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L–MATRICES WITH LACUNARY COEFFICIENTS

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Abstract. We show that an L-matrices $A = [a_n]$, with lacunary coefficients (a_n) is a bounded operator on ℓ^2, provided that (a_n) satisfy an explicit decay rate. Moreover, by a concrete example, we see that the decay restriction is optimal. The extension to operators on ℓ^p spaces, for $p > 1$, is also discussed.

1. Introduction

Let $(a_n)_{n\geqslant 0}$ be a sequence of complex numbers. Then the infinite matrix

$$A = [a_n] = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_1 & a_1 & a_2 & a_3 & \cdots \\ a_2 & a_2 & a_2 & a_3 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

is called an *L-matrix*. These matrices were introduced in [1]. Infinite matrices have been the center of several recent studies. A very incomplete list is as follows: Bozkurt [2], Solak [14], Solak–Bozkurt [15] and Orr [13] studied the norm of infinite matrices. Ismail–Štampach [10] and Dai–Ismail–Wang [5] provided a complete spectral analysis of self-adjoint operators action on $\ell^2(\mathbb{Z})$ and studied their connections to difference equations. van de Mee–Seatzu [16] gave an algorithm to generate infinite multi-index positive self-adjoint Toeplitz matrices. For further on history and relevant literature of infinite matrices and in particular L-matrices, we refer to [11, 12, 1]. In [1], by providing two results, one necessary and the other sufficient, we studied the boundedness of A as an operator on ℓ^2. In particular, we could precisely evaluate the norm of

$$A_s = \begin{pmatrix} \frac{1}{s} & \frac{1}{s+1} & \frac{1}{s+2} & \cdots \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{1}{s+2} & \cdots \\ \frac{1}{s+2} & \frac{1}{s+2} & \frac{1}{s+2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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by showing that $\|A_s\| = 4$, for all $s \geqslant 1/2$. This is an interesting addition to a short list of infinite matrices for which we can precisely determine the norm, e.g., the Hilbert matrix H [9] with $\|H\| = \pi$ [4, 7], the Cesàro matrix C [8] with $\|C\| = 2$ [3, 18], the Bergman–Hilbert matrix [6], Hankel matrices [17]. However, for the general setting, an estimation formula was provided. As a necessary condition, we showed that

$$a_n = O\Big(\frac{1}{\sqrt{n}}\Big), \qquad (n \to \infty), \tag{1}$$

is required and, by providing a set of examples, we justified the sharpness of this condition. One of the explicit examples provided was

$$a_{4^n} = \frac{1}{n2^n}, \qquad (n \geqslant 1),$$

and $a_j = 0$ for other values of index. Then A is a Hilbert–Schmidt operator on ℓ^2. Furthermore,

$$\sqrt{m}\, a_m = \begin{cases} \frac{\log 4}{\log m} & \text{if} \qquad m = 4^n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, using similar technique, the decay rate $1/\log m$ can be decreased as fast as required.

This work started with an attempt to show that the operator $A = [a_n]$ with

$$a_{4^n} = \frac{1}{2^n}, \qquad (n \geqslant 1), \tag{2}$$

and $a_j = 0$ for other values of index, is bounded on ℓ^2, and thus the condition $O(1/\sqrt{n})$ in (1) is the best possible. Surprisingly enough, even this simple looking matrix was not easy to handle. As a matter of fact, verification of the boundedness of A with coefficients (2) took a long period and eventually led to a more general result which is discussed in Section 2. Briefly speaking, we will see that if (a_n) is a sparse sequence, then, up to certain decay rate which is optimal, the operator A is bounded on ℓ^2. The proof is direct, but somehow nontrivial, and in particular requires a judicial application of Cauchy-Schwarz and Hölder inequalities to different patterns in the formula of the norm.

In the following, we will write $a_n \asymp b_n$ whenever there are positive constants c and C such that

$$c|a_n| \leqslant |b_n| \leqslant C|b_n|, \qquad n \geqslant 1.$$

2. The boundedness on ℓ^2

We say that the sequence (a_n) is lacunary if there is a constant $\rho > 1$ and a subsequence (n_j) such that $n_{j+1}/n_j \geqslant \rho$ and $a_n = 0$ except possibly for indices $n \in \{n_j : j \geqslant 1\}$. We were initially interested in the exponential case $n_j = 4^j$, for which the formulas in the following theorem are simpler. See Corollary 1. However, the result can be extended to a more general setting as described below.

THEOREM 1. *Let $A = [a_n]$ be an L-matrix with lacunary coefficient (a_n) satisfying*

$$\sum_{s=j}^{\infty} \sqrt{n_s}\,|a_{n_s}|^2 = O\big(1/\sqrt{n_j}\big), \qquad (as\ j \to \infty).$$

Then A maps ℓ^2 to itself as a bounded operator.

Proof. Since $\big\|[a_n]\big\|_{\ell^2\to\ell^2} \leqslant \big\|[|a_n|]\big\|_{\ell^2\to\ell^2}$, without loss of generality, we assume that $a_n \geqslant 0$, for all $n \geqslant 0$. Then we can write $A = B + B^* - D$, where

$$B = \begin{pmatrix} a_0 & 0 & 0 & 0 & \cdots \\ a_1 & a_1 & 0 & 0 & \cdots \\ a_2 & a_2 & a_2 & 0 & \cdots \\ a_3 & a_3 & a_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and $D = \operatorname{diag}(a_0, a_1, \ldots)$ is the diagonal matrix with entries (a_n). Clearly, D is a bounded operator. Hence, it is enough to show that B is bounded and then the result follows. For this, we directly estimate $\|Bx\|$, where $x = (x_n) \in \ell^2$ and again, without loss of generality, we assume that $x_n \geqslant 0$.

Write $y = Bx$. Hence,

$$y_n = a_n(x_0 + x_1 + \cdots + x_n), \qquad (n \geqslant 0),$$

which immediately shows that (y_n) is also a lacunary series. The delicate part starts from here where we estimate y_n. Assuming $n = n_s$, $a_{n_s} \neq 0$ and for simplicity $x_0 = 0$, write

$$y_{n_s}/a_{n_s} = \sum_{j=0}^{n_1} x_j + \sum_{j=n_1+1}^{n_2} x_j + \cdots + \sum_{j=n_{s-1}+1}^{n_s} x_j.$$

Hence, by Cauchy–Schwarz inequality,

$$\begin{aligned} y_{n_s}/a_{n_s} &\leqslant n_1^{1/2}\left(\sum_{j=1}^{n_1} x_j^2\right)^{1/2} + (n_2 - n_1)^{1/2}\left(\sum_{j=n_1+1}^{n_2} x_j^2\right)^{1/2} \\ &\quad + \cdots + (n_s - n_{s-1})^{1/2}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^2\right)^{1/2} \\ &\leqslant n_1^{1/2}\left(\sum_{j=1}^{n_1} x_j^2\right)^{1/2} + n_2^{1/2}\left(\sum_{j=n_1+1}^{n_2} x_j^2\right)^{1/2} + \cdots + n_s^{1/2}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^2\right)^{1/2}. \end{aligned}$$

Note that since

$$n_j - n_{j-1} = n_j\left(1 - \frac{n_{j-1}}{n_j}\right) \geqslant n_j\left(1 - \frac{1}{\rho}\right),$$

the above estimation, up to a multiplicative constant, is optimal. We need to apply the Cauchy-Schwarz inequality one more time, but to the combination

$$n_1^{1/4}\cdot n_1^{1/4}\left(\sum_{j=1}^{n_1} x_j^2\right)^{1/2} + n_2^{1/4}\cdot n_2^{1/4}\left(\sum_{j=n_1+1}^{n_2} x_j^2\right)^{1/2} + \cdots + n_s^{1/4}\cdot n_s^{1/4}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^2\right)^{1/2}.$$

Hence,

$$\begin{aligned}\left(y_{n_s}/a_{n_s}\right)^2 \leqslant &\left(n_1^{1/2}+n_2^{1/2}+\cdots+n_s^{1/2}\right)\\ &\left[n_1^{1/2}\left(\sum_{j=1}^{n_1} x_j^2\right)+n_2^{1/2}\left(\sum_{j=n_1+1}^{n_2} x_j^2\right)+\cdots+n_s^{1/2}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^2\right)\right].\end{aligned}$$

That (n_j) is a lacunary series is used here one more time to get

$$n_1^{1/2}+n_2^{1/2}+\cdots+n_s^{1/2} \asymp n_s^{1/2},$$

and thus

$$y_{n_s}^2 \leqslant C a_{n_s}^2 n_s^{1/2}\left[n_1^{1/2}\left(\sum_{j=1}^{n_1} x_j^2\right)+n_2^{1/2}\left(\sum_{j=n_1+1}^{n_2} x_j^2\right)+\cdots+n_s^{1/2}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^2\right)\right],$$

where $C=C(\rho)$ is a constant. Therefore,

$$\begin{aligned}\|Bx\|^2 &= \sum_{n=0}^{\infty} y_n^2 = \sum_{s=1}^{\infty} y_{n_s}^2\\ &\leqslant C\left(\eta_1\sum_{j=1}^{n_1} x_j^2+\eta_2\sum_{j=n_1+1}^{n_2} x_j^2+\cdots\right),\end{aligned}$$

where

$$\eta_j = n_j^{1/2}\sum_{s=j}^{\infty} a_{n_s}^2 n_s^{1/2}.$$

By assumption η_js are uniformly bounded, i.e., $\eta_j \leqslant C'$, for all $j \geqslant 1$. Therefore, $\|Bx\|^2 \leqslant CC'\,\|x\|^2$, for all $x\in\ell^2$. In other words, B is bounded, which in return shows that A is bounded. □

3. Application

Fix an integer $N \geqslant 2$ and put

$$n_j = N^j, \qquad (j \geqslant 1).$$

We also assume that

$$a_{n_j} = R^j, \qquad (j \geqslant 1),$$

where R is a fixed ratio. To verify the required condition in Theorem 1, note that

$$\begin{aligned}\eta_j = \sqrt{n_j}\sum_{s=j}^{\infty}\sqrt{n_s}\,a_{n_s}^2 &= N^{j/2}\sum_{s=j}^{\infty}N^{s/2}R^{2s}\\ &= N^{j/2}\sum_{s=j}^{\infty}\left(\sqrt{N}R^2\right)^s\\ &= N^{j/2}\frac{\left(\sqrt{N}R^2\right)^j}{1-\sqrt{N}R^2}\\ &= \frac{(NR^2)^j}{1-\sqrt{N}R^2},\end{aligned}$$

provided that $\sqrt{N}R^2 < 1$. However, η_j s remain uniformly bounded if we put the stronger assumption $NR^2 \leqslant 1$.

COROLLARY 1. *Let $N \geqslant 2$ be a positive integer and let $0 \leqslant R \leqslant 1/\sqrt{N}$. Let $A = [a_n]$ be the L-matrix with lacunary coefficient*

$$a_{N^j} = R^j, \qquad (j \geqslant 1),$$

and $a_n = 0$ for other values of n. Then A is a bounded operator on ℓ^2.

As a very special case, by taking $N = 4$ and $R = 1/2$, we see that the operator $A = [a_n]$ with coefficients (2) is bounded on ℓ^2.

4. The boundedness on ℓ^p

With a similar techniques, but using the Hölder inequality, we can prove the following more general version of Theorem 1. Below, we provide a sketch of proof. In the following, given $1 < p < \infty$, its exponent conjugate q is the unique real number satisfying $1/p + 1/q = 1$.

THEOREM 2. *Let $p > 1$, with exponent conjugate q, and let $A = [a_n]$ be an L-matrix with lacunary coefficient (a_n) satisfying*

$$\sum_{s=j}^{\infty}|a_{n_s}|^p n_s^{(1-t)p/q} = O\big(n_j^{-tp/q}\big), \qquad (j \to \infty),$$

for some $t \in [0,1)$. Then A maps ℓ^p to itself as a bounded operator.

Proof. As in the proof of Theorem 1, write $y = Bx$ and

$$y_{n_s}/a_{n_s} = \sum_{j=0}^{n_1}x_j + \sum_{j=n_1+1}^{n_2}x_j + \cdots + \sum_{j=n_{s-1}+1}^{n_s}x_j.$$

Hence, by Hölder's inequality,

$$\begin{aligned} y_{n_s}/a_{n_s} &\leqslant n_1^{1/q}\left(\sum_{j=1}^{n_1} x_j^p\right)^{1/p} + (n_2-n_1)^{1/q}\left(\sum_{j=n_1+1}^{n_2} x_j^p\right)^{1/p} \\ &\quad + \cdots + (n_s-n_{s-1})^{1/q}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^p\right)^{1/p} \\ &\leqslant n_1^{1/q}\left(\sum_{j=1}^{n_1} x_j^p\right)^{1/p} + n_2^{1/q}\left(\sum_{j=n_1+1}^{n_2} x_j^p\right)^{1/p} + \cdots + n_s^{1/q}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^p\right)^{1/p}. \end{aligned}$$

We apply the Hölder inequality one more time, but to the combination

$$n_1^{(1-t)/q}\cdot n_1^{t/q}\left(\sum_{j=1}^{n_1} x_j^p\right)^{1/p} + \cdots + n_s^{(1-t)/q}\cdot n_s^{t/q}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^p\right)^{1/p}.$$

Hence,

$$\begin{aligned} y_{n_s}/a_{n_s} &\leqslant \left(n_1^{1-t}+n_2^{1-t}+\cdots+n_s^{1-t}\right)^{1/q} \\ &\quad \left[n_1^{tp/q}\left(\sum_{j=1}^{n_1} x_j^p\right) + n_2^{tp/q}\left(\sum_{j=n_1+1}^{n_2} x_j^p\right) + \cdots + n_s^{tp/q}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^p\right)\right]^{1/p}. \end{aligned} \tag{3}$$

Since (n_j) is a lacunary series, we have

$$\left(n_1^{1-t}+n_2^{1-t}+\cdots+n_s^{1-t}\right)^{1/q} \asymp n_s^{(1-t)/q} \tag{4}$$

and thus

$$y_{n_s}^p \leqslant C a_{n_s}^p n_s^{(1-t)p/q}\left[n_1^{tp/q}\left(\sum_{j=1}^{n_1} x_j^p\right) + n_2^{tp/q}\left(\sum_{j=n_1+1}^{n_2} x_j^p\right) + \cdots + n_s^{tp/q}\left(\sum_{j=n_{s-1}+1}^{n_s} x_j^p\right)\right],$$

where $C = C(\rho)$ is a constant. Therefore,

$$\begin{aligned} \|Bx\|_p^p &= \sum_{n=0}^{\infty} y_n^p = \sum_{s=1}^{\infty} y_{n_s}^p \\ &\leqslant C\left(\eta_1 \sum_{j=1}^{n_1} x_j^p + \eta_2 \sum_{j=n_1+1}^{n_2} x_j^p + \cdots\right), \end{aligned}$$

where

$$\eta_j = n_j^{tp/q}\sum_{s=j}^{\infty} a_{n_s}^p n_s^{(1-t)p/q} = O(1). \tag{5}$$

We are done. $\square$

A similar result holds for the case $t=1$. In fact, in the above proof, the estimation (4) should be replaced with

$$\left(n_1^{1-t}+n_2^{1-t}+\cdots+n_s^{1-t}\right)^{1/q} \asymp s^{1/q}, \qquad \text{if } t=1.$$

The rest of proof is the same, and thus the required condition of Theorem 2 becomes

$$\sum_{s=j}^{\infty} |a_{n_s}|^p s^{p/q} = O\left(n_j^{-p/q}\right), \qquad (j\to\infty).$$

If so, A maps ℓ^p to itself as a bounded operator.

5. Quantitative estimations

Under the assumptions of Theorem 2, we proceed to find an upper bound for $\|A\|_{\ell^p\to\ell^p}$. Hence, fix $p>1$ and $t\in(0,1)$. Since (n_j) is a lacunary series with ratio $\rho>1$, we have

$$n_1^{1-t}+n_2^{1-t}+\cdots+n_s^{1-t} \leqslant \left(1+\frac{1}{\rho^{1-t}}+\frac{1}{\rho^{2(1-t)}}+\cdots\right)n_s^{1-t} = \frac{\rho^{1-t}}{\rho^{1-t}-1}n_s^{1-t}.$$

By (3), this estimation implies

$$\|Bx\|_p^p \leqslant \left(\frac{\rho^{1-t}}{\rho^{1-t}-1}\right)^{p/q}\left(\eta_1\sum_{j=1}^{n_1}x_j^p+\eta_2\sum_{j=n_1+1}^{n_2}x_j^p+\cdots\right), \tag{6}$$

where η_j are given by (5). Define

$$\eta := \sup_{j\geqslant 1}\left(n_j^{tp/q}\sum_{s=j}^{\infty}a_{n_s}^p n_s^{(1-t)p/q}\right)^{1/p}.$$

Plugging back to (6), we deduce

$$\|Bx\|_p \leqslant \eta\left(\frac{\rho^{1-t}}{\rho^{1-t}-1}\right)^{1/q}\|x\|_p,$$

or equivalently

$$\|B\|_{\ell^p\to\ell^p} \leqslant \eta\left(\frac{\rho^{1-t}}{\rho^{1-t}-1}\right)^{1/q}. \tag{7}$$

Since $A=B+B^*-D$, we conclude

$$\|A\|_{\ell^p\to\ell^p} \leqslant 2\eta\left(\frac{\rho^{1-t}}{\rho^{1-t}-1}\right)^{1/q}+\|(a_n)\|_\infty.$$

If we apply the estimation (7) to the matrix introduced in Corollary 1, with $t = 1/2$, we get

$$\|B\|_{\ell^2\to\ell^2} \leqslant \frac{\sqrt{N}}{\sqrt{N}-1}.$$

On the other hand, fixing J, let $x = (1,1,\ldots,1,0,0,0,\ldots)$, where 1 repeats N^J times. Then

$$\|x\|^2 = N^J$$

while, by considering the coordinates of $y = Bx$,

$$\|Bx\|^2 \geqslant N + N^2 + \cdots + N^J = \frac{N^{J+1}-N}{N-1}.$$

Thus,

$$\|B\|^2 \geqslant \frac{N-N^{1-J}}{N-1}.$$

Since J is arbitrary, letting $J \to \infty$, we get

$$\|B\|^2 \geqslant \frac{N}{N-1}.$$

Therefore, we have the estimation

$$\frac{\sqrt{N}}{\sqrt{N-1}} \leqslant \|B\| \leqslant \frac{\sqrt{N}}{\sqrt{N}-1}.$$

Therefore, the estimation (7) is not far from being optimal.

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