) &= A^{\pm}(\lambda; a_n, b_n) A^{\pm}(\lambda; a_{n-1}, b_{n-1}) \ldots A^{\pm}(\lambda; a_1, b_1) \\ &= V_{\pm}(\lambda) V^{\lambda_q-\lambda_{q-1}-1}(\lambda) V_{\pm}(\lambda) V^{\lambda_{q-1}-\lambda_{q-2}-1}(\lambda) \ldots V_{\pm}(\lambda) V^{\lambda_2-\lambda_1-1}(\lambda) V_{\pm}(\lambda). \end{aligned}$$

The matrix V is diagonalizable. Indeed, for $-2 < \lambda < 2$

$$|V(\lambda) - zI_2| = z^2 - \lambda z + 1 \;\Rightarrow\; z_{\pm}(\lambda) = \frac{\lambda \pm i\sqrt{4-\lambda^2}}{2} \in \mathbb{T},$$

the eigenvalues are unimodular. So, if

$$U(\lambda) = \begin{bmatrix} z_+(\lambda) & z_-(\lambda) \\ 1 & 1 \end{bmatrix}, \quad U^{-1}(\lambda) = \frac{1}{z_+(\lambda) - z_-(\lambda)} \begin{bmatrix} 1 & -z_-(\lambda) \\ -1 & z_+(\lambda) \end{bmatrix},$$

we have

$$V^k(\lambda) = U(\lambda) \begin{bmatrix} z_+^k(\lambda) & 0 \\ 0 & z_-^k(\lambda) \end{bmatrix} U^{-1}(\lambda), \quad k \in \mathbb{N}.$$

Hence, for each $k \in \mathbb{N}$ and $\lambda \in (-2,2)$ the bounds

$$\|V^k(\lambda)\| \leqslant \|U(\lambda)\| \, \|U^{-1}(\lambda)\| \leqslant \frac{\|U(\lambda)\|^2}{\sqrt{4-\lambda^2}} = C_1(\lambda)$$

hold. The latter gives the bound for the norm of the transfer matrices

$$\begin{aligned}
&\|\mathscr{T}^{\pm}_{\lambda_q}(\lambda)\| \leqslant C_1^{q-1}(\lambda)\, \|V_{\pm}(\lambda)\|^q, \\
&\|\mathscr{T}^{\pm}_{j}(\lambda)\| \leqslant C_1^{q}(\lambda)\, \|V_{\pm}(\lambda)\|^q, \quad j = \lambda_q + 1, \dots, \lambda_{q+1} - 1,
\end{aligned}$$

so

$$\begin{aligned}
\sum_{j=\lambda_q+1}^{\lambda_{q+1}-1} \|\mathscr{T}^{\pm}_{j}(\lambda)\|^{-2} &\geqslant C_2^{-q}(\lambda)(\lambda_{q+1} - \lambda_q - 1), \\
\sum_{q=1}^{\infty} \sum_{j=\lambda_q+1}^{\lambda_{q+1}-1} \|\mathscr{T}^{\pm}_{j}(\lambda)\|^{-2} &\geqslant \sum_{q=1}^{\infty} C_2^{-q}(\lambda)(\lambda_{q+1} - \lambda_q - 1).
\end{aligned}$$

The series on the right side diverges in view of (4.16). By the Simon–Stolz theorem, the matrices $J^{\pm}$ have no eigenvalues in $(-2,2)$, which means that the spectrum of the strongly sparse Simon's ladder is *purely singular continuous* on $(-2,2)$.

EXAMPLE 4.4. (cont.)

A chain of cycles Γ is said to be sparse if $\{n_k\}_{k\geqslant 1}$ is sparse. The Jacobi component of the adjacency operator $A(\Gamma)$ (4.11) takes the form of the Jacobi matrix J in Example 2.7. The sets of all right limits $RL(J)$ is available, see (2.43), so the Last–Simon theorem applies for the description of its essential spectrum. The nonzero right limits for J have the same spectrum. Take $k=0$ and compute the spectrum of J_{right} (2.43) by using the perturbation determinant (2.2) with $T_0 = J_0(\mathbb{Z})$, $T = J_{right}$, and

$$T - T_0 = (\cdot, g_1)h_1 + (\cdot, g_2)h_2, \quad g_1 = h_2 = e_1, \quad g_2 = h_1 = (\sqrt{2}-1)(e_0+e_2).$$

We have with $\kappa = \sqrt{2} - 1$

$$\begin{aligned}
L\Big(z + \frac{1}{z}, J_{right}\Big) &= \begin{vmatrix} 1 + \kappa\big(r_{01}(z) + r_{12}(z)\big) & \kappa^2\big(r_{00}(z) + r_{22}(z) + r_{02}(z) + r_{20}(z)\big) \\ r_{11}(z) & 1 + \kappa\big(r_{01}(z) + r_{21}(z)\big) \end{vmatrix} \\
&= \Big(1 + 2\kappa \frac{z^2}{z^2-1}\Big)^2 - 2\kappa^2 z^2 \frac{z^2+1}{(z^2-1)^2} = \frac{3z^2-1}{z^2-1}.
\end{aligned}$$

Now both roots

$$z_{\pm} = \pm \frac{1}{\sqrt{3}} \in (-1,1),$$

so

$$\sigma_{ess}(\Gamma) = [-2,2] \cup \Big\{\pm \frac{4}{\sqrt{3}}\Big\}. \tag{4.17}$$

Again, two isolated points $\pm\frac{4}{\sqrt{3}}$ of the essential spectrum are accumulation points for the eigenvalues. The endpoints ± 2 can also attract some eigenvalues of Γ.

Following the line of reasoning in the above example, we see that the spectrum of the sparse chain of cycles is purely singular. Moreover, for strongly sparse chains of cycles

$$\limsup_{j\to\infty} \frac{\log n_j}{j} = +\infty$$

the spectrum on $(-2,2)$ is a combination of purely point one (from the finite component) lying on a purely singular continuous spectrum of the Jacobi component.

4.3. Toeplitz graphs and a comb graph

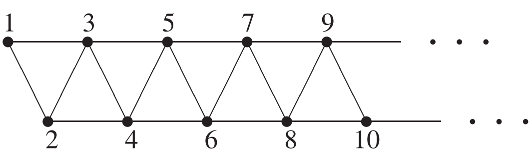
EXAMPLE 4.5. "The Toeplitz graphs".

We say that an infinite graph Γ is a Toeplitz graph if for some enumeration of the vertex set with positive integers the adjacency matrix $A(\Gamma)$ is Toeplitz, i.e.,

$$A(\Gamma) = [\alpha_{i-k}]_{i,k\geqslant 1}, \qquad \alpha_j = a_{-j} = 0,1.$$

The simplest Toeplitz graph is the infinite path $\mathbb{P}_\infty$: $A(\mathbb{P}_\infty) = J_0$, the discrete Laplacian.

Here is another, a bit more sophisticated, Toeplitz graph.



Clearly, its adjacency matrix is

$$A(\Gamma) = [\alpha_{i-k}]_{i,k\geqslant 1}, \qquad \alpha_j = \alpha_{-j} = \begin{cases} 1, & j = 1,2; \\ 0, & \text{otherwise.} \end{cases}$$

To find the spectrum of this matrix, we proceed in a standard way. Write the symbol

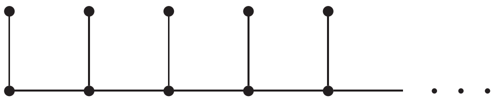
$$\varphi(t) = t^{-2} + t^{-1} + t + t^2 = p(\cos\theta), \quad p(x) = 4x^2 + 2x - 2.$$

The Toeplitz operator $A(\Gamma)$ is selfadjoint, so, by [2, Theorem 1.27], the spectrum is

$$\sigma(\Gamma) = p([-1,1]) = \left[-\frac{9}{4}, 4\right].$$

PROBLEM 1. Describe all Toeplitz graphs.

EXAMPLE 4.6. "A complete comb graph".



The Hilbert space in question is $\ell^2(\mathscr{V}(\Gamma)) = \ell^2 \oplus \ell^2$, and the adjacency operator in the block form looks

$$A(\Gamma) = \begin{bmatrix} J_0 & I \\ I & 0 \end{bmatrix}.$$

Let us compute the resolvent $R(A(\Gamma),z) = (A(\Gamma) - z)^{-1}$. Clearly,

$$\begin{bmatrix} J_0 - z & I \\ I & -zI \end{bmatrix} \begin{bmatrix} zI & I \\ I & z - J_0 \end{bmatrix} = \begin{bmatrix} z(J_0 - \zeta(z))I & 0 \\ 0 & z(J_0 - \zeta(z))I \end{bmatrix}, \quad \zeta(z) = z - \frac{1}{z},$$

and so

$$R(A(\Gamma),z) = \frac{1}{z} R(J_0, \zeta(z)) \begin{bmatrix} zI & I \\ I & z - J_0 \end{bmatrix}, \quad z \neq 0, \quad \zeta(z) \notin [-2,2].$$

Since for $z = 0$ we have

$$A(\Gamma)^{-1} = \begin{bmatrix} 0 & I \\ I & -J_0 \end{bmatrix},$$

the spectrum is

$$\sigma(\Gamma) = \zeta^{(-1)}[-2,2] = [-\sqrt{2} - 1, -\sqrt{2} + 1] \cup [\sqrt{2} - 1, \sqrt{2} + 1].$$

PROBLEM 2. As in Example 4.2, one can consider the complete comb graph with some teeth missing. Analyze the spectral properties of such graphs.

For the description of the spectra of finite comb graphs with tails see [14].

5. Spectra of graphs via Schur complement

Our second method relies on the block representation (1.5) for the adjacency matrix $A(\Gamma)$ of the coupling $\Gamma = \Gamma_1 + \Gamma_2$ by means of the bridge of weight d. The simplest case is $|\Gamma_1| = n$, and $d = 1$

$$A(\Gamma) = \begin{bmatrix} A(\Gamma_1) & E_1 \\ E_1^* & A(\Gamma_2) \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \end{bmatrix}. \tag{5.1}$$

Proposition 1.7 applies with $A_{22} = A(\Gamma_2)$, $A_{11} = A(\Gamma_1)$, so for $\lambda \notin \sigma(A(\Gamma_2))$ we have $\lambda \in \sigma(A(\Gamma))$ if and only if the matrix (finite) $C_{11}(\lambda)$ (1.28) is degenerate.

Denote by $P(\cdot,F)$ the characteristic polynomial of a finite graph F

$$P(\lambda,F)=\det(\lambda-A(F)).$$

A key ingredient in the argument is the so-called spectral Green's function

$$G_1(\lambda,\Gamma_2)=\big(\lambda-A(\Gamma_2)^{-1}\big)_{11}, \qquad \lambda\notin\sigma(A(\Gamma_2)).$$

Given a graph F and a set of vertices $V\subset\mathscr{V}(F)$, under $F\backslash V$ we mean the graph induced by the vertices $\mathscr{V}(F)\backslash V$.

We come to the following result, see [13, Theorem 1].

PROPOSITION 5.1. *Let $\Gamma=\Gamma_1+\Gamma_2$ be the coupling of a finite graph Γ_1 and an arbitrary graph Γ_2 by means of the bridge $\{n,n+1\}$, and let $\lambda\notin\sigma(\Gamma_2)$. The point λ belongs to the spectrum of Γ if and only if it solves the equation*

$$P(\lambda,\Gamma_1)-G_1(\lambda,\Gamma_2)P(\lambda,\Gamma_1\backslash\{n\})=0. \tag{5.2}$$

The result in Proposition 5.1 is effective as long as both Green's function and characteristic polynomials are available. In the case of infinite tail attached to a finite graph, we have $\Gamma_2=\mathbb{P}_\infty$, and, see (2.3) (and note the negative sign),

$$G_1(\lambda,\mathbb{P}_\infty)=-r_{11}(z)=z, \quad \lambda=z+\frac{1}{z}.$$

The spectrum $\sigma(\Gamma)$ in this case is

$$\sigma(\Gamma)=[-2,2]\cup\sigma_d(\Gamma),$$

and the discrete spectrum agrees with the roots of the basic equation (5.2)

$$\lambda\in\sigma_d(\Gamma)\Leftrightarrow P(\lambda,\Gamma_1)-xP(\lambda,\Gamma_1\backslash\{n\})=0,\ \ \lambda=x+\frac{1}{x},\ \ x\in(-1,1). \tag{5.3}$$

Let

$$\sigma(\Gamma_1):=\{\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_n\}, \quad \sigma(\Gamma_1\backslash\{n\}):=\{\mu_1\geqslant\mu_2\geqslant\ldots\geqslant\mu_{n-1}\}$$

be the spectra of Γ_1 and $\Gamma_1\backslash\{n\}$, respectively. By the Cauchy interlacing theorem,

$$\lambda_1\geqslant\mu_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_{n-1}\geqslant\mu_{n-1}\geqslant\lambda_n, \tag{5.4}$$

(as a matter of fact, the Perron–Frobenius theorem claims that $\lambda_1>\mu_1$). A quick analysis of the main equation (5.3), in view of (5.4), shows that each multiple eigenvalue of Γ_1 off $[-2,2]$ solves (5.3), and so belongs to $\sigma_d(\Gamma)$. One can rewrite (5.3) for $\lambda>2$ as

$$\frac{1}{x}=\frac{\lambda+\sqrt{\lambda^2-4}}{2}=\frac{P(\lambda,\Gamma_1\backslash\{n\})}{P(\lambda,\Gamma_1)}=\big(\lambda-A(\Gamma_1))^{-1}\big)_{nn}=G_n(\lambda,\Gamma_1), \tag{5.5}$$

the n-th spectral Green's function of Γ_1. The function on the right side is monotone decreasing on each interval of regularity. For instance, if the interlacing in (5.4) is strict, i.e.,

$$\lambda_1 > \mu_1 > \lambda_2 > \ldots > \lambda_{n-1} > \mu_{n-1} > \lambda_n,$$

G_n is monotone decreasing on each interval $(-\infty, \lambda_n)$, $(\lambda_j, \lambda_{j-1})$, and (λ_1, ∞). Assume that

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \ldots > \mu_{k-1} > \lambda_k > 2 > \mu_k.$$

Then there is exactly one root of (5.5) on each interval (λ_1, ∞), $(\lambda_{j+1}, \lambda_j)$, $j = 1, \ldots, k-1$, and there is no such root on $(2, \lambda_k)$. Next, if

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \ldots > \mu_{k-1} > 2 > \lambda_k > \mu_k,$$

then there is exactly one root of (5.5) on each interval (λ_1, ∞), $(\lambda_{j+1}, \lambda_j)$, $j = 1, \ldots, k-2$. The existence of the root on $(\lambda_k, \lambda_{k-1})$ depends on whether $G_n(2, \Gamma_1) > 1$ (there is a root) or $G_n(2, \Gamma_1) \leqslant 1$ (there are no roots).

The situation for $\lambda < -2$ can be analyzed in exactly the same way.

To compute the characteristic polynomials in (5.3), the following result of Schwenk [30], [7, Problem 2.7.9], proves helpful.

THEOREM. (Schwenk). *For a given finite graph F and $v \in \mathscr{V}(F)$, let $\mathscr{C}(v)$ denote the set of all simple cycles Z containing v. Then*

$$P(\lambda, F) = \lambda P(\lambda, F \backslash v) - \sum_{v' \sim v} P(\lambda, F \backslash \{v', v\}) - 2 \sum_{Z \in \mathscr{C}(v)} P(\lambda, F \backslash Z).$$

EXAMPLE 5.2. "A flower with n petals" [13].

In this example Γ_1 is composed of $n \geqslant 2$ cycles $\{\mathbb{C}_j\}_{j=1}^n$, glued together at one common vertex (root) $\mathscr{O}$. Put $\Gamma = \Gamma_1 + \mathbb{P}_\infty$ with the infinite path attached to the root $\mathscr{O}$. Assume that the cycle $\mathbb{C}_j$ contains $k_j + 1 \geqslant 3$ vertices.

For the standard Chebyshev polynomial of the second kind U_k (2.22) we denote

$$Q(\lambda, k) := U_k\left(\frac{\lambda}{2}\right) = P(\lambda, \mathbb{P}_k), \quad Q(\lambda) := \prod_{j=1}^n Q(\lambda, k_j) = P(\lambda, \Gamma_1 \backslash \mathscr{O}),$$

the characteristic polynomials of the finite path $\mathbb{P}_k$ and the graph $\Gamma_1 \backslash \mathscr{O}$, respectively.

The result of Schwenk applied to the flower graph gives

$$P(\lambda, \Gamma_1) = \lambda P(\lambda, \Gamma_1 \backslash \mathscr{O}) - \sum_{v' \sim \mathscr{O}} P(\lambda, \Gamma_1 \backslash \{v', \mathscr{O}\}) - 2 \sum_{Z \in \mathscr{C}(\mathscr{O})} P(\lambda, \Gamma_1 \backslash Z).$$

Now

$$P(\lambda, \Gamma_1 \backslash \mathscr{O}) = \prod_{j=1}^n P(\lambda, \mathbb{P}_{k_j}) = \prod_{j=1}^n Q(\lambda, k_j) = Q(\lambda),$$

$$P(\lambda, \Gamma_1 \backslash \{v', \mathscr{O}\}) = Q(\lambda, k_m - 1) \prod_{j \neq m} Q(\lambda, k_j) = Q(\lambda) \frac{Q(\lambda, k_m - 1)}{Q(\lambda, k_m)},$$

and so

$$\sum_{v' \sim \mathcal{O}} P(\lambda, \Gamma_1 \backslash \{v', \mathcal{O}\}) = 2Q(\lambda) \sum_{j=1}^{n} \frac{Q(\lambda, k_j - 1)}{Q(\lambda, k_j)}$$

(the factor 2 arises since each cycle enters twice). Next,

$$P(\lambda, \Gamma_1 \backslash Z_m) = \prod_{j \neq m} Q(\lambda, k_j) = \frac{Q(\lambda)}{Q(\lambda, k_m)},$$

so

$$\sum_{m=1}^{n} P(\lambda, \Gamma_1 \backslash Z_m) = Q(\lambda) \sum_{j=1}^{n} \frac{1}{Q(\lambda, k_j)},$$

and finally,

$$P(\lambda, \Gamma_1) = Q(\lambda) \Big\{ \lambda - 2 \sum_{j=1}^{n} \frac{Q(\lambda, k_j - 1) + 1}{Q(\lambda, k_j)} \Big\}. \tag{5.6}$$

Since $Q \neq 0$ off $[-2, 2]$, the main equation (5.3) looks

$$\lambda - 2 \sum_{j=1}^{n} \frac{Q(\lambda, k_j - 1) + 1}{Q(\lambda, k_j)} = x, \quad \lambda = x + \frac{1}{x}$$

or

$$2 \sum_{j=1}^{n} \frac{Q(\lambda, k_j - 1) + 1}{Q(\lambda, k_j)} = \frac{1}{x}, \quad x \in (-1, 1). \tag{5.7}$$

Let first $x = e^{-t}$, $t > 0$. We have

$$Q(\lambda, k) = Q\Big(x + \frac{1}{x}, k\Big) = U_k(\cosh) = \frac{\sinh(k+1)t}{\sinh kt},$$

so [1]

$$\varphi_1(t) := 2 \sum_{j=1}^{n} \frac{\sinh k_j t + \sinh t}{\sinh(k_j + 1)t} = e^t. \tag{5.8}$$

Note that the functions

$$f_s(t) := \frac{\sinh at}{\sinh bt}, \quad f_c(t) := \frac{\cosh at}{\cosh bt}, \quad 0 < a < b$$

are monotone decreasing for $t > 0$. The latter can be seen, e.g., from the product expansions

$$\sinh t = t \prod_{k=1}^{\infty} \Big(1 + \frac{t^2}{k^2 \pi^2}\Big), \quad \cosh t = \prod_{k=0}^{\infty} \Big(1 + \frac{4t^2}{(2k+1)^2 \pi^2}\Big),$$

[1]There is a misprint in [13, formula (1.9)]

(or by elementary calculus). Hence φ_1 in (5.8) is monotone decreasing, vanishing at infinity, and $\varphi(+0)=2n>1$, so (5.8) has a unique solution $t_+>0$ with

$$\lambda_+ = 2\cosh t_+ \in \sigma_d(\Gamma). \tag{5.9}$$

Similarly, for $x=-e^{-t}$, $t>0$, the main equation takes the form

$$\varphi_2(t) := 2\sum_{j=1}^{n} \frac{\sinh k_j t + (-1)^{k_j+1}\sinh t}{\sinh(k_j+1)t} = e^t. \tag{5.10}$$

As

$$\frac{\sinh kt \pm \sinh t}{\sinh(k+1)t} = \frac{\sinh \frac{k\pm 1}{2}t \, \cosh \frac{k\mp 1}{2}t}{\sinh \frac{k+1}{2}t \, \cosh \frac{k+1}{2}t},$$

the function φ_2 in (5.10) is monotone decreasing (for whatever parity of k_j), vanishing at infinity,

$$\varphi_2(+0) \geqslant 2\sum_{j=1}^{n} \frac{k_j-1}{k_j+1} > 1,$$

and (5.10) has a unique solution $t_->0$, so

$$\lambda_- = -2\cosh t_- \in \sigma_d(\Gamma). \tag{5.11}$$

Thereby, the discrete spectrum consists of two points

$$\sigma_d(\Gamma) = \{\pm 2\cosh t_\pm\}. \tag{5.12}$$

REMARK 5.3. The method of Section 3 does not seem to work properly in the general situation of Example 5.2. However, the spectral theory of Jacobi matrices applies in some particular instances. For example, for $n=2$, $k_1=k_2$ (the propeller graph with equal blades [23]), the spectrum is calculated in [12, Example 5.2]. For the case $n\geqslant 2$, $k_1=\ldots=k_n=2$, the finite rank Jacobi matrices are still effective, and the spectrum is

$$\sigma(\Gamma) = [-2,2] \cup \left\{x_- + \frac{1}{x_-},\ x_+ + \frac{1}{x_+}\right\} \cup \{(-1)_n,\ 1_{n-1}\},$$
$$x_\pm = \frac{-1\pm\sqrt{1+4(2n-1)}}{2(2n-1)}, \quad -1<x_-<0<x_+<1.$$

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Leonid Golinskii
B. Verkin Institute for
Low Temperature Physics and Engineering
47 Nauky ave., Kharkiv 61103, Ukraine
e-mail: golinskii@ilt.kharkov.ua