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ON m–QUASI–TOTALLY–(α,β)–NORMAL OPERATORS

R. Pradeep*, P. Maheswari Naik and Sid Ahmed Ould Ahmed Mahmoud

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Abstract. An operator $\mathscr{S}$ acting on a Hilbert space is called m-quasi-totally-(α,β)-normal $(0 \leqslant \alpha \leqslant 1 \leqslant \beta)$ if

$$\alpha^2 \mathscr{S}^{m*}(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\mathscr{S}^m \leqslant \mathscr{S}^{m*}(\mathscr{S}-\lambda)(\mathscr{S}-\lambda)^*\mathscr{S}^m \leqslant \beta^2 \mathscr{S}^{m*}(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\mathscr{S}^m$$

for a natural number m and for all $\lambda \in \mathbb{C}$. m-quasi-totally-(α,β)-normal operator is equivalent to the study of mutual majorization between $(\mathscr{S}-\lambda)\mathscr{S}^m$ and $(\mathscr{S}-\lambda)^*\mathscr{S}^m$ for a natural number m and for all $\lambda \in \mathbb{C}$. This article aims to establish various inequalities between the operator norm and the numerical radius of m-quasi-totally-(α,β)-normal operators in Hilbert spaces. Further, this article analyzes spectral and algebraic properties of m-quasi-totally-(α,β)-normal operators.

1. Introduction

One of the current trends in operator theory is studying natural generalization of normal operators. Let $\mathscr{H}$ be a non zero complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. Let m be a natural number. An operator $\mathscr{S} \in \mathscr{B}(\mathscr{H})$ is called m-quasi-totally-(α,β)-normal $(0 \leqslant \alpha \leqslant 1 \leqslant \beta)$ if

$$\begin{aligned}\alpha^2 \mathscr{S}^{m*}(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\mathscr{S}^m &\leqslant \mathscr{S}^{m*}(\mathscr{S}-\lambda)(\mathscr{S}-\lambda)^*\mathscr{S}^m \\ &\leqslant \beta^2 \mathscr{S}^{m*}(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\mathscr{S}^m\end{aligned}$$

for all $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}\alpha^2 \langle \mathscr{S}^{m*}(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\mathscr{S}^m x,x\rangle &\leqslant \langle \mathscr{S}^{m*}(\mathscr{S}-\lambda)(\mathscr{S}-\lambda)^*\mathscr{S}^m x,x\rangle \\ &\leqslant \beta^2 \langle \mathscr{S}^{m*}(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\mathscr{S}^m x,x\rangle,\end{aligned}$$

whence

$$\begin{aligned}\alpha\|(\mathscr{S}-\lambda)\mathscr{S}^m x\| &\leqslant \|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\| \\ &\leqslant \beta\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|\end{aligned}$$

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* Corresponding author.

for all $\lambda \in \mathbb{C}$ and for all $x \in \mathscr{H}$.

In 1966, R. G. Douglas [12] proved an equivalence of factorization, range inclusion and majorization of operators, known as Douglas lemma. Inspired by the Douglas lemma, V. Manuilov, M. S. Moslehian, and Q. Xu [21] investigated the solvability of the operator equation $AX = C$ for operators on Hilbert C^*-modules. M.S. Moslehian, M. Kian, and Q. Xu [23] characterized the positivity of 2×2 block matrices of operators on Hilbert space. Note that m-quasi-totally-(α, β)-normal operator is equivalent to the study of mutual majorization between $(\mathscr{S} - \lambda)\mathscr{S}^m$ and $(\mathscr{S} - \lambda)^*\mathscr{S}^m$. It can be said that both $(\mathscr{S} - \lambda)\mathscr{S}^m$ majorizes $(\mathscr{S} - \lambda)^*\mathscr{S}^m$ and $(\mathscr{S} - \lambda)^*\mathscr{S}^m$ majorizes $(\mathscr{S} - \lambda)\mathscr{S}^m$ for a natural number m. Using Douglas' result, it is observed that $\mathscr{S}$ is m-quasi-totally-(α, β)-normal if and only if $ran((\mathscr{S} - \lambda)\mathscr{S}^m) = ran((\mathscr{S} - \lambda)^*\mathscr{S}^m)$ or equivalently $ker((\mathscr{S} - \lambda)\mathscr{S}^m) = ker((\mathscr{S} - \lambda)^*\mathscr{S}^m)$. In particular (choose $\lambda = 0$), an operator $\mathscr{S}$ is called m-quasi-(α, β)-normal $(0 \leqslant \alpha \leqslant 1 \leqslant \beta)$ if

$$\begin{aligned} \alpha^2 \mathscr{S}^{*m+1}\mathscr{S}^{m+1} &\leqslant \mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m \\ &\leqslant \beta^2 \mathscr{S}^{*m+1}\mathscr{S}^{m+1}. \end{aligned}$$

Equivalently

$$\begin{aligned} \alpha \|\mathscr{S}^{m+1}x\| &\leqslant \|\mathscr{S}^*\mathscr{S}^m x\| \\ &\leqslant \beta \|\mathscr{S}^{m+1}x\| \end{aligned}$$

for all $x \in \mathscr{H}$.

The numerical radius $\omega(\mathscr{S})$ of an operator $\mathscr{S}$ on $(\mathscr{H}; \langle ., . \rangle)$ is given as

$$\omega(\mathscr{S}) = \sup\{|\langle \mathscr{S}x, x\rangle| : \|x\| = 1\}.$$

S. S. Dragomir and M. S. Moslehian [10] studied various inequlities between the operator norm and the numerical radius of (α, β)-normal operators in Hilbert space. According to them, an operator $\mathscr{S}$ is called (α, β)-normal $(0 \leqslant \alpha \leqslant 1 \leqslant \beta)$ if $\alpha^2 \mathscr{S}^*\mathscr{S} \leqslant \mathscr{S}\mathscr{S}^* \leqslant \beta^2 \mathscr{S}^*\mathscr{S}$. It is true from the definition that, m-quasi-totally-(α, β)-normal operator coincides with (α, β)-normal operator if $m = 0$ and $\lambda = 0$. A. Benali and O. A. M. Sid Ahmed [4] studied structural properties of (α, β)-A-normal operators in semi-Hilbertian spaces.

EXAMPLE 1. It is obvious that m-quasi-hyponormal operators are m-quasi-(α, β)-normal for some appropriate values of α and β. The following operator $\mathscr{S}$ in $\mathscr{B}(\mathbb{C}^2)$ is 2-quasi-(α, β)-normal for $\alpha = \sqrt{(15 - \sqrt{221})/2}$ and $\beta = \sqrt{(15 + \sqrt{221})/2}$, which is not normal, quasi-normal, hyponormal and quasi-hyponormal.

$$\mathscr{S} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

2. Inequalities involving numerical radius and operator norm

In this section, the study of some inequalities concerning the numerical radius and norm of m-quasi-totally-(α, β)-normal operators form the substance. It is followed by several inequalities refer the articles [6, 7, 8, 9, 11, 13, 14].

THEOREM 1. *If $\mathscr{S} \in \mathscr{B}(\mathscr{H})$ is m-quasi-totally-(α,β)-normal operator, then*

$$\left(1+\frac{\alpha^{2r}}{\beta^{2r}}\right)\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^2 \leqslant \begin{cases} \frac{2}{\beta}\omega\left[\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m\right]+\frac{r^2}{\beta^2}\|\beta(\mathscr{S}-\lambda)\mathscr{S}^m-(\mathscr{S}-\lambda)^*\mathscr{S}^m\|^2 & \text{if } r \geqslant 1, \\ \frac{2}{\beta}\omega\left[\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m\right]+\frac{1}{\beta^2}\|\beta(\mathscr{S}-\lambda)\mathscr{S}^m-(\mathscr{S}-\lambda)^*\mathscr{S}^m\|^2 & \text{if } r < 1. \end{cases}$$

In particular,

$$\left(1+\frac{\alpha^{2r}}{\beta^{2r}}\right)\|\mathscr{S}^{m+1}\|^2 \leqslant \begin{cases} \frac{2}{\beta}\omega\left[\mathscr{S}^{m*}\mathscr{S}^{m+2}\right]+\frac{r^2}{\beta^2}\|\beta\mathscr{S}^{m+1}-\mathscr{S}^*\mathscr{S}^m\|^2 & \text{if } r \geqslant 1, \\ \frac{2}{\beta}\omega\left[\mathscr{S}^{m*}\mathscr{S}^{m+2}\right]+\frac{1}{\beta^2}\|\beta\mathscr{S}^{m+1}-\mathscr{S}^*\mathscr{S}^m\|^2 & \text{if } r < 1. \end{cases}$$

Proof. We use the following inequality [14],

$$\|a\|^{2r}+\|b\|^{2r}-2\|a\|^{r-1}\|b\|^{r-1}.Re\langle a,b\rangle \leqslant \begin{cases} r^2\|a\|^{2r-2}\|a-b\|^2 & \text{if } r \geqslant 1, \\ \|b\|^{2r-2}\|a-b\|^2 & \text{if } r < 1, \end{cases}$$

provided $r \in \mathbb{R}$ and $a,b \in \mathscr{H}$ with $\|a\| \geqslant \|b\|$.

Assume that $r \geqslant 1$. Let $x \in \mathscr{H}$ with $\|x\| = 1$. Applying the above inequality for the choices $a = \beta(\mathscr{S}-\lambda)\mathscr{S}^m x$, $b = (\mathscr{S}-\lambda)^*\mathscr{S}^m x$ we get

$$\begin{aligned} &\|\beta(\mathscr{S}-\lambda)\mathscr{S}^m x\|^{2r}+\|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^{2r} \\ \leqslant\ & r^2\|\beta(\mathscr{S}-\lambda)\mathscr{S}^m x\|^{2r-2}\|\beta(\mathscr{S}-\lambda)\mathscr{S}^m x-(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2 \\ &+2\|\beta(\mathscr{S}-\lambda)\mathscr{S}^m x\|^{r-1}\|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^{r-1} \\ &\times Re\left\langle\beta(\mathscr{S}-\lambda)\mathscr{S}^m x,(\mathscr{S}-\lambda)^*\mathscr{S}^m x\right\rangle \end{aligned}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$ and $r \geqslant 1$.

Therefore

$$\begin{aligned} &\left(\alpha^{2r}+\beta^{2r}\right)\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^{2r} \\ \leqslant\ & r^2\beta^{2r-2}\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^{2r-2}\|\beta(\mathscr{S}-\lambda)\mathscr{S}^m x-(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2 \\ &+2\beta^{2r-1}\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^{2r-2}|\left\langle(\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m x,x\right\rangle|. \end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$, $\|x\| = 1$, we deduce

$$\begin{aligned} &\left(\alpha^{2r}+\beta^{2r}\right)\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^{2r} \\ \leqslant\ & r^2\beta^{2r-2}\|\beta(\mathscr{S}-\lambda)\mathscr{S}^m-(\mathscr{S}-\lambda)^*\mathscr{S}^m\|^2+2\beta^{2r-1}\omega\left[\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m\right], \end{aligned}$$

which is the first inequality.

If $r<1$, then similar substitution yields the second inequality. □

THEOREM 2. *If $\mathscr{S}\in\mathscr{B}(\mathscr{H})$ is m-quasi-totally-(α,β)-normal operator and if $k\in\mathbb{C}$, then*

$$\alpha\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^2\leqslant\omega\left[\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m\right]+\frac{2\beta\|(\mathscr{S}-\lambda)\mathscr{S}^m-k(\mathscr{S}-\lambda)^*\mathscr{S}^m\|^2}{(1+|k|\alpha)^2}.$$

In particular,

$$\alpha\|\mathscr{S}^{m+1}\|^2\leqslant\omega\left[\mathscr{S}^{m*}\mathscr{S}^{m+2}\right]+\frac{2\beta\|\mathscr{S}^{m+1}-k\mathscr{S}^*\mathscr{S}^m\|^2}{(1+|k|\alpha)^2}.$$

Proof. We use the following inequality [13],

$$\|a\|\|b\|\leqslant|\langle a,b\rangle|+\frac{2\|a\|\|b\|\|a-b\|^2}{(\|a\|+\|b\|)^2}\ \text{ for } a,b\in\mathscr{H}\setminus\{0\}.$$

Take $a=(\mathscr{S}-\lambda)\mathscr{S}^m x$ and $b=k(\mathscr{S}-\lambda)^*\mathscr{S}^m x$ to get

$$\begin{aligned}&\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|\|k(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|\\ \leqslant\ &|\langle(\mathscr{S}-\lambda)\mathscr{S}^m x,k(\mathscr{S}-\lambda)^*\mathscr{S}^m x\rangle|\\ &+\frac{2\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|\|k(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|\|(\mathscr{S}-\lambda)\mathscr{S}^m x-k(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2}{\left(\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|+\|k(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|\right)^2}.\end{aligned}$$

From this we deduce that

$$\begin{aligned}&\alpha\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2\\ \leqslant\ &\left|\left\langle\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m x,x\right\rangle\right|+\frac{2\beta\|(\mathscr{S}-\lambda)\mathscr{S}^m x-k(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2}{(1+|k|\alpha)^2}.\end{aligned}$$

Taking the supremum over $x\in\mathscr{H}$, $\|x\|=1$, we get the desired result. □

THEOREM 3. *If $\mathscr{S}\in\mathscr{B}(\mathscr{H})$ is m-quasi-totally-(α,β)-normal operator and if $k\in\mathbb{C}\setminus\{0\}$, then*

$$\left[\alpha^2-\left(\frac{1}{|k|}+\beta\right)^2\right]\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^4\leqslant\omega\left[\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m\right]^2.$$

In particualr,

$$\left[\alpha^2-\left(\frac{1}{|k|}+\beta\right)^2\right]\|\mathscr{S}^{m+1}\|^4\leqslant\omega\left[\mathscr{S}^{m*}\mathscr{S}^{m+2}\right]^2.$$

Proof. We use the following inequality [9],

$$\|a\|^2\|b\|^2 \leqslant |\langle a,b\rangle|^2 + \frac{1}{|k|^2}\|a\|^2\|a-kb\|^2,$$

provided $a,b \in \mathscr{H}$ and $k \in \mathbb{C}\setminus\{0\}$.

Choose $a = (\mathscr{S}-\lambda)\mathscr{S}^m x, b = (\mathscr{S}-\lambda)^*\mathscr{S}^m x$, to get

$$\begin{aligned}
&\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2\|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2\\
&\leqslant |\left\langle(\mathscr{S}-\lambda)\mathscr{S}^m x,(\mathscr{S}-\lambda)^*\mathscr{S}^m x\right\rangle|^2\\
&\quad+\frac{1}{|k|^2}\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2\|(\mathscr{S}-\lambda)\mathscr{S}^m x-k(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2.
\end{aligned}$$

This gives

$$\begin{aligned}
&\alpha^2\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^4-\frac{1}{|k|^2}\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^4(1+|k|\beta)^2\\
&\leqslant |\left\langle\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m x,x\right\rangle|^2.
\end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$, $\|x\|=1$, we get the desired result. $\square$

THEOREM 4. *If $\mathscr{S}\in\mathscr{B}(\mathscr{H})$ is m-quasi-totally-(α,β)-normal operator, then*

$$2\omega[(\mathscr{S}-\lambda)\mathscr{S}^m]\,\omega\left[(\mathscr{S}-\lambda)^*\mathscr{S}^m\right]\leqslant\beta\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^2+\omega\left[\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m\right].$$

In particular,

$$2\omega\left[\mathscr{S}^{m+1}\right]\omega\left[\mathscr{S}^*\mathscr{S}^m\right]\leqslant\beta\|\mathscr{S}^{m+1}\|^2+\omega\left[\mathscr{S}^{m*}\mathscr{S}^{m+2}\right].$$

Proof. We use the following inequality [6],

$$2|\langle a,e\rangle\langle e,b\rangle|\leqslant\|a\|\|b\|+|\langle a,b\rangle|$$

for any $a,b,e\in\mathscr{H}$ with $\|e\|=1$.

Let $x\in\mathscr{H}$ with $\|x\|=1$. Put $e=x$, $a=(\mathscr{S}-\lambda)\mathscr{S}^m x$, $b=(\mathscr{S}-\lambda)^*\mathscr{S}^m x$ in the above inequality to get

$$\begin{aligned}
&2|\langle(\mathscr{S}-\lambda)\mathscr{S}^m x,x\rangle\left\langle x,(\mathscr{S}-\lambda)^*\mathscr{S}^m x\right\rangle|\\
&\leqslant\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|\|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|+|\left\langle(\mathscr{S}-\lambda)\mathscr{S}^m x,(\mathscr{S}-\lambda)^*\mathscr{S}^m x\right\rangle|.
\end{aligned}$$

Hence

$$\begin{aligned}
&2|\langle(\mathscr{S}-\lambda)\mathscr{S}^m x,x\rangle\langle\mathscr{S}^{m*}(\mathscr{S}-\lambda)x,x\rangle|\\
&\leqslant\beta\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^2+\left|\left\langle\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m x,x\right\rangle\right|.
\end{aligned}$$

Taking the supremum over $x\in\mathscr{H}$, $\|x\|=1$, we obtain the required inequality. $\square$

THEOREM 5. *If $\mathscr{S} \in \mathscr{B}(\mathscr{H})$ is m-quasi-totally-(α,β)-normal operator and if $p \geqslant 2$, then*

$$\begin{aligned}&(1+\alpha^p)\,\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m\|^p\\ \leqslant\;&\frac{1}{2}\left(\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m+(\mathscr{S}-\lambda)^*\,\mathscr{S}^m\|^p+\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m-(\mathscr{S}-\lambda)^*\,\mathscr{S}^m\|^p\right).\end{aligned}$$

In particular,

$$(1+\alpha^p)\,\|\mathscr{S}^{m+1}\|^p \leqslant \frac{1}{2}\left(\|\mathscr{S}^{m+1}+\mathscr{S}^*\mathscr{S}^m\|^p+\|\mathscr{S}^{m+1}-\mathscr{S}^*\mathscr{S}^m\|^p\right).$$

Proof. We use the following inequality [11],

$$\|a\|^p+\|b\|^p \leqslant \frac{1}{2}\left(\|a+b\|^p+\|a-b\|^p\right)$$

for any $a,b \in \mathscr{H}$ and $p \geqslant 2$.

Now, if we choose $a=(\mathscr{S}-\lambda)\,\mathscr{S}^m x, b=(\mathscr{S}-\lambda)^*\,\mathscr{S}^m x$, then we get

$$\begin{aligned}&\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m x\|^p+\|\,(\mathscr{S}-\lambda)^*\,\mathscr{S}^m x\|^p\\ \leqslant\;&\frac{1}{2}\left(\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m x+(\mathscr{S}-\lambda)^*\,\mathscr{S}^m x\|^p+\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m x-(\mathscr{S}-\lambda)^*\,\mathscr{S}^m x\|^p\right).\end{aligned}$$

Since $\mathscr{S}$ is m-quasi-totally-(α,β)-normal operator, we have

$$\begin{aligned}&(1+\alpha^p)\,\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m x\|^p\\ \leqslant\;&\frac{1}{2}\left(\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m x+(\mathscr{S}-\lambda)^*\,\mathscr{S}^m x\|^p+\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m x-(\mathscr{S}-\lambda)^*\,\mathscr{S}^m x\|^p\right).\end{aligned}$$

Taking the supremum over $\|x\|=1$ in this inequality, we get the desired result. □

THEOREM 6. *If $\mathscr{S} \in \mathscr{B}(\mathscr{H})$ is m-quasi-totally-(α,β)-normal operator. If $p \geqslant 2$, then*

$$\begin{aligned}&\omega\left[\frac{\mathscr{S}^{m*}\,(\mathscr{S}-\lambda)^*\,(\mathscr{S}-\lambda)\,\mathscr{S}^m+\mathscr{S}^{m*}\,(\mathscr{S}-\lambda)\,(\mathscr{S}-\lambda)^*\,\mathscr{S}^m}{2}\right]^{\frac{p}{2}}\\ \leqslant\;&\frac{1}{4}\left(\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m+(\mathscr{S}-\lambda)^*\,\mathscr{S}^m\|^p+\|\,(\mathscr{S}-\lambda)\,\mathscr{S}^m-(\mathscr{S}-\lambda)^*\,\mathscr{S}^m\|^p\right).\end{aligned}$$

In particular,

$$\begin{aligned}&\omega\left[\frac{\mathscr{S}^{m+1*}\,\mathscr{S}^{m+1}+\mathscr{S}^{m*}\,\mathscr{S}\,\mathscr{S}^*\,\mathscr{S}^m}{2}\right]^{\frac{p}{2}}\\ \leqslant\;&\frac{1}{4}\left(\|\mathscr{S}^{m+1}+\mathscr{S}^*\mathscr{S}^m\|^p+\|\mathscr{S}^{m+1}-\mathscr{S}^*\mathscr{S}^m\|^p\right).\end{aligned}$$

Proof. We use the following elementary inequality,

$$a^q+b^q \geqslant 2^{1-q}(a+b)^q$$

for $a,b \geqslant 0$ and $q \geqslant 1$.

Substitute $a=\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2$, $b=\|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2$ and $q=\frac{p}{2}$ to get,

$$\begin{aligned}&\left(\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2\right)^{\frac{p}{2}}+\left(\|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2\right)^{\frac{p}{2}}\\ &\geqslant 2^{1-\frac{p}{2}}\left(\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2+\|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2\right)^{\frac{p}{2}}\end{aligned}$$

whence

$$\begin{aligned}&\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^p+\|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^p\\ &\geqslant 2^{1-\frac{p}{2}}\left(\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2+\|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^2\right)^{\frac{p}{2}}.\end{aligned}$$

Applying Theorem 5,

$$\begin{aligned}&\frac{1}{4}\left(\|(\mathscr{S}-\lambda)\mathscr{S}^m x+(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^p+\|(\mathscr{S}-\lambda)\mathscr{S}^m x-(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^p\right)\\ &\geqslant\left|\left\langle\left(\frac{\mathscr{S}^{m*}(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\mathscr{S}^m+\mathscr{S}^{m*}(\mathscr{S}-\lambda)(\mathscr{S}-\lambda)^*\mathscr{S}^m}{2}\right)x,x\right\rangle\right|^{\frac{p}{2}}.\end{aligned}$$

Taking the supremum over $x\in\mathscr{H}$, $\|x\|=1$ gives the desired result. □

THEOREM 7. *Let $\mathscr{S}\in\mathscr{B}(\mathscr{H})$ be an m-quasi-totally-(α,β)-normal operator. If $p\in(1,2)$ and if $k,l\in\mathbb{C}$, then*

$$\begin{aligned}&(|k|+\beta|l|)^p\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^p+\max\left(\{|k|-|l|\beta,\alpha|l|-|k|\}\right)\|\mathscr{S}-\lambda\mathscr{S}^m\|^p\\ &\leqslant\|k(\mathscr{S}-\lambda)\mathscr{S}^m+l(\mathscr{S}-\lambda)^*\mathscr{S}^m\|^p+\|k(\mathscr{S}-\lambda)\mathscr{S}^m-l(\mathscr{S}-\lambda)^*\mathscr{S}^m\|^p.\end{aligned}$$

Proof. We use the following inequality [11],

$$(\|a\|+\|b\|)^p+(\|a\|-\|b\|)^p\leqslant\|a+b\|^p+\|a-b\|^p$$

for any $a,b\in\mathscr{H}$ and $p\in(1,2)$.

Put $a=k(\mathscr{S}-\lambda)\mathscr{S}^m x$ and $b=l(\mathscr{S}-\lambda)^*\mathscr{S}^m x$ to get

$$\begin{aligned}&\left(\|k(\mathscr{S}-\lambda)\mathscr{S}^m x\|+\|l(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|\right)^p\\ &+\left(\|k(\mathscr{S}-\lambda)\mathscr{S}^m x\|-\|l(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|\right)^p\\ &\leqslant\|k(\mathscr{S}-\lambda)\mathscr{S}^m x+l(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^p+\|k(\mathscr{S}-\lambda)\mathscr{S}^m x-l(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^p.\end{aligned}$$

Since $\mathscr{S}$ is m-quasi-totally-(α,β)-normal operator, it follows that

$$\begin{aligned}&[(|k|+\beta|l|)^p\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^p+\max\{|k|-|l|\beta,\alpha|l|-|k|\}]\|\mathscr{S}-\lambda\mathscr{S}^m x\|^p\\ &\leqslant\|k(\mathscr{S}-\lambda)\mathscr{S}^m x+l(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^p+\|k(\mathscr{S}-\lambda)\mathscr{S}^m x-l(\mathscr{S}-\lambda)^*\mathscr{S}^m x\|^p.\end{aligned}$$

Taking the supremum over $x\in\mathscr{H}$, $\|x\|=1$ gives the desired result. □

THEOREM 8. *Let $\mathscr{S} \in \mathscr{B}(\mathscr{H})$ be an m-quasi-totally-(α,β)-normal operator, $r \geqslant 0$ and $\lambda \in \mathbb{C} \setminus \{0\}$. If*

$$\|k(\mathscr{S}-\lambda)^* \mathscr{S}^m - (\mathscr{S}-\lambda)\mathscr{S}^m\| \leqslant r$$

and

$$\frac{r}{|k|} \leqslant \inf\{\|(\mathscr{S}-\lambda)^* \mathscr{S}^m x\| : \|x\| = 1\},$$

then

$$\alpha^2\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^4 \leqslant \omega\left[\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m\right]^2 + \frac{r^2}{|k|^2}\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^2.$$

In particular,

$$\alpha^2\|\mathscr{S}^{m+1}\|^4 \leqslant \omega\left[\mathscr{S}^{m*}\mathscr{S}^{m+2}\right]^2 + \frac{r^2}{|k|^2}\|\mathscr{S}^{m+1}\|^2.$$

Proof. We use the following inequality [7],

$$\|y\|^2\|a\|^2 \leqslant [Re\langle y,a\rangle]^2 + r^2\|y\|^2$$

provided $\|y-a\| \leqslant r \leqslant \|a\|$.

Setting $a = k(\mathscr{S}-\lambda)^* \mathscr{S}^m x$ and $y = (\mathscr{S}-\lambda)\mathscr{S}^m x$ to get,

$$\begin{aligned}&\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2\|k(\mathscr{S}-\lambda)^* \mathscr{S}^m x\|^2\\ &\leqslant \left[Re\left\langle(\mathscr{S}-\lambda)\mathscr{S}^m x, k(\mathscr{S}-\lambda)^* \mathscr{S}^m x\right\rangle\right]^2 + r^2\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2.\end{aligned}$$

Hence

$$|k|^2\alpha^2\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^4 \leqslant |k|^2\left[Re\left\langle \mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m x, x\right\rangle\right]^2 + r^2\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^2.$$

Taking the supremum over $x \in \mathscr{H}$, $\|x\| = 1$, we get the desired result. □

THEOREM 9. *Let $\mathscr{S} \in \mathscr{B}(\mathscr{H})$ be an m-quasi-totally-(α,β)-normal operator, $r \geqslant 0$ and $\lambda \in \mathbb{C} \setminus \{0\}$. If $\|k(\mathscr{S}-\lambda)^* \mathscr{S}^m - (\mathscr{S}-\lambda)\mathscr{S}^m\| \leqslant r$, then*

$$\alpha\|(\mathscr{S}-\lambda)\mathscr{S}^m\|^2 \leqslant \omega\left[\mathscr{S}^{m*}(\mathscr{S}-\lambda)^2\mathscr{S}^m\right] + \frac{r^2}{2|k|}.$$

In particular,

$$\alpha\|\mathscr{S}^{m+1}\|^2 \leqslant \omega\left[\mathscr{S}^{m*}\mathscr{S}^{m+2}\right] + \frac{r^2}{2|k|}.$$

Proof. We use the following reverse of the Schwarz inequality [8],

$$\|y\|\| a\| \leqslant [Re\langle y,a\rangle] + \frac{r^2}{2}$$

provided $\|y-a\| \leqslant r$.

Setting $a = k(\mathscr{S}-\lambda)^* \mathscr{S}^m x$ and $y = (\mathscr{S}-\lambda)\mathscr{S}^m x$ to get

$$\|(\mathscr{S}-\lambda)\mathscr{S}^m x\| \|k(\mathscr{S}-\lambda)^* \mathscr{S}^m x\| \leqslant \left[Re\left\langle(\mathscr{S}-\lambda)\mathscr{S}^m x, k(\mathscr{S}-\lambda)^* \mathscr{S}^m x\right\rangle\right] + \frac{r^2}{2}.$$

From this we obtain

$$|k|\alpha\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|^2 \leqslant |k||Re\left\langle \mathscr{S}^{m*}(\mathscr{S}-\lambda)^2 \mathscr{S}^m x, x\right\rangle| + \frac{r^2}{2}.$$

Taking the supremum over $\|x\| = 1$ in this inequality, we get the desired result. $\square$

3. Algebraic and spectral properties

The following theorem gives a characterization of m-quasi-totally-(α,β)-normal operators. Using this result we obtained several important properties of this class of operators.

THEOREM 10. *Let $\mathscr{S} \in \mathscr{B}(\mathscr{H})$ such that $\mathscr{S}^m$ does not have a dense range, then the following statements are equivalent.*

(1) *$\mathscr{S}$ is a m-quasi-totally-(α,β)-normal operator.*

(2) *$\mathscr{S} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ on $\mathscr{H} = \overline{ran(\mathscr{S}^m)} \oplus ker(\mathscr{S}^{*m})$, where $A = \mathscr{S}|_{\overline{ran(\mathscr{S})}}$ is a totally (α,β)-normal operator and $C^m = 0$. Furthermore $\sigma(\mathscr{S}) = \sigma(A) \cup \{0\}$.*

Proof. $(1) \Rightarrow (2)$. Consider the matrix representation of $\mathscr{S}$ with respect to the decomposition $\mathscr{H} = \overline{ran(\mathscr{S}^m)} \oplus ker(\mathscr{S}^{*m})$: $\mathscr{S} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Let P be the projection onto $\overline{ran(\mathscr{S}^m)}$. Then $\mathscr{S} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \mathscr{S}P = P\mathscr{S}P$. Since $\mathscr{S}$ is an m-quasi totally-(α,β)-normal operator, we have then

$$\begin{aligned} \alpha^2 P\Big(\mathscr{S}^{*m}(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\mathscr{S}^m\Big)P &\leqslant P\Big(\mathscr{S}^{*m}(\mathscr{S}-\lambda)(\mathscr{S}-\lambda)^*\mathscr{S}^m\Big)P \\ &\leqslant \beta^2 P\Big(\mathscr{S}^{*m}(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\mathscr{S}^m\Big)P. \end{aligned}$$

That is

$$\alpha^2(A-\lambda)^*(A-\lambda) \leqslant (A-\lambda)(A-\lambda)^* \leqslant \beta^2\Big((A-\lambda)^*(A-\lambda)\Big),$$

for all $\lambda \in \mathbb{C}$. Hence A is a totally-(α,β)-normal.

On the other hand, let $x = x_1 + x_2 \in \mathscr{H} = \overline{ran(\mathscr{S}^m)} \oplus ker(\mathscr{S}^{*m})$. A simple computation shows that

$$\begin{aligned}\langle C^m x_2, x_2\rangle &= \langle \mathscr{S}^m (I-P)x, (I-P)x\rangle \\ &= \langle (I-P)x, \mathscr{S}^{*m}(I-P)x\rangle = 0.\end{aligned}$$

So, $C^m = 0$.

Since $\sigma(\mathscr{S}) \cup \mathscr{T} = \sigma(A) \cup \sigma(C)$, where $\mathscr{T}$ is the union of the holes in $\sigma(\mathscr{S})$ which happen to be subset of $\sigma(A) \cap \sigma(C)$ by Corollary 7 of [16], and $\sigma(A) \cap \sigma(C)$ has no interior point and C is nilpotent, we have $\sigma(\mathscr{S}) = \sigma(A) \cup \{0\}$.

$(2) \Rightarrow (1)$ Suppose that $\mathscr{S} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ onto $\mathscr{H} = \overline{ran(\mathscr{S}^m)} \oplus ker(\mathscr{S}^{*m})$, with

$$\alpha^2 (A-\lambda)^*(A-\lambda) \leqslant (A-\lambda)(A-\lambda)^* \leqslant \beta^2 \Big((A-\lambda)^*(A-\lambda) \Big),$$

for all $\lambda \in \mathbb{C}$ and $C^m = 0$.

Since $\mathscr{S}^m = \begin{pmatrix} A^m & \sum\limits_{j=0}^{m-1} A^j B C^{k-1-j} \\ 0 & 0 \end{pmatrix}$,

$$(\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda) = \begin{pmatrix} (A-\lambda)^*(A-\lambda) & (A-\lambda)^* B \\ B^*(A-\lambda) & B^*B + (C-\lambda)^*(C-\lambda) \end{pmatrix}$$

and

$$(\mathscr{S}-\lambda)(\mathscr{S}-\lambda)^* = \begin{pmatrix} (A-\lambda)(A-\lambda)^* & B(A-\lambda)^* \\ (A-\lambda)B^* & BB^* + (C-\lambda)(C-\lambda)B^* \end{pmatrix}.$$

Further

$$\begin{aligned}\mathscr{S}^m \mathscr{S}^{*m} &= \begin{pmatrix} A^k A^{*k} + \left(\sum\limits_{j=0}^{m-1} A^j B C^{k-1-j}\right)\left(\sum\limits_{j=0}^{m-1} A^j B C^{k-1-j}\right)^* & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.\end{aligned}$$

where $D = A^k A^{*k} + \left(\sum\limits_{j=0}^{m-1} A^j B C^{k-1-j}\right)\left(\sum\limits_{j=0}^{m-1} A^j B C^{k-1-j}\right)^* = D^*$.

Hence for all $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
&\alpha^2 \mathscr{S}^m \mathscr{S}^{*m}\big((\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\big)\mathscr{S}^m \mathscr{S}^{*m} \\
&= \begin{pmatrix} \alpha^2 D(A-\lambda)^*(A-\lambda)D & 0 \\ 0 & 0 \end{pmatrix} \\
&\leqslant \begin{pmatrix} D(A-\lambda)(A-\lambda)^*D & 0 \\ 0 & 0 \end{pmatrix} = \mathscr{S}^m \mathscr{S}^{*m}\big((\mathscr{S}-\lambda)(\mathscr{S}-\lambda)^*\big)\mathscr{S}^m \mathscr{S}^{*m} \\
&\leqslant \begin{pmatrix} \beta^2 D(A-\lambda)^*(A-\lambda)D & 0 \\ 0 & 0 \end{pmatrix} = \beta^2 \mathscr{S}^m \mathscr{S}^{*m}\big((\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\big)\mathscr{S}^m \mathscr{S}^{*m}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\alpha^2 \mathscr{S}^m \mathscr{S}^{*m}\big((\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\big)\mathscr{S}^m \mathscr{S}^{*m} \\
\leqslant\ & \mathscr{S}^m \mathscr{S}^{*m}\big((\mathscr{S}-\lambda)(\mathscr{S}-\lambda)^*\big)\mathscr{S}^m \mathscr{S}^{*m} \\
\leqslant\ & \beta^2 \mathscr{S}^m \mathscr{S}^{*m}\big((\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\big)\mathscr{S}^m \mathscr{S}^{*m}.
\end{aligned}$$

This means that

$$\begin{aligned}
\alpha^2 \mathscr{S}^{*m}\big((\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\big)\mathscr{S}^m &\leqslant \mathscr{S}^{*m}\big((\mathscr{S}-\lambda)(\mathscr{S}-\lambda)^*\big)\mathscr{S}^m \\
&\leqslant \beta^2 \mathscr{S}^{*m}\big((\mathscr{S}-\lambda)^*(\mathscr{S}-\lambda)\big)\mathscr{S}^m,
\end{aligned}$$

on $\mathscr{H} = ran(\mathscr{S}^{*m}) \oplus ker(\mathscr{S}^m)$.

Consequently, $\mathscr{S}$ is a m-quasi-totally-(α,β)-normal. $\square$

COROLLARY 1. *Let $\mathscr{S} \in \mathscr{L}(\mathscr{H})$ be an m-quasi totally-(α,β)-normal operator. If $A = \mathscr{S}_{/\overline{ran(\mathscr{S}^m)}}$ is invertible, then $\mathscr{S}$ is similar to a direct sum of a totally-(α,β)-normal operator and a nilpotent operator.*

Proof. By Theorem 10 we write the matrix representation of $\mathscr{S}$ on $\mathscr{H} = \overline{ran(\mathscr{S}^m)} \oplus ker(\mathscr{S}^{*m})$ as follows $\mathscr{S} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ where $A = \mathscr{S}_{/\overline{ran(\mathscr{S}^m)}}$ is a totally-(α,β)-normal operator and $C^m = 0$. Since A is invertible, we have $\sigma(A) \cap \sigma(C) = \emptyset$. Then there exists an operator X such that $AX - XC = B$. Hence

$$\mathscr{S} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}. \quad \square$$

We say that an operator $\mathscr{S}$ doubly commutes with $\mathscr{T}$ if $\mathscr{S}$ commutes with $\mathscr{T}$ and $\mathscr{T}^*$.

THEOREM 11. *Let $\mathscr{S}_1, \mathscr{S}_2 \in \mathscr{B}(\mathscr{H})$ are doubly commuting. If $\mathscr{S}_1$ is an m-quasi-(α,β)-normal and $\mathscr{S}_2$ is an m-quasi-(α',β')-normal, then $\mathscr{S}_1\mathscr{S}_2$ is an m-quasi-$(\alpha\alpha',\beta\beta')$-normal*

Proof.

$$\begin{aligned}\alpha\alpha'\|(\mathscr{S}_1\mathscr{S}_2)^{m+1}x\| = \alpha\alpha'\|\mathscr{S}_1^{m+1}\mathscr{S}_2^{m+1}x\| &\leqslant \alpha'\|\mathscr{S}_1^*\mathscr{S}_1^m\mathscr{S}_2^{m+1}x\| \\ &\leqslant \|\mathscr{S}_2^*\mathscr{S}_1^*\mathscr{S}_1^m\mathscr{S}_2^m x\|\end{aligned}$$

and

$$\begin{aligned}\|\mathscr{S}_2^*\mathscr{S}_2^m\mathscr{S}_1^*\mathscr{S}_1^m x\| &\leqslant \beta'\|\mathscr{S}_2\mathscr{S}_2^m\mathscr{S}_1^*\mathscr{S}_1^m x\| = \beta'\|\mathscr{S}_1^*\mathscr{S}_1^m\mathscr{S}_2\mathscr{S}_2^m x\| \\ &\leqslant \beta\beta'\|\|\mathscr{S}_1^{m+1}\mathscr{S}_2^{m+1}x\|.\end{aligned}$$

$$\alpha\alpha'\|(\mathscr{S}_1\mathscr{S}_2)^{m+1}x\| \leqslant \|(\mathscr{S}_2\mathscr{S}_1)^*\mathscr{S}_1^m\mathscr{S}_2^m x\| \leqslant \beta\beta'\|\|\mathscr{S}_1^{m+1}\mathscr{S}_2^{m+1}x\|. \quad \square$$

THEOREM 12. *Let $(\alpha,\beta)\in\mathbb{R}^2$ such that $0<\alpha\leqslant 1\leqslant\beta$ and let $\mathscr{S}\in\mathscr{B}(\mathscr{H})$ such that $ran(\mathscr{S}^m)=ran(\mathscr{S}^{*m})$. If $\mathscr{S}$ is an m-quasi-(α,β)-normal, then $\mathscr{S}^*$ is m-quasi-$\left(\frac{1}{\beta},\frac{1}{\alpha}\right)$-normal.*

Proof. Since $\mathscr{S}$ is m-quasi-(α,β)-normal, it follows that

$$\alpha\|\mathscr{S}\mathscr{S}^m x\| \leqslant \|\mathscr{S}^*\mathscr{S}^m x\| \leqslant \beta\|\mathscr{S}\mathscr{S}^m x\|, \quad \forall\, x\in\mathscr{H}.$$

This means that

$$\alpha\|\mathscr{S}\mathscr{S}^{*m} x\| \leqslant \|\mathscr{S}^*\mathscr{S}^{*m} x\| \leqslant \beta\|\mathscr{S}\mathscr{S}^{*m} x\|, \quad \forall\, x\in\mathscr{H}.$$

Combining these inequalities,

$$\frac{1}{\beta}\|\mathscr{S}^*\mathscr{S}^{*m}x\| \leqslant \|\mathscr{S}\mathscr{S}^{*m}x\| \leqslant \frac{1}{\alpha}\|\mathscr{S}^*\mathscr{S}^{*m}x\|.$$

So, $\mathscr{S}^*$ is m-quasi-$\left(\frac{1}{\beta},\frac{1}{\alpha}\right)$-normal. $\square$

COROLLARY 2. *Under the same conditions of Theorem 12, if $\alpha\beta=1$ then $\mathscr{S}$ is m-quasi-(α,β)-normal if and only if $\mathscr{S}^*$ is m-quasi-(α,β)-normal.*

THEOREM 13. *Let $\mathscr{S}$ be an m-quasi-totally-(α,β)-normal operator. If $\mathscr{S}^m$ has dense range, then $\mathscr{S}$ is totally-(α,β)-normal.*

Proof. Since $\mathscr{S}^m$ has a dense range, it follows that $\overline{ran(\mathscr{S}^m)}=\mathscr{H}$. Let $y\in\mathscr{H}$. Then there exists a sequence (x_n) in $\mathscr{H}$ such that $\mathscr{S}^m(x_n)\to y$ as $n\to\infty$.

Since $\mathscr{S}$ is m-quasi-totally-(α,β)-normal operator, we have

$$\alpha\|(\mathscr{S}-\lambda)\mathscr{S}^m x\| \leqslant \|(\mathscr{S}-\lambda)^*\mathscr{S}^m x\| \leqslant \beta\|(\mathscr{S}-\lambda)\mathscr{S}^m x\|$$

for all $x\in\mathscr{H}$ and for all $\lambda\in\mathbb{C}$.

In particular,

$$\alpha\|(\mathscr{S}-\lambda)\mathscr{S}^m x_n\| \leqslant \|(\mathscr{S}-\lambda)^*\mathscr{S}^m x_n\| \leqslant \beta\|(\mathscr{S}-\lambda)\mathscr{S}^m x_n\|$$

for all $x_n \in \mathscr{H}$ and for all $\lambda \in \mathbb{C}$.

It follows that

$$\alpha\|(\mathscr{S}-\lambda)y\| \leqslant \|(\mathscr{S}-\lambda)^* y\| \leqslant \beta\|(\mathscr{S}-\lambda)y\|$$

for all $y \in \mathscr{H}$ and for all $\lambda \in \mathbb{C}$. Therefore $\mathscr{S}$ is totally-(α,β)-normal operator. □

COROLLARY 3. *Let $\mathscr{S}$ be an m-quasi-totally-(α,β)-normal operator. If $\mathscr{S}^m \neq 0$ and if $\mathscr{S}$ has no nontrivial $\mathscr{S}^m$-invariant closed subspace, then $\mathscr{S}$ is totally-(α,β)-normal.*

Proof. Since $\mathscr{S}^m$ has no nontrivial invariant closed subspace, it has no nontrivial hyperinvariant subspace. But $ker(\mathscr{S}^m)$ and $\overline{ran(\mathscr{S}^m)}$ are hyperinvariant subspaces, and $\mathscr{S}^m \neq 0$, hence $ker(\mathscr{S}^m) = 0$ and $\overline{ran(\mathscr{S}^m)} = \mathscr{H}$. Therefore $\mathscr{S}$ is totally-(α,β)-normal operator. □

EXAMPLE 2. Let $\mathscr{S}$ be the diagonal positive operator define on the Hilbert space $\mathscr{H} = \ell^2_{\mathbb{N}}(\mathbb{C})$ by $\mathscr{S}e_j = \frac{1}{j!}e_j$, $\forall\, j \in \mathbb{N} = \{1,2,\cdots\}$, where $\{e_j,\ J = 1,2,\cdots\}$ denotes the canonical basis of $\ell^2_{\mathbb{N}}(\mathbb{C})$. It easily to show that $\mathscr{S}$ is an m-quasi-totally-(α,β)-normal operator.

By observing that $\mathscr{S}^m e_j - \left(\frac{1}{j!}\right)^m e_j$ and hence, $e_j = (j!)^m\mathscr{S}^m e_j$, for all $j \in \mathbb{N}$, it easily to see that $ran(\mathscr{S}^m)$ contains the span of $\{e_j,\ j = 1,2,\ \cdots,\}$ and

$$\overline{span\{e_j,\ j=1,2,\ \cdots,\}} = \ell^2_{\mathbb{N}}(\mathbf{C}).$$

However $\overline{ran(\mathscr{S}^m)} = \ell^2_{\mathbb{N}}(\mathbb{C})$. Therefore $\mathscr{S}$ is totally-(α,β)-normal.

Note that $ran(\mathscr{S}^m)$ is not closed. In fact, by choosing $u_0 = \left(\frac{1}{j^m}\right)_{j\geqslant 1} \in \ell^2_{\mathbb{N}}(\mathbb{C})$ and $x = (x_j)_{j\geqslant 1}$ such that $u_0 = \mathscr{S}^m x$, we obtain $\frac{1}{j^m} = \frac{1}{j^m!}x_j$ and hence $x_j = (j^m - 1)!$ for $j = 1,2,\cdots$. So $x \notin \ell^2_{\mathbb{N}}(\mathbb{C})$ and therefore $\mathscr{S}^m$ is not surjective. Hence $\overline{ran(\mathscr{S}^m)} \neq ran(\mathscr{S}^m)$ i.e, $ran(\mathscr{S}^m)$ is not closed. Since $\overline{ran(\mathscr{S}^m)} = \ell^2_{\mathbb{N}}(\mathbb{C})$, it follows that $\mathscr{S}$ has no nontrivial $\mathscr{S}^m$-invariant closed subspace.

COROLLARY 4. *If $\mathscr{S}$ is such that $a+b\mathscr{S}$ is m-quasi-totally-(α,β)-normal operator for all scalars a and b, then $\mathscr{S}$ is totally-(α,β)-normal.*

Proof. If $\mathscr{S}$ is m-quasi-totally-(α,β)-normal operator but not totally-(α,β)-normal operator, then $\mathscr{S}^m$ is not invertible. It is possible to find scalars a and $b \neq 0$

such that $\mathscr{T} = a + b\mathscr{S}$ is invertible m-quasi-totally-(α,β)-normal operator. Therefore $\mathscr{T}$ is totally-(α,β)-normal operators.

$$\mathscr{T} = a + b\mathscr{S} \Rightarrow \mathscr{S} = \frac{1}{b}(\mathscr{T} - a).$$

Therefore $\mathscr{S}$ is also totally-(α,β)-normal. $\square$

EXAMPLE 3. Let $\mathscr{S} = \begin{pmatrix} 1 & 0 & 1 \\ \frac{1}{2} & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathscr{B}(\mathbb{C}^3)$. Then it is 3-quasi-(α,β)-normal and (α,β)-normal for $\alpha = 0.03$ and $\beta = 1.8$.

EXAMPLE 4. Let $\mathscr{S} = \begin{pmatrix} I & 0 \\ I & 0 \end{pmatrix} \in \mathscr{B}(l_2 \oplus l_2)$. Then it is 1-quasi-(α,β)-normal but not (α,β)-normal for $\alpha = 0.5$ and $\beta = 1.5$.

PROPOSITION 1. *Let $\mathscr{S}$ be an m-quasi-totally-(α,β)-normal operator. If a,b are non-zero eigenvalues of $\mathscr{S}$ such that $a \neq b$, then $ker(\mathscr{S} - a) \perp ker(\mathscr{S} - b)$.*

Proof. Let $x \in ker(\mathscr{S} - a)$ and $y \in ker(\mathscr{S} - b)$. Then $\mathscr{S}x = ax$ and $\mathscr{S}y = by$. Therefore $a\langle x,y\rangle = b\langle x,y\rangle$, and so $(a-b)\langle x,y\rangle = 0$. Hence $ker(\mathscr{S} - a) \perp ker(\mathscr{S} - b)$. $\square$

THEOREM 14. *Let $\mathscr{S}$ be an m-quasi-totally-(α,β)-normal operator. If k is a complex number, then $ker(\mathscr{S} - k)$ reduces $\mathscr{S}$ and $\mathscr{S}$ is normal on $ker(\mathscr{S} - k)$.*

Proof. We first prove that $ker(\mathscr{S} - k) \subseteq ker\left(\mathscr{S}^* - \overline{k}\right)$ for each $k \neq 0$.

Suppose $\mathscr{S}x = kx$. Since $\mathscr{S}$ is m-quasi-totally-(α,β)-normal,

$$\alpha\|(\mathscr{S} - \lambda)\mathscr{S}^m y\| \leqslant \|(\mathscr{S} - \lambda)^* \mathscr{S}^m y\| \leqslant \beta\|(\mathscr{S} - \lambda)\mathscr{S}^m y\|$$

for all $y \in \mathscr{H}$ and for all $\lambda \in \mathbb{C}$. In particular,

$$\alpha\|(\mathscr{S} - k)\mathscr{S}^m x\| \leqslant \|(\mathscr{S} - k)^* \mathscr{S}^m x\| \leqslant \beta\|(\mathscr{S} - k)\mathscr{S}^m x\|.$$

It follows that

$$\alpha\|(\mathscr{S} - k)k^m x\| \leqslant \|(\mathscr{S} - k)^* k^m x\| \leqslant \beta\|(\mathscr{S} - k)k^m x\|.$$

Therefore

$$\alpha\|(\mathscr{S} - k)x\| \leqslant \|(\mathscr{S} - k)^* x\| \leqslant \beta\|(\mathscr{S} - k)x\|.$$

This clearly forces $x \in ker(\mathscr{S} - k)^*$.

For any $x \in ker(\mathscr{S} - k)$, we have $(\mathscr{S} - k)(\mathscr{S}x) = 0$, which implies that $\mathscr{S}x \in ker(\mathscr{S} - k)$.

Therefore $\mathscr{S}(ker(\mathscr{S} - k)) \subset ker(\mathscr{S} - k)$.

Also, for $x \in ker(\mathscr{S}-k)$, we have $(\mathscr{S}-k)(\mathscr{S}^*x)=0$, which means that $\mathscr{S}^*x \in ker(\mathscr{S}-k)$.

This gives $\mathscr{S}^*(ker(\mathscr{S}-k)) \subset ker(\mathscr{S}-k)$. Hence $ker(\mathscr{S}-k)$ reduces $\mathscr{S}$.

For $x \in ker(\mathscr{S}-k)$, we have $\mathscr{S}\mathscr{S}^*x = |k|^2x = \mathscr{S}^*\mathscr{S}x$, i.e. $\mathscr{S}$ is normal on $ker(\mathscr{S}-k)$. □

Let $\sigma_{jp}(\mathscr{S})$ and $\sigma_p(\mathscr{S})$ denote the joint point spectrum and point spectrum of $\mathscr{S}$, respectively. It is known that, if $\mathscr{S}$ is normal, then $\sigma_{jp}(\mathscr{S})=\sigma_p(\mathscr{S})$. Of course, if $\mathscr{S}$ is m-quasi-totally-(α,β)-normal operator, then $\sigma_{jp}(\mathscr{S})\setminus\{0\}=\sigma_p(\mathscr{S})\setminus\{0\}$.

COROLLARY 5. *If $\mathscr{S}$ is a pure m-quasi-totally-(α,β)-normal operator, then* $\sigma_p(\mathscr{S})=\phi$

Proof. An operator $\mathscr{S}$ is pure if it has no reducing subsapce on which it is normal. Suppose that $\sigma_p(\mathscr{S})\neq\phi$. By Theorem 14, we have $\mathscr{S}$ is normal on $ker(\mathscr{S}-k)$, it is a contradiction to pure. □

We say that an operator $\mathscr{S}$ has the single valued extension property at $\lambda_0\in\mathbb{C}$, if $f\equiv 0$ is the only solution to $(\mathscr{S}-\lambda)f(\lambda)=0$ that is analytic in an open neighborhood of λ_0. The operator $\mathscr{S}$ is said to have SVEP if it has SVEP at every point λ_0 in the complex plane.

THEOREM 15. *If $\mathscr{S}$ is quasi-(α,β)-normal operator, then $\mathscr{S}$ has SVEP.*

Proof. Suppose $x\in ker(\mathscr{S}^2)$. Then $\mathscr{S}^2x=0$. Since $\mathscr{S}$ is quasi-(α,β)-normal operator,

$$\alpha\|\mathscr{S}^2x\| \leqslant \|\mathscr{S}^*\mathscr{S}x\| \leqslant \beta\|\mathscr{S}^2x\|$$

for all $x\in\mathscr{H}$. Therefore $\|\mathscr{S}^*\mathscr{S}x\|=0$, and hence $x\in ker(\mathscr{S}^*\mathscr{S})$. But $ker(\mathscr{S}^*\mathscr{S})=ker(\mathscr{S})$, hence $x\in ker(\mathscr{S})$. Hence $\mathscr{S}$ has finite ascent. Therefore $\mathscr{S}$ has SVEP by Theorem 3.8 of [1]. □

THEOREM 16. *If $\mathscr{S}$ is m-quasi-(α,β)-normal such that $\alpha\beta=1$, then*

$$\alpha^2\mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m \leqslant \mathscr{S}^{*m+1}\mathscr{S}^{m+1} \leqslant \beta^2\mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m.$$

Proof. $\mathscr{S}$ is m-quasi-totally-(α,β)-normal if and only if

$$\alpha^2\mathscr{S}^{*m+1}\mathscr{S}m+1 \leqslant \mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m \leqslant \beta^2\mathscr{S}^{*m+1}\mathscr{S}^{m+1}.$$

Therefore

$$\alpha^4\mathscr{S}^{*m+1}\mathscr{S}^{m+1} \leqslant \alpha^2\mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m \leqslant \alpha^2\beta^2\mathscr{S}^{*m+1}\mathscr{S}^{m+1}$$

and

$$\beta^2\alpha^2\mathscr{S}^{*m+1}\mathscr{S}^{m+1} \leqslant \beta^2\mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m \leqslant \beta^4\mathscr{S}^{*m+1}\mathscr{S}^{m+1}.$$

Combining these inequalities,

$$\alpha^2\mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m \leqslant \mathscr{S}^{*m+1}\mathscr{S}^{m+1} \leqslant \beta^2\mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m. \quad □$$

PROPOSITION 2. *Direct sum of two m-quasi-(α,β)-normal is also m-quasi-(α,β)-normal but tensor product of two m-quasi-(α,β)-normal is m-quasi-$\left(\alpha^2,\beta^2\right)$-normal.*

Proof. Suppose that $\mathscr{S}_1$ and $\mathscr{S}_2$ are m-quasi-(α,β)-normal.
Let $x = x_1 \oplus x_2 \in \mathscr{H} \oplus \mathscr{H}$ such that $\|x\| = 1$. Then

$$\alpha\|(\mathscr{S}_1 \oplus \mathscr{S}_2)^{m+1}x\| \leqslant \|(\mathscr{S}_1 \oplus \mathscr{S}_2)^*(\mathscr{S}_1 \oplus \mathscr{S}_2)^m x\| \leqslant \beta\|(\mathscr{S}_1 \oplus \mathscr{S}_2)^{m+1}x\|.$$

Let $x = x_1 \times x_2 \in \mathscr{H} \otimes \mathscr{H}$ such that $\|x\| = 1$. Then

$$\alpha^2\|(\mathscr{S}_1 \otimes \mathscr{S}_2)^{m+1}x\| \leqslant \|(\mathscr{S}_1 \otimes \mathscr{S}_2)^*(\mathscr{S}_1 \otimes \mathscr{S}_2)^m x\| \leqslant \beta^2\|(\mathscr{S}_1 \otimes \mathscr{S}_2)^{m+1}x\|. \quad \square$$

Let $\mathscr{S},\mathscr{T} \in \mathscr{B}(\mathscr{H})$. $\mathscr{S}$ and $\mathscr{T}$ have mutually majorized each other if $\alpha^2\mathscr{S}\mathscr{S}^* \leqslant \mathscr{T}\mathscr{T}^* \leqslant \beta^2\mathscr{S}\mathscr{S}^*(0 \leqslant \alpha \leqslant 1 \leqslant \beta)$.

THEOREM 17. *If $\mathscr{S}^2$ and $\mathscr{S}^*\mathscr{S}$ have mutually majorized each other and if $\mathscr{T}$ is an unitary operator such that $\mathscr{S}\mathscr{T} = \mathscr{T}\mathscr{S}$ and $\mathscr{S}^*\mathscr{T} = \mathscr{T}\mathscr{S}^*$, then $(\mathscr{S}\mathscr{T})^2$ and $(\mathscr{S}\mathscr{T})^*(\mathscr{S}\mathscr{T})$ have mutually majorized each other.*

Proof. Since $\mathscr{T}$ is unitary operator, $\mathscr{T}^*\mathscr{T} = \mathscr{T}\mathscr{T}^* = I$. Therefore

$$\alpha^2(\mathscr{S}\mathscr{T})^{*2}(\mathscr{S}\mathscr{T})^2 = \alpha^2\mathscr{S}^{*2}\mathscr{T}^{*2}\mathscr{T}^2\mathscr{S}^2 = \alpha^2\mathscr{S}^{*2}\mathscr{S}^2$$

and

$$(\mathscr{S}\mathscr{T})^*(\mathscr{S}\mathscr{T})(\mathscr{S}\mathscr{T})^*(\mathscr{S}\mathscr{T}) = \mathscr{S}^*\mathscr{S}\mathscr{T}^*\mathscr{T}\mathscr{S}^*\mathscr{S} = \mathscr{S}^*\mathscr{S}\mathscr{S}^*\mathscr{S}.$$

Hence $(\mathscr{S}\mathscr{T})^2$ and $(\mathscr{S}\mathscr{T})^*(\mathscr{S}\mathscr{T})$ have mutually majorized each other. $\square$

THEOREM 18. *If $\mathscr{S}^{m+1}$ and $\mathscr{S}^*\mathscr{S}^m$ have mutually majorized each other and if $\mathscr{T}$ is self-adjoint such that $\mathscr{S}\mathscr{T} = \mathscr{T}\mathscr{S}$, then $(\mathscr{S}\mathscr{T})^{m+1}$ and $(\mathscr{S}\mathscr{T})^*(\mathscr{S}\mathscr{T})^m$ have mutually majorized each other for a natural number m.*

Proof. Since $\mathscr{S}^{m+1}$ and $\mathscr{S}^*\mathscr{S}^m$ have mutually majorized each other, we have for $x \in \mathscr{H}$,

$$\alpha\|\mathscr{S}^{m+1}\mathscr{T}^{m+1}x\| \leqslant \|\mathscr{S}^*\mathscr{S}^m\mathscr{T}^{m+1}x\| \leqslant \beta\|\mathscr{S}^{m+1}\mathscr{T}^{m+1}x\|.$$

On the other hand,

$$\begin{aligned}\|\mathscr{S}^*\mathscr{S}^m\mathscr{T}^{m+1}x\|^2 &= \left\langle \mathscr{S}^*\mathscr{S}^m\mathscr{T}^{m+1}x, \mathscr{S}^*\mathscr{S}^m\mathscr{T}^{m+1}x \right\rangle \\ &= \langle \mathscr{T}^*\mathscr{S}^*(\mathscr{S}\mathscr{T})^m x, \mathscr{T}^*\mathscr{S}^*(\mathscr{S}\mathscr{T})^m x \rangle \\ &= \|(\mathscr{S}\mathscr{T})^*(\mathscr{S}\mathscr{T})^m x\|^2.\end{aligned}$$

This implies $\alpha\|(\mathscr{S}\mathscr{T})^{m+1}x\| \leqslant \|(\mathscr{S}\mathscr{T})^*(\mathscr{S}\mathscr{T})^m x\| \leqslant \beta\|(\mathscr{S}\mathscr{T})^{m+1}x\|$. $\square$

THEOREM 19. *If $\mathscr{S}^{m+1}$ and $\mathscr{S}^*\mathscr{S}^m$ have mutually majorized each other and if $\mathscr{T}$ is unitary equivalent to $\mathscr{S}$, then $\mathscr{T}^{m+1}$ and $\mathscr{T}^*\mathscr{T}^m$ have mutually majorized each other for a natural number m.*

Proof. Let $\mathscr{T}$ be an operator unitary equivalent to $\mathscr{S}$. Then $\mathscr{T} = \mathscr{U}^*\mathscr{S}\mathscr{U}$ for some unitary operator $\mathscr{U}$. Therefore

$$\begin{aligned}\alpha\|\mathscr{T}^{m+1}x\| &= \alpha\|(\mathscr{U}^*\mathscr{S}\mathscr{U})^{m+1}x\| = \alpha\|\mathscr{U}^*\mathscr{S}^{m+1}\mathscr{U}x\| = \alpha\|\mathscr{S}^{m+1}\mathscr{U}x\| \\ &\leqslant \|\mathscr{S}^*\mathscr{S}^m\mathscr{U}x\| = \|\mathscr{U}^*\mathscr{S}^*\mathscr{S}^m\mathscr{U}x\| = \|(\mathscr{U}^*\mathscr{S}\mathscr{U})^*(\mathscr{U}^*\mathscr{S}\mathscr{U})^m x\| \\ &= \|\mathscr{T}^*\mathscr{T}^m x\|\end{aligned}$$

and

$$\begin{aligned}\|\mathscr{T}^*\mathscr{T}^m x\| &= \|(\mathscr{U}^*\mathscr{S}\mathscr{U})^*(\mathscr{U}^*\mathscr{S}\mathscr{U})^m x\| = \|\mathscr{U}^*\mathscr{S}^*\mathscr{S}^m\mathscr{U}x\| = \|\mathscr{S}^*\mathscr{S}^m\mathscr{U}x\| \\ &\leqslant \beta\|\mathscr{S}^{m+1}\mathscr{U}x\| = \beta\|\mathscr{U}^*\mathscr{S}^{m+1}\mathscr{U}x\| = \beta\|(\mathscr{U}^*\mathscr{S}\mathscr{U})^{m+1}x\| \\ &= \beta\|\mathscr{T}^{m+1}x\|.\end{aligned}$$

Hence $\mathscr{T}^{m+1}$ and $\mathscr{T}^*\mathscr{T}^m$ have mutually majorized each other for a natural number m. $\square$

THEOREM 20. *The set $\{\mathscr{S} \in \mathscr{B}(\mathscr{H}) : \alpha^2\mathscr{S}^{*m+1}\mathscr{S}^{m+1} \leqslant \mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m \leqslant \beta^2\mathscr{S}^{*m+1}\mathscr{S}^{m+1}$ and $(0 \leqslant \alpha \leqslant 1 \leqslant \beta)\}$ is arcwise connected for $m \in \mathbb{N}$.*

Proof. It is enough to prove that $k\mathscr{S}$ is m-quasi-(α,β)-normal operator for every non zero complex number k. Now for $x \in \mathscr{H}$,

$$\begin{aligned}\left\langle \alpha^2(k\mathscr{S})^{*m+1}(k\mathscr{S})^{m+1}x,x,\right\rangle &\leqslant \langle (k\mathscr{S})^{m*}(k\mathscr{S})(k\mathscr{S})^*(k\mathscr{S})^m x,x\rangle \\ &\leqslant \left\langle \beta^2(k\mathscr{S})^{*m+1}(k\mathscr{S})^{m+1}x,x\right\rangle.\end{aligned}$$

Therefore $\alpha^2\mathscr{S}^{*m+1}\mathscr{S}^{m+1} \leqslant \mathscr{S}^{m*}\mathscr{S}\mathscr{S}^*\mathscr{S}^m \leqslant \beta^2\mathscr{S}^{*m+1}\mathscr{S}^{m+1}$.

This implies that the class of m-quasi-(α,β)-normal operator is arcwise connected. $\square$

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R. Pradeep
Department of Mathematics
PSG College of Arts and Science
Coimbatore-641 014, Tamil Nadu, India
e-mail: pradeeph123@gmail.com

P. Maheswari Naik
Department of Mathematics
Sri Ramakrishna Engineering College
Coimbatore-641 022, Tamil Nadu, India
e-mail: maheswarinaik21@gmail.com

Sid Ahmed Ould Ahmed Mahmoud
Mathematics Department, College of Science
Jouf University
Sakaka P. O. Box 2014, Saudi Arabia
e-mail: sidahmed@ju.edu.sa, sidahmed.sidha@gmail.com