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NEW BEREZIN SYMBOL INEQUALITIES FOR OPERATORS ON THE REPRODUCING KERNEL HILBERT SPACE

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Abstract. We use Kittaneh and Manasrah inequality and Kian's functional calculus method to prove some new inequalities for Berezin symbols and Berezin numbers of operators. In particular, we prove that

$$\operatorname{ber}\left(f(A)^2\right) \leqslant \operatorname{ber}\left(\frac{f(A)^p}{p}+\frac{f(A)^q}{q}\right)$$

for all self-adjoint operators A on the reproducing kernel Hilbert space $\mathscr{H}(\Omega)$ with spectrum in $J \subset (-\infty,+\infty)$ and all continuous nonnegative functions f defined on J. We also prove new upper and lower bounds for Berezin numbers of reproducing kernel Hilbert space operators. Among our results, we prove that if $A : \mathscr{H}(\Omega) \to \mathscr{H}(\Omega)$ is a bounded pseudo-hypornormal operator on the reproducing kernel Hilbert space $\mathscr{H}(\Omega)$, then for all non-negative non-decreasing pseudo-operator convex function f on $[0,\infty)$, we have

$$f(\operatorname{ber}(A)) \leqslant \frac{1}{2}\left\|f\left(\frac{|A|}{1+\frac{\xi_{|A|}^2}{8}}\right)+f\left(\frac{|A^*|}{1+\frac{\xi_{|A|}^2}{8}}\right)\right\|_{\mathrm{Ber}},$$

where $\|.\|_{\mathrm{Ber}}$ denotes the Berezin norm of operator.

1. Introduction

Let Ω be a subset of a topological space X such that the boundary $\partial\Omega$ is nonempty. Let $\mathscr{H}$ be an infinite-dimensional Hilbert space of complex-valued functions defined on Ω. We say that $\mathscr{H}$ is a reproducing kernel Hilbert space if the following two conditions are satisfied:

(i) for any $\lambda \in \Omega$, the evaluation functionals $f \to f(\lambda)$ are continuous on $\mathscr{H}$;

(ii) for any $\lambda \in \Omega$, there exists $f_\lambda \in \mathscr{H}$ such that $f_\lambda(\lambda) \neq 0$ (or equivalently, there is no $\lambda_0 \in \Omega$ such that $f(\lambda_0) = 0$ for every $f \in \mathscr{H}$).

According to the classical Riesz representation theorem, the assumption (i) implies that, for every $\lambda \in \Omega$ there exists a unique function $k_\lambda \in \mathscr{H}$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{\mathscr{H}}, \; f \in \mathscr{H}.$$

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The function $k_\lambda(z)$ is called the reproducing kernel of $\mathscr{H}$ at point λ. It is well known that every reproducing kernel Hilbert space is separable. So, if $\{e_n(z)\}_{n\geqslant 0}$ is any orthonormal basis of $\mathscr{H}$, then (see Aronzajn [2])

$$k_\lambda(z) = \sum_{n=0}^{\infty} \overline{e_n(\lambda)} e_n(z).$$

By virtue of assumption (ii), we surely have $k_\lambda \neq 0$ and we denote by $\widehat{k}_\lambda$ the normalized reproducing kernel, that is $\widehat{k}_\lambda := \frac{k_\lambda}{\|k_\lambda\|_{\mathscr{H}}}$. Recall that if $\mathscr{B}(\mathscr{H})$ is the Banach algebra of all bounded linear operator on $\mathscr{H}$, then the Berezin symbol $\widetilde{A}$ of any operator $A \in \mathscr{B}(\mathscr{H})$ is the complex-valued function defined on the Ω by the formula (see, Nordgren and Rosenthal [24] and Berezin [6])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathscr{H}}, \ \lambda \in \Omega.$$

The Berezin set of operator A is defined by

$$\mathrm{Ber}(A) = \left\{ \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle : \lambda \in \Omega \right\} = \mathrm{Range}\left(\widetilde{A}\right),$$

and Berezin number $\mathrm{ber}(A)$ of operator A is the following number (see [16, 17])

$$\mathrm{ber}(A) := \sup_{\lambda\in\Omega} \left|\widetilde{A}(\lambda)\right|.$$

Since, $\left|\widetilde{A}(\lambda)\right| \leqslant \|A\|$, Berezin symbol is a bounded function on Ω. Also, it is trivial by Cauchy-Schwarz inequality that $\mathrm{ber}(A) \leqslant \|A\|$. The Berezin norm $\|A\|_{\mathrm{Ber}}$ of operator $A \in \mathscr{B}(\mathscr{H}(\Omega))$ is defined by (see, [13])

$$\|A\|_{\mathrm{Ber}} := \sup_{\lambda\in\Omega} \left\|A\widehat{k}_\lambda\right\|_{\mathscr{H}}.$$

It is easy to verify that $\|A\|_{\mathrm{Ber}}$ determines a new norm in the algebra $\mathscr{B}(\mathscr{H}(\Omega))$, since the set of reproducing kernels span $\mathscr{H}$ (see [2]). It is also known that the Berezin norm $\|A\|_{\mathrm{Ber}}$ is not equivalent to the usual operator norm $\|A\|$ (see Engliś [7] and [23]), while the inequality $\|A\|_{\mathrm{Ber}} \leqslant \|A\|$ is trivial; also it is obvious that $\mathrm{ber}(A) \leqslant \|A\|_{\mathrm{Ber}}$.

In the present article, we prove some new inequalities for the Berezin number of operators, and also we give some inequality between $\|A\|_{\mathrm{Ber}}$ and $\mathrm{ber}(A)$ for some operators A. Related results are contained in [3, 4, 5, 8, 9, 10, 13, 11, 12, 27, 28, 29].

2. Berezin symbol inequalities via refinements of the scalar Young inequality and related inequalities

Recall that an operator $A \in \mathscr{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. In this case we will write $A \geqslant 0$. The classical operator Jensen inequality for the positive operators $S \in \mathscr{B}(\mathscr{H})$ is

$$\langle Sx, x\rangle^r \leqslant (\geqslant) \langle S^r x, x\rangle, \ r \geqslant 1 \ (0 \leqslant r \leqslant 1).$$

Kittaneh and Manasrah [22] obtained the following result which is a refinement of scalar Young inequality.

LEMMA 1. *Let* $a, b \geqslant 0$, *and* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$ab + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(a^{\frac{p}{2}} - b^{\frac{q}{2}}\right)^2 \leqslant \frac{a^p}{p} + \frac{b^q}{q}. \tag{1}$$

Note that for $p = 2$, we have equality in (1).

In this section, we will use inequality (1) and a method of the paper [20] to prove new inequalities for the Berezin number of some operators on the reproducing kernel Hilbert space $\mathscr{H}(\Omega)$.

Let $\mathscr{B}(\mathscr{H})_+$ denote the set of all positive operators in $\mathscr{B}(\mathscr{H})$, and let $\mathscr{B}(\mathscr{H})_h$ be the real space of all self-adjoint operators on $\mathscr{H}$. It is well known (and easy to verify) that every bounded positive operator on $\mathscr{H}$ is self-adjoint. So, always $\mathscr{B}(\mathscr{H})_+ \subset \mathscr{B}(\mathscr{H})_h$.

THEOREM 1. *Let* f, g *be two continuous functions defined on an interval* $J \subset (-\infty, +\infty)$ *and* $f, g \geqslant 0$. *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, *then:*

(i) $\widetilde{(fg)(A)}(\lambda) + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left[\left(f(A)^{\frac{p}{2}} - g(A)^{\frac{q}{2}}\right)^2\right]^{\sim}(\lambda) \leqslant \left[\frac{f(A)^p}{p} + \frac{g(A)^q}{q}\right]^{\sim}(\lambda)$ *for all operators* $A \in \mathscr{B}(\mathscr{H})_h$ *with spectrum contained in* J *and all* $\lambda \in \Omega$;

(ii) $\operatorname{ber}\left(f(A)^2\right) \leqslant \operatorname{ber}\left(\frac{f(A)^p}{p} + \frac{f(A)^q}{q}\right)$ *for all* $A \in \mathscr{B}(\mathscr{H})_h$ *with spectrum contained in* J *and all* $\lambda \in \Omega$.

(iii) $\operatorname{ber}\left(A + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(f(A)^{p/2} - g(A)^{q/2}\right)^2\right) \leqslant \operatorname{ber}\left(\frac{f(A)^p}{p} + \frac{g(A)^q}{q}\right)$ *for all* $A \in \mathscr{B}(\mathscr{H})_h$ *with spectrum contained in* J.

Proof. Let $t, s \in J$. Following Kian [20], noticing that $f(t) \geqslant 0$ and $g(t) \geqslant 0$ for all $t \in J$, and putting $a = f(t)$ and $b = g(t)$ in inequality (1), we obtain

$$f(t)g(t) + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(f(t)^{\frac{p}{2}} - g(t)^{\frac{q}{2}}\right)^2 \leqslant \frac{f(t)^p}{p} + \frac{g(t)^q}{q} \tag{2}$$

for all $t \in J$. Using the functional calculus for A to inequality (2), we have that

$$f(A)g(A) + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left(f(A)^{\frac{p}{2}} - g(A)^{\frac{q}{2}}\right)^2 \leqslant \frac{f(A)^p}{p} + \frac{g(A)^q}{q}.$$

Whence

$$\begin{aligned}&\left\langle f(A)g(A)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \min\left\{\frac{1}{p}, \frac{1}{q}\right\}\left\langle\left(f(A)^{\frac{p}{2}} - g(A)^{\frac{q}{2}}\right)^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\ &\leqslant \left\langle\left(\frac{f(A)^p}{p} + \frac{g(A)^q}{q}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\end{aligned}$$

for all $\lambda \in \Omega$, which means that

$$\widetilde{f(A)g(A)}(\lambda)+\min\left\{\frac{1}{p},\frac{1}{q}\right\}\widetilde{\left[\left(f(A)^{\frac{p}{2}}-g(A)^{\frac{q}{2}}\right)^{2}\right]}(\lambda)\leqslant\widetilde{\left[\frac{f(A)^{p}}{p}+\frac{g(A)^{q}}{q}\right]}(\lambda),$$

or equivalently,

$$\widetilde{(fg)(A)}(\lambda)+\min\left\{\frac{1}{p},\frac{1}{q}\right\}\widetilde{\left[\left(f(A)^{\frac{p}{2}}-g(A)^{\frac{q}{2}}\right)^{2}\right]}(\lambda)\leqslant\widetilde{\left[\frac{f(A)^{p}}{p}+\frac{g(A)^{q}}{q}\right]}(\lambda) \quad (3)$$

for all $\lambda \in \Omega$, which proves (i).

Replacing g by f in (3), we get

$$\widetilde{f(A)^{2}}(\lambda)\leqslant\widetilde{\left[\frac{f(A)^{p}}{p}+\frac{f(A)^{q}}{q}\right]}(\lambda),\ \forall\lambda\in\Omega,$$

which implies by taking sup in both side that

$$\operatorname{ber}\left(f(A)^{2}\right)\leqslant\operatorname{ber}\left(\frac{f(A)^{p}}{p}+\frac{f(A)^{q}}{q}\right),$$

which proves (ii). For the proof of (iii), it suffices to take functions f and g in (i) satisfying $f(t)g(t)=t$. The theorem is proven. □

Manasrah and Kittaneh [1] have generalized inequality (1) as follows:

LEMMA 2. *If $a,b>0$ and $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, then for $m=1,2,\ldots,$*

$$\left(a^{\frac{1}{p}}b^{\frac{1}{q}}\right)^{m}+r_{0}^{m}\left(a^{\frac{m}{2}}-b^{\frac{m}{2}}\right)^{2}\leqslant\left(\frac{a^{r}}{p}+\frac{b^{r}}{q}\right)^{\frac{m}{r}},\ r\geqslant 1, \quad (4)$$

where $r_0=\min\left\{\frac{1}{p},\frac{1}{q}\right\}$. In particular, if $p=q=2$, then

$$\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right)^{m}+\frac{1}{2^{m}}\left(a^{\frac{m}{2}}-b^{\frac{m}{2}}\right)^{2}\leqslant 2^{-\frac{m}{r}}\left(a^{r}+b^{r}\right)^{\frac{m}{r}}.$$

For $m=1$

$$\left(a^{\frac{1}{2}}b^{\frac{1}{2}}\right)+\frac{1}{2}\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)\leqslant 2^{-\frac{1}{r}}\left(a^{r}+b^{r}\right)^{\frac{1}{r}}.$$

THEOREM 2. *Let f,g be two continuous functions defined on an interval $J\subset(-\infty,+\infty)$ and $f,g\geqslant 0$, and let $p,q>1$ be numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Then, for $m=1,2,\ldots,$ we have:*

(i)

$$\widetilde{\left[\left(f(A)^{\frac{1}{p}}g(A)^{\frac{1}{q}}\right)^{m}\right]}(\lambda)+r_{0}^{m}\widetilde{\left[\left(f(A)^{\frac{m}{2}}-g(A)^{\frac{m}{2}}\right)^{2}\right]}(\lambda)$$
$$\leqslant\widetilde{\left[\left(\frac{f(A)^{r}}{p}+\frac{g(A)^{r}}{q}\right)^{\frac{m}{r}}\right]}(\lambda),r\geqslant 1, \quad (5)$$

where $r_0 = \min\left\{\frac{1}{p}, \frac{1}{q}\right\}$, *for all operators* $A \in \mathscr{B}(\mathscr{H})_+$ *with* $\sigma(A) \subset J \subset [0, +\infty)$ *and all* $\lambda \in \Omega$;

(ii)

$$\operatorname{ber}\left(A^{\frac{m}{2}} + \frac{1}{2^m}\left(f(A)^{\frac{m}{2}} - g(A)^{\frac{m}{2}}\right)^2\right) \leqslant \inf_{r \geqslant 1} \operatorname{ber}\left(\left(\frac{f(A)^r + g(A)^r}{2}\right)^{\frac{m}{r}}\right),$$

for every operator $A \in \mathscr{B}(\mathscr{H})_+$ *with* $\sigma(A) \subset J \subset [0, +\infty)$.

Proof. The proof of this theorem uses Lemma 2 and it is quite similar to the proof of Theorem 1, and therefore, we omit details only saying that the proof of (i) is obtained from inequality (4) by applying the functional calculus arguments of Kian and for the proof of (ii), it is enough to choose in (i) functions f and g such that $f(t)g(t) = t$ (for example $f(t) = t^r$ and $g(t) = t^{1-r}$ with $0 < r < 1$) and $p = q = 2$. $\square$

Recall that the Schwarz inequality for positive operators reads that

$$|\langle Ax, y\rangle|^2 \leqslant \langle Ax, x\rangle \langle Ay, y\rangle \tag{6}$$

for all vectors $x, y \in \mathscr{H}$.

Reid [26] proved that for all operators A, B such that $A \in \mathscr{B}(\mathscr{H})_+$ and $AB \in \mathscr{B}(\mathscr{H})_h$, we have

$$|\langle ABx, y\rangle| \leqslant \|B\| \langle Ax, x\rangle, \ \forall x, y \in \mathscr{H}, \tag{7}$$

which is in some sense a variant of Schwarz inequality. Halmos [15] presented his stronger version of inequality (7) by replacing $r(B)$, the spectral radius, instead of $\|B\|$. Kato [18] introduced a companion inequality of Schwarz inequality (6), called the mixed Schwarz inequality, which asserts

$$|\langle ABx, y\rangle|^2 \leqslant \left\langle |A|^{2\alpha} x, x\right\rangle \left\langle |A^*|^{2(1-\alpha)} y, y\right\rangle, \ 0 \leqslant \alpha \leqslant 1, \tag{8}$$

for every operators $A \in \mathscr{B}(\mathscr{H})$ and any vectors $x, y \in \mathscr{H}$.

Finally, Kittaneh [21] proved an interesting extension combining both the Halmos-Reid inequality (7) and the mixed Schwarz inequality (8), which reads that

$$|\langle ABx, y\rangle| \leqslant r(B) \|f(|A|)x\| \|g(|A^*|)y\| \tag{9}$$

for all vectors $x, y \in \mathscr{H}$, where $A, B \in \mathscr{B}(\mathscr{H})$ such that $|A|B = B^*|A|$ and f, g are non-negative continuous functions defined on $[0, \infty)$ satisfying that $f(t)g(t) = t$ $(t \geqslant 0)$.

Now by putting $x = \widehat{k}_\lambda$ and $y = \widehat{k}_\lambda$ in Halmos version of (7), (8) and (9), we obtain the following important inequalities for Berezin numbers and Berezin norms of operators:

$$\operatorname{ber}(AB) \leqslant r(B) \operatorname{ber}(A).$$

$$(\operatorname{ber}(A))^2 \leqslant \operatorname{ber}\left(|A|^{2\alpha}\right) \operatorname{ber}\left(|A^*|^{2(1-\alpha)}\right), \ (0 \leqslant \alpha \leqslant 1).$$

$$\operatorname{ber}(AB) \leqslant r(B) \|f(|A|)\|_{\operatorname{Ber}} \|g(|A^*|)\|_{\operatorname{Ber}}.$$

3. Other inequalities

3.1. Upper bounds for the Berezin numbers

For $A \in \mathscr{B}(\mathscr{H})$, we denote by $|A|$ the absolute value operator of A, that is, $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the adjoint operator of A. So, $|A^*| = (AA^*)^{\frac{1}{2}}$. A continuous real-valued function f defined on an interval Δ is said to be pseudo-operator convex if

$$f(\alpha A + (1-\alpha) B) \leqslant \alpha f(A) + (1-\alpha) f(B)$$

in the sense that

$$(\alpha f(A) + (1-\alpha) f(B))^{\sim}(\lambda) \leqslant (\alpha f(A) + (1-\alpha) f(B))^{\sim}(\lambda)$$

for all $\lambda \in \Omega$, for all self-adjoint operators A, B with spectra contained in Δ and all $\alpha \in [0,1]$.

The following lemma is well known, which is called the mixed Schwarz inequality (see Halmos [15]).

LEMMA 3. *If $A \in \mathscr{B}(\mathscr{H})$, then*

$$|\langle Ax, y\rangle| \leqslant \langle |A| x, x\rangle^{\frac{1}{2}} \langle |A^*| y, y\rangle^{\frac{1}{2}}$$

for all $x, y \in \mathscr{H}$.

LEMMA 4. *([25]) For each $\alpha \geqslant 1$, we have*

$$\frac{\alpha - 1}{\alpha + 1} \leqslant \ln \alpha. \tag{10}$$

Recall that an operator $A \in \mathscr{B}(\mathscr{H})$ is said to be hyponormal, if $[A^*, A] \in \mathscr{B}(\mathscr{H})_+$, i.e., $A^*A - AA^* \geqslant 0$, or equivalently $\|A^*x\| \leqslant \|Ax\|$ for every $x \in \mathscr{H}$.

Typical examples are subnormal operators and singular integral operators on the line with Cauchy type integral (see [14]).

DEFINITION 1. We say that an operator $A \in \mathscr{B}(\mathscr{H})$ is a pseudo-hyponormal operator if $\left\|A^*\widehat{k}_\lambda\right\| \leqslant \left\|A\widehat{k}_\lambda\right\|$, or equivalenty, if $\widetilde{[A^*, A]}(\lambda) \geqslant 0$ for all $\lambda \in \Omega$.

It is clear that every hyponormal operator is pseudo-hyponormal. The following example shows that the set of hyponormal operators is a proper subset in the set of pseudo-hyponormal operators on $\mathscr{H}$.

EXAMPLE 1. Let θ be a non-constant inner function and $N_\theta := T_\theta \left(I - T_\theta T_\theta^*\right)$ be an operator on the Hardy space $H^2 := H^2(\mathbb{D})$ over the unit disc $\mathbb{D}$. Then N_θ is a pseudo-hyponormal operator, but it is not hyponormal.

Proof. Since θ is an inner function (i.e., $|\theta(z)| \leqslant 1$ for all $z \in \mathbb{D}$ and $|\theta(\xi)| = 1$ for almost all $\xi \in \partial\mathbb{D}$), an analytic Toeplitz operator T_θ is isometry on H^2, that is $T_\theta^* T_\theta = 1$. So, it is easy to verify that $N_\theta^2 = 0$ and $\|N_\theta\| = 1$, i.e., N_θ is a nonzero nilpotent operator on H^2. Therefore N_θ is not hyponormal operator, because for hyponormal operators their norm and spectral radius coincide (see, for instance, [14]).

However, N_θ is pseudo-hyponormal. In fact, for every $\lambda \in \mathbb{D}$ we have:

$$\begin{aligned}
\widetilde{\left[N_\theta^*, N_\theta\right]}(\lambda) &= \left\langle (N_\theta^* N_\theta - N_\theta N_\theta^*) \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \\
&= \left\langle [(I - T_\theta T_\theta^*) T_\theta^* T_\theta (I - T_\theta T_\theta^*) - T_\theta (I - T_\theta T_\theta^*)(I - T_\theta T_\theta^*) T_\theta^*] \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \\
&= \left\langle \left(I - 2T_\theta T_\theta^* + T_\theta^2 T_\theta^{*2}\right) \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \\
&= 1 - 2|\theta(\lambda)|^2 + |\theta(\lambda)|^4 = \left(1 - |\theta(\lambda)|^2\right)^2 \geqslant 0,
\end{aligned}$$

which shows that N_θ is a pseudo-hyponormal operator on H^2, as desired. $\square$

This section motivated by the paper [25], where upper bounds for the numerical radii are proved. In the present section, we prove the similar results for the Berezin numbers.

THEOREM 3. *Let $A \in \mathscr{B}(\mathscr{H})$ be a pseudo-hyponormal operator. Then*

$$f(\mathrm{ber}(A)) \leqslant \frac{1}{2} \left\| f\left(\frac{1}{1 + \frac{\xi_{|A|}^2}{8}} |A|\right) + f\left(\frac{1}{1 + \frac{\xi_{|A|}^2}{8}} |A^*|\right) \right\|_{\mathrm{Ber}}$$

for all nonnegative non-decreasing pseudo-operator convex f on $[0,\infty)$, where $\xi_{|A|} = \inf_{\lambda \in \Omega} \left\{ \frac{\widetilde{|A| - |A^*|}(\lambda)}{\widetilde{|A| + |A^*|}(\lambda)} \right\}$.

Proof. Since A is pseudo-hyponormal, we have

$$1 \leqslant \frac{\left\langle |A| \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle}{\left\langle |A^*| \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle} = \frac{\widetilde{|A|}(\lambda)}{\widetilde{|A^*|}(\lambda)}$$

for each $\lambda \in \Omega$. Putting $\alpha = \frac{\widetilde{|A|}(\lambda)}{\widetilde{|A^*|}(\lambda)}$ in (10), we get

$$(0 \leqslant) \frac{\widetilde{|A| - |A^*|}(\lambda)}{\widetilde{|A| + |A^*|}(\lambda)} \leqslant \ln \frac{\widetilde{|A|}(\lambda)}{\widetilde{|A^*|}(\lambda)},$$

hence

$$\inf_{\lambda \in \Omega} \frac{\widetilde{|A| - |A^*|}(\lambda)}{\widetilde{|A| + |A^*|}(\lambda)} \leqslant \ln \frac{\widetilde{|A|}(\lambda)}{\widetilde{|A^*|}(\lambda)}. \tag{11}$$

We set

$$\xi_{|A|} := \inf_{\lambda\in\Omega} \frac{\widetilde{|A|-|A^*|}(\lambda)}{\widetilde{|A|+|A^*|}(\lambda)}.$$

On the other hand, it is known that (see [30])

$$\left(1+\frac{(\ln a-\ln b)^2}{8}\right)\sqrt{ab} \leqslant \frac{a+b}{2}$$

for each $a,b>0$.

By taking $a=\widetilde{|A|}(\lambda)$ and $b=\widetilde{|A^*|}(\lambda)$, and by considering that $\xi_{|A|} \leqslant \ln \frac{\widetilde{|A|}(\lambda)}{\widetilde{|A^*|}(\lambda)}$, we infer that

$$\left(\widetilde{|A|}(\lambda)\widetilde{|A^*|}(\lambda)\right)^{1/2} \leqslant \frac{\widetilde{|A|+|A^*|}(\lambda)}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)}.$$

By applying Lemma 3, we obtain

$$\widetilde{|A|}(\lambda) \leqslant \frac{\widetilde{|A|+|A^*|}(\lambda)}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)}.$$

Now, by taking supremum over $\lambda\in\Omega$, we get

$$\operatorname{ber}(A) \leqslant \frac{1}{2\left(1+\xi_{|A|}^2/8\right)}\operatorname{ber}(|A|+|A^*|) \leqslant \frac{1}{2\left(1+\xi_{|A|}^2/8\right)}\,\||A|+|A^*|\|_{\mathrm{Ber}}.$$

Therefore,

$$\begin{aligned} f(\operatorname{ber}(A)) &\leqslant f\left(\frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)}\,\||A|+|A^*|\|_{\mathrm{Ber}}\right) \\ &= \left\| f\left(\frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)}|A| + \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)}|A^*|\right)\right\|_{\mathrm{Ber}} \\ &\leqslant \frac{1}{2}\left\| f\left(\frac{1}{1+\frac{\xi_{|A|}^2}{8}}|A|\right) + f\left(\frac{1}{1+\frac{\xi_{|A|}^2}{8}}|A^*|\right)\right\|_{\mathrm{Ber}}. \end{aligned}$$

This proves the theorem. □

Since $\xi_{|A|}=0$ for any normal operator $A\in\mathscr{B}(\mathscr{H})$, then we have from Theorem 3 the following.

COROLLARY 1.

$$f(\mathrm{ber}(A)) \leqslant \frac{1}{2}\left\| f(|A|)+f(|A^*|)\right\|_{\mathrm{Ber}}$$

for every normal operator A on $\mathscr{H}(\Omega)$.

COROLLARY 2. *Let $A \in \mathscr{B}(\mathscr{H})$ be a pseudo-hyponormal operator. Then, for all $1 \leqslant r \leqslant 2$ we have*

$$\mathrm{ber}^r(A) \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)} \left\| |A|^r + |A^*|^r \right\|_{\mathrm{Ber}},$$

where

$$\xi_{|A|} = \inf_{\lambda\in\Omega} \frac{\widetilde{|A|-|A^*|}(\lambda)}{\widetilde{|A|+|A^*|}(\lambda)}.$$

In particular,

$$\mathrm{ber}(A) \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)} \left\| |A| + |A^*| \right\|_{\mathrm{Ber}},$$

and

$$\mathrm{ber}^2(A) \leqslant \frac{1}{2\left(1+\frac{\xi_{|A|}^2}{8}\right)} \left\| A^*A + AA^* \right\|_{\mathrm{Ber}}.$$

The following lemma can be proved by slight modification of the proof of Example 3.6 of the paper [19] (the proof is omitted).

LEMMA 5. *Let $A_1,..A_n$ be positive operators, then*

$$\begin{aligned} &\left\| \sum_{i=1}^n c_i A_i \right\|_{\mathrm{Ber}}^r \\ &\leqslant \left\| \sum_{i=1}^n c_i A_i^r \right\|_{\mathrm{Ber}} - \inf_{\lambda\in\Omega}\left\{ \sum_{i=1}^n c_i \left\langle \left| A_i - \sum_{j=1}^n c_j \left\langle A_j \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^r \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right\},\ r \geqslant 2, \end{aligned} \tag{12}$$

for each $c_1,..c_n$ with $\sum_{i=1}^n c_i = 1$.

THEOREM 4. *Let $A \in \mathscr{B}(\mathscr{H})$, then*

$$\mathrm{ber}^2(A) \leqslant \frac{1}{2}\left(\left\| |A|^2 + |A^*|^2 \right\|_{\mathrm{Ber}} - \inf_{\lambda\in\Omega} \xi(\lambda) \right),$$

where

$$\xi(\lambda) := \left\langle \left(\left| |A| - \frac{1}{2}\left(\widetilde{|A|+|A^*|}(\lambda) I \right) \right|^2 + \left| |A^*| - \frac{1}{2}\left(\widetilde{|A|+|A^*|}(\lambda) I \right) \right|^2 \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle,\ \lambda \in \Omega.$$

Proof. It is easy to verify that for any $A \in \mathscr{B}(\mathscr{H})$

$$\operatorname{ber}(A) \leqslant \frac{1}{2} \left\| |A| + |A^*| \right\|_{\mathrm{Ber}},$$

or equivalently,

$$\operatorname{ber}^2(A) \leqslant \frac{1}{4} \left\| |A| + |A^*| \right\|_{\mathrm{Ber}}^2. \tag{13}$$

Taking $n,\ r = 2,\ c_1 = c_2 = \frac{1}{2},\ A_1 = |A|$ and $A_2 = |A^*|$ in inequality (12) in Lemma 5, we infer

$$\begin{aligned}\left\| |A| + |A^*| \right\|_{\mathrm{Ber}}^2 \leqslant 2 \Bigg(\left\| |A|^2 + |A^*|^2 \right\|_{\mathrm{Ber}} - \inf_{\lambda \in \Omega} \Bigg\{ \left(\left| |A| - \frac{1}{2}\left(\widetilde{|A|}(\lambda) + \widetilde{|A^*|}(\lambda) \right) \right|^2 \right)^{\sim} (\lambda) \\ + \left(\left| |A^*| - \frac{1}{2}\left(\widetilde{|A|}(\lambda) + \widetilde{|A^*|}(\lambda) \right) \right|^2 \right)^{\sim} (\lambda) \Bigg\} \Bigg).\end{aligned}$$

It follows from (13) that

$$\begin{aligned}\operatorname{ber}^2(A) &\leqslant \frac{1}{4} \left\| |A| + |A^*| \right\|_{\mathrm{Ber}}^2 \\ &\leqslant \frac{1}{2} \Bigg(\left\| |A| + |A^*| \right\|_{\mathrm{Ber}}^2 - \inf_{\lambda \in \Omega} \Bigg\{ \left(\left| |A| - \frac{1}{2}\left(\widetilde{|A|}(\lambda) + \widetilde{|A^*|}(\lambda) \right) \right|^2 \right)^{\sim} (\lambda) \\ &\quad + \left(\left| |A^*| - \frac{1}{2}\left(\widetilde{|A|}(\lambda) + \widetilde{|A^*|}(\lambda) \right) \right|^2 \right)^{\sim} (\lambda) \Bigg\} \Bigg).\end{aligned}$$

This proves the theorem. $\square$

Note that $\inf_{\lambda \in \Omega} \xi(\lambda) > 0$ if and only if

$$0 \notin \operatorname{closBer}\left(\left| |A| - \frac{1}{2}\left(\widetilde{|A|}(\lambda) + \widetilde{|A^*|}(\lambda) \right)^{\sim} (\lambda) \right|^2 + \left| |A^*| - \frac{1}{2}\left(\widetilde{|A|}(\lambda) + \widetilde{|A^*|}(\lambda) \right)^{\sim} (\lambda) \right|^2 \right).$$

3.2. Lower bounds for the Berezin numbers

Here we give some lower bounds for the Berezin numbers of operators.

THEOREM 5. *Let $A \in \mathscr{B}(\mathscr{H})$, then*

$$\|A\|_{\mathrm{Ber}} \left(1 - \frac{1}{2} \left\| I - \frac{A}{\|A\|} \right\|_{\mathrm{Ber}}^2 \right) \leqslant \operatorname{ber}(A). \tag{14}$$

Proof. It is easy to see that

$$1 - \frac{1}{2} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 \leqslant \frac{1}{\|x\| \, \|y\|} \left| \langle x, y \rangle \right| \tag{15}$$

for every $x,y \in \mathscr{H}$. If we choose $\|x\| = \|y\| = 1$ in (15), we have

$$1 - \frac{1}{2}\|x-y\|^2 \leqslant |\langle x,y\rangle|. \tag{16}$$

Now taking $x = \widehat{k}_\lambda$ and $y = \frac{A\widehat{k}_\lambda}{\|A\widehat{k}_\lambda\|}$ in (16), we infer

$$1 - \frac{1}{2}\left\|\widehat{k}_\lambda - \frac{A\widehat{k}_\lambda}{\left\|A\widehat{k}_\lambda\right\|}\right\|^2 \leqslant \left|\left\langle \widehat{k}_\lambda, \frac{A\widehat{k}_\lambda}{\left\|A\widehat{k}_\lambda\right\|}\right\rangle\right|,$$

or equivalently,

$$\left\|A\widehat{k}_\lambda\right\|\left(1 - \frac{1}{2}\left\|\widehat{k}_\lambda - \frac{A\widehat{k}_\lambda}{\left\|A\widehat{k}_\lambda\right\|}\right\|^2 \leqslant \left|\widetilde{A}(\lambda)\right|\right) \tag{17}$$

for all $\lambda \in \Omega$. Since $\left\|A\widehat{k}\right\| \leqslant \|A\|_{\mathrm{Ber}}$, we get from (17) that

$$\left\|A\widehat{k}_\lambda\right\|\left(1 - \frac{1}{2}\left\|I - \frac{A}{\|A\|}\right\|_{\mathrm{Ber}}^2\right) \leqslant \left|\widetilde{A}(\lambda)\right|$$

for all $\lambda \in \Omega$. Hence, by taking supremum over $\lambda \in \Omega$, we deduce the required inequality (14). The theorem is proved. $\square$

The following lemma is proved in [25].

LEMMA 6. *Let $x,y,z_i,\ i = 1,\dots,n$ be nonzero vectors and $\langle z_j, z_i\rangle \neq 0$, then*

$$\left|\left\langle x - \sum_i \frac{\langle x, z_i\rangle}{\sum_j \left|\langle z_j, z_i\rangle\right|} z_i, y\right\rangle\right|^2 \leqslant \|y\|^2\left(\|x\|^2 - \sum_i \frac{\left|\langle x, z_j\rangle\right|^2}{\sum_j \left|\langle z_i, z_j\rangle\right|}\right). \tag{17}$$

THEOREM 6. *Let $A \in \mathscr{B}(\mathscr{H})$ be an invertible operator. Then*

$$\inf_{\lambda\in\Omega} \xi^2(\lambda) + \mathrm{ber}^2(A) \leqslant \|A\|_{\mathrm{Ber}}^2, \tag{18}$$

where

$$\xi(\lambda) := \frac{\left|\left\langle A^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle - \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^2\right|}{\left\|A^*\widehat{k}_\lambda\right\|}.$$

Proof. Simplifying (17) for the case $n = 1$, we find that (see [25])

$$\left|\langle x,y\rangle - \frac{\langle x,z\rangle}{\|z\|^2}\langle z,y\rangle\right|^2 + \frac{|\langle x,z\rangle|^2}{\|z\|^2}\|y\|^2 \leqslant \|x\|^2\|y\|^2.$$

Apply these consideration to $x = Ax,\ y = A^*x$ and $z = x$ with $\|x\| = 1,$ we obtain

$$\left(\frac{\left|\langle A^2x,x\rangle - \langle Ax,x\rangle^2\right|}{\|A^*x\|}\right)^2 + |\langle Ax,x\rangle|^2 \leqslant \|Ax\|^2. \tag{19}$$

Now let us put $x = \widehat{k}_\lambda$ in (19). Then, we have that

$$\left(\frac{\left|\widetilde{A^2}(\lambda) - \widetilde{A}(\lambda)^2\right|}{\left\|A^*\widehat{k}_\lambda\right\|}\right)^2 + \left|\widetilde{A}(\lambda)\right|^2 \leqslant \left\|A\widehat{k}_\lambda\right\|^2 \tag{20}$$

for all $\lambda \in \Omega$. We denote $\xi(\lambda) := \frac{\left|\widetilde{A^2}(\lambda) - \widetilde{A}(\lambda)^2\right|}{\left\|A^*\widehat{k}_\lambda\right\|},\ \lambda \in \Omega.$

Then, inequality (20) implies that

$$\inf_{\lambda\in\Omega} \xi^2(\lambda) + \left|\widetilde{A}(\lambda)\right|^2 \leqslant \left\|A\widehat{k}_\lambda\right\|^2,\ \forall\lambda \in \Omega,$$

and thus, by taking the supremum over all points λ in Ω, we have the desired inequality (18). The theorem is proven. □

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