Operators and Matrices
Volume 15, Number 3 (2021), 1089–1108
doi:10.7153/oam-2021-15-68

ON SEMIMONOTONE STAR MATRICES AND LINEAR COMPLEMENTARITY PROBLEM

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(*Communicated by L. Zhang*)

Abstract. In this article, we introduce the class of semimonotone star (E_0^s) matrices. We establish the importance of the class of E_0^s-matrices in the context of complementarity theory. We show that the principal pivot transform of an E_0^s-matrix is not necessarily E_0^s in general. However, we prove that $\tilde{E}_0^s$-matrices, a subclass of the E_0^s-matrices with some additional conditions, is fully semimonotone matrix by showing this class is in P_0. We prove that $\mathrm{LCP}(q,A)$ can be processable by Lemke's algorithm if $A \in \tilde{E}_0^s \cap P_0$. We find some conditions for which the solution set of $\mathrm{LCP}(q,A)$ is bounded and stable under the $\tilde{E}_0^s$-property. We propose an algorithm based on an interior point method to solve $\mathrm{LCP}(q,A)$ given $A \in \tilde{E}_0^s$.

1. Introduction

The concept of pseudomonotone or copositive star matrices on a closed convex cone with respect to complementarity condition was studied by Gowda [14]. The properties of copositive star matrices are well studied in the literature of the linear complementarity problem. A matrix A is said to be a star matrix [13] if for any x from the solution set of $\mathrm{LCP}(0,A)$ implies $A^T x \leqslant 0$. Bazan and Lopez [13] studied F_1-matrices in the context of star matrices and proved the necessary and sufficient conditions of F_1-properties. In linear complementarity theory, much of the research is devoted to finding a constructive characterization of Q_0 and Q-matrices. The linear complementarity problem is a combination of linear and nonlinear systems of inequalities and equations.

The problem may be stated as follows: Given $A \in R^{n\times n}$ and a vector $q \in R^n$, the *linear complementarity problem* is the problem of finding a solution $w \in R^n$ and $z \in R^n$ to the following system:

$$w - Az = q, \;\; w \geqslant 0, \, z \geqslant 0 \tag{1.1}$$

$$w^T z = 0 \tag{1.2}$$

The *linear complementarity problem* is denoted as $\mathrm{LCP}(q,A)$. Let $\mathrm{FEA}(q,A) = \{z \geqslant 0 : q + Az \geqslant 0\}$ and $\mathrm{SOL}(q,A) = \{z \in \mathrm{FEA}(q,A) : z^T(q+Az) = 0\}$ denote the

Mathematics subject classification (2020): 90C33, 90C90, 15B99.
Keywords and phrases: Linear complementarity problem, principal pivot transform, Lemke's algorithm, interior point method, semimonotone star matrix, $\tilde{E}_0^s$-matrix.
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Presently working in an integrated steel plant.

feasible and solution set of LCP(q,A) respectively. In this article, we introduce the class of semimonotone star (E_0^s) matrices. We call the property of E_0^s-matrices as E_0^s-property. We establish the importance of E_0^s-matrix in the light of complementarity theory, principal pivot transform and Lemke's algorithm.

In linear complementarity theory, the copositive star (C_0^*) matrix plays an important role. Extending the applicability of Lemke's algorithm and the potential future applications have motivated to introduce the class of E_0^s-matrices. We characterize the properties of E_0^s-matrices in view of matrix theory and establish the importance of this class in linear complementarity theory. We state some matrix theoretic results which are needed in the sequel. As an alternative approach, we propose an iterative method to find a solution of LCP(q,A) on the assumption that the matrix A belongs to E_0^s-matrix. The class of semimonotone matrices (E_0) introduced by Eaves [12] (denoted by L_1 also) consists of all real square matrices A such that LCP(q,A) has a unique solution for every $q > 0$.

Many of the concepts and algorithms in optimization theory are developed based on the principal pivot transform (PPT). The notion of the PPT is originally motivated by the well known linear complementarity problem. Cottle and Stone [9] introduced the notion of a fully semimonotone matrix (E_0^f) by requiring that every PPT of such a matrix is a semimonotone matrix. Stone studied various properties of E_0^f-matrices and conjectured that E_0^f with Q_0-property are contained in P_0. We illustrate that the principal pivot transform of E_0^s is not necessarily E_0^s. However, the class of E_0^s-matrices with some additional conditions is in E_0^f by showing this class in P_0. Suppose $K(A)$ is the set of all $q \in R^n$ such that LCP(q,A) has a solution. Eaves [12] showed that $A \in Q_0$ if and only if $K(A)$ is convex. A subclass Q of Q_0 is defined by the property that $A \in Q$ if and only if $K(A) = R^n$. If $A \in E_0^s \cap P_0$, then LCP(q,A) can be processed by Lemke's algorithm and the solution set of LCP(q,A) is bounded.

The outline of the article is as follows. In Section 2, some notations, definitions, and results are presented that are used in the next sections. In Section 3, we introduce semimonotone star (E_0^s)-matrix and study some properties of this class in connection with complementarity theory, principal pivot transform. Section 4 deals with PPT based matrix classes under the E_0^s-property. In Section 5, we consider the SOL(q,A) under E_0^s-property. In this connection, we partially settle an open problem raised by Jones and Gowda [17]. We propose an iterative algorithm [11] to process LCP(q,A) where $A \in \tilde{E}_0^s$, a subclass of E_0^s-matrix in Section 6. A numerical example is presented to show the performance of the proposed algorithm in Section 7.

2. Preliminaries

We denote the n dimensional real space by R^n where R^n_+ and R^n_{++} denote the nonnegative and positive orthant of R^n respectively. We consider vectors and matrices with real entries. For any set $\beta \subseteq \{1,2,\ldots,n\}$, $\bar{\beta}$ denotes its complement in $\{1,2,\ldots,n\}$. Any vector $x \in R^n$ is a column vector unless otherwise specified. For any matrix $A \in R^{n\times n}$, a_{ij} denotes its ith row and jth column entry, $A_{\cdot j}$ denotes the jth column and $A_{i\cdot}$ denotes the ith row of A. If A is a matrix of order n, $\emptyset \neq \alpha \subseteq \{1,2,\ldots,n\}$

and $\emptyset \neq \beta \subseteq \{1,2,\dots,n\}$, then $A_{\alpha\beta}$ denotes the submatrix of A consisting of only the rows and columns of A whose indices are in α and β, respectively. The interior of a set S is the union of all open sets contained in it and it is denoted by int S. A matrix $A \geqslant 0$ or $A \leqslant 0$ implies that either the matrix A is non-negative or non-positive respectively. For any set α, $|\alpha|$ denotes its cardinality. $\|A\|$ and $\|q\|$ denote the norms of a matrix A and a vector q respectively.

The *principal pivot transform* (PPT) of A, a real $n\times n$ matrix, with respect to $\alpha \subseteq \{1,2,\dots,n\}$ is defined as the matrix given by

$$M = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha} & M_{\bar{\alpha}\bar{\alpha}} \end{bmatrix}$$

where $M_{\alpha\alpha} = (A_{\alpha\alpha})^{-1}$, $M_{\alpha\bar{\alpha}} = -(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$, $M_{\bar{\alpha}\alpha} = A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}$, $M_{\bar{\alpha}\bar{\alpha}} = A_{\bar{\alpha}\bar{\alpha}} - A_{\bar{\alpha}\alpha}(A_{\alpha\alpha})^{-1}A_{\alpha\bar{\alpha}}$. Note that PPT is only defined with respect to those α for which $\det A_{\alpha\alpha} \neq 0$. By a *legitimate principal pivot transform* we mean the PPT obtained from A by performing a principal pivot on its nonsingular principal submatrices. When $\alpha = \emptyset$, by convention $\det A_{\alpha\alpha} = 1$ and $M = A$. For further details see [6], [8], [23] and [25] in this connection. The PPT of LCP(q,A) with respect to α (obtained by pivoting on $A_{\alpha\alpha}$) is given by LCP(q',M) where M has the same structure already mentioned with $q'_{\alpha} = -A^{-1}_{\alpha\alpha}q_{\alpha}$ and $q'_{\bar{\alpha}} = q_{\bar{\alpha}} - A_{\bar{\alpha}\alpha}A^{-1}_{\alpha\alpha}q_{\alpha}$.

We say that $A \in R^{n\times n}$ is

- *positive definite* (PD) matrix if $x^TAx > 0$, $\forall\, 0 \neq x \in R^n$.
- *positive semidefinite* (PSD) matrix if $x^TAx \geqslant 0$, $\forall\, x \in R^n$.
- *column sufficient* matrix if $x_i(Ax)_i \leqslant 0\ \forall\, i \implies x_i(Ax)_i = 0\ \forall\, i$.
- *row sufficient* matrix if A^T is column sufficient.
- *sufficient* matrix if A is both column and row sufficient.
- $P\,(P_0)$-matrix if all its principal minors are positive (nonnegative).
- $N(N_0)$-matrix if all its principal minors are negative (nonpositive).
- *copositive* (C_0) matrix if $x^TAx \geqslant 0$, $\forall\, x \geqslant 0$.
- *strictly copositive* (C) matrix if $x^TAx > 0$, $\forall\, 0 \neq x \geqslant 0$.
- *copositive plus* (C_0^+) matrix if A is copositive and $x^TAx = 0$, $x \geqslant 0 \implies (A + A^T)x = 0$.
- *copositive star* (C_0^*) matrix if A is copositive and $x^TAx = 0$, $Ax \geqslant 0$, $x \geqslant 0 \implies A^Tx \leqslant 0$.
- *semimonotone* (E_0) matrix if for every $0 \neq x \geqslant 0$, $\exists$ an i such that $x_i > 0$ and $(Ax)_i \geqslant 0$.
- L_2-matrix if for every $0 \neq x \geqslant 0$, $x \in R^n$, such that $Ax \geqslant 0$, $x^TAx = 0$, $\exists$ two diagonal matrices $D_1 \geqslant 0$ and $D_2 \geqslant 0$ such that $D_2x \neq 0$ and $(D_1A + A^TD_2)x = 0$.
- L-matrix if it is $E_0 \cap L_2$.
- *strictly semimonotone* (E) matrix if for every $0 \neq x \geqslant 0$, $\exists$ an i such that $x_i > 0$ and $(Ax)_i > 0$.
- *pseudomonotone* matrix if for all $x, y \geqslant 0$, $(y-x)^TAx \geqslant 0 \implies (y-x)^TAy \geqslant 0$.
- *positive subdefinite matrix* (PSBD) if $\forall\, x \in R^n$, $x^TAx < 0 \implies$ either $A^Tx \leqslant 0$ or $A^Tx \geqslant 0$.
- *fully copositive* (C_0^f) matrix if every legitimate PPT of A is C_0-matrix.

- *fully semimonotone* (E_0^f) matrix if every legitimate PPT of A is E_0-matrix.
- *almost* $P_0(P)$-matrix if $\det A_{\alpha\alpha} \geqslant 0 \ (>0) \ \forall\, \alpha \subset \{1,2,\ldots,n\}$ and $\det A < 0$.
- *an almost* $N_0(N)$-matrix if $\det A_{\alpha\alpha} \leqslant 0 \ (<0) \ \forall\, \alpha \subset \{1,2,\ldots,n\}$ and $\det A > 0$.
- *almost copositive* matrix if it is copositive of order $n-1$ but not of order n.
- *almost E* matrix if it is E of order $n-1$ but not of order n.
- *almost fully copositive* (*almost* C_0^f) matrix if its PPTs are either C_0 or almost C_0 and there exists at least one PPT M of A for some $\alpha \subset \{1,2,\ldots,n\}$ that is almost C_0.
- *copositive of exact order* k matrix if it is copositive up to order $n-k$.
- Z-matrix if $a_{ij} \leqslant 0$ for $i \neq j$.
- K_0-matrix [4] if it is Z-matrix as well as P_0-matrix.
- *connected* (E_c) matrix if $\forall\, q$, LCP(q,A) has a connected solution set.
- R-matrix if $\nexists\ z \in R^n_+,\ t(\geqslant 0) \in R$ satisfying

$$A_{i.}z + t = 0 \text{ if } z_i > 0,$$
$$A_{i.}z + t \geqslant 0 \text{ if } z_i = 0.$$

- R_0-matrix if LCP$(0,A)$ has unique solution.
- Q_b-matrix if SOL(q,A) is nonempty and compact $\forall\, q \in R^n$.
- Q-matrix if for every $q \in R^n$, LCP(q,A) has a solution.
- Q_0-matrix if for any $q \in R^n$, (1.1) has a solution implies that LCP(q,A) has a solution.
- *completely* Q-matrix $(\bar{Q})$ if all its principal submatrices are Q-matrices.
- *completely* Q_0-matrix $(\bar{Q}_0)$ if all its principal submatrices are Q_0-matrices.

Several matrix classes arise in the literature of linear complementarity problem. We use the terms namely fully, complete and invariance to indicate the properties of matrix classes in the context of LCP(q,A). For summary of matrix classes, see [5] and [24]. Now we state some game theoretic results due to von Neumann [31] which are needed in the sequel. The results say that there exist $x^* \in R^m, y^* \in R^n$ and $v \in R$ such that

$$\textstyle\sum_{i=1}^m x_i^* a_{ij} \leqslant v,\ \forall\, j = 1,2,\cdots,n,$$
$$\textstyle\sum_{j=1}^n y_j^* a_{ij} \geqslant v,\ \forall\, i = 1,2,\cdots,m.$$

The strategies (x^*, y^*) are said to be optimal strategies for player I and player II and v is said to be minimax value of game. In a two person zero-sum matrix game, let $v(A)$ denote the value of the game corresponding to the pay-off matrix A. The value of the game $v(A)$ is *positive* (*nonnegative*) if there exists a $0 \neq x \geqslant 0$ such that $Ax > 0$ $(Ax \geqslant 0)$. Similarly, $v(A)$ is *negative* (*nonpositive*) if there exists a $0 \neq y \geqslant 0$ such that $A^T y < 0$ $(A^T y \leqslant 0)$.

The following result was proved by Väliaho [30] for symmetric almost copositive matrices. However this is true for nonsymmetric almost copositive matrices as well.

THEOREM 2.1. ([10], Theorem 2.2) *Let* $A \in R^{n\times n}$ *be almost copositive matrix. Then* A *is PSD of order* $n-1$, *and* A *is PD of order* $n-2$.

THEOREM 2.2. ([19], Theorem 2.2) *Suppose* $A \in R^{n\times n}$ *is a PSBD matrix and* $rank(A) \geqslant 2$. *Then* A^T *is PSBD and at least one of the following conditions hold:*

(i) A *is a PSD matrix.*

(ii) $(A+A^T) \leqslant 0$.

(iii) $A \in C_0^*$.

THEOREM 2.3. ([19], Lemma 3.2) *Suppose* $A \in R^{n\times n}$ *is a PSBD matrix and* $rank(A) \geqslant 2$ *and* $A+A^T \leqslant 0$. *If* A *is not a skew-symmetric matrix, then* $A \leqslant 0$.

Here we consider some more results which will be required in the next section.

THEOREM 2.4. ([32], Lemma 1) *The matrix* A *is a* P_0*-matrix of order* $n-1$ *and* $A \notin P_0$ *if and only if* $A^{-1} \in N_0$.

Now we give a result on $(++)$-property along with the definition which will be required in the subsequent section.

DEFINITION 2.1. [4] A matrix A is said to satisfy $(++)$-property if there exists a matrix $X \in K_0$ such that AX is a Z-matrix.

THEOREM 2.5. ([4], Theorem 5) *Suppose* $A \in R^{n\times n}$ *with* A *satisfies* $(++)$*-property. If* $A \in E_0$ *then* $A \in P_0$.

We state the notion of stability to a linear complementarity problem at solution point. For details, see [8].

DEFINITION 2.2. A solution x^* is said to be stable if there are neighborhoods V of x^* and U of (q,A) such that

(i) for all $(\bar{q},\bar{A}) \in U$, the set $\mathrm{SOL}(\bar{q},\bar{A}) \cap V \neq \emptyset$.

(ii) $\sup\{\|y-x^*\| : y \in \mathrm{SOL}(\bar{q},\bar{A}) \cap V \neq \emptyset\}$ goes to zero as $(\bar{q},\bar{A})$ approaches (q,A).

THEOREM 2.6. ([15], Theorem 2) *Let* $A \in R^{n\times n}$ *be given. Consider the statements*

(i) $A \in R$.

(ii) $A \in int(Q) \cap R_0$.

(iii) the zero vector is a stable solution of the LCP $(0,A)$.

(iv) $A \in Q \cap R_0$.

(v) $A \in R_0$.

Then the following implications hold: (i) $\implies$ *(ii)* $\implies$ *(iii)* $\implies$ *(iv)* $\implies$ *(v). Moreover, if* $A \in E_0$, *then all five statements are equivalent.*

THEOREM 2.7. ([15], Theorem 3) *Let* $A \in int(Q) \cap R_0$. *If the LCP* (q,A) *has a unique solution* x^*, *then LCP* (q,A) *is stable at* x^*.

THEOREM 2.8. ([27], Theorem 2.5) *Let $A \in R^{n\times n}$ be such that for some index set α (possibly empty), $A_{\bar{\alpha}\bar{\alpha}} = 0$. If $A_{\alpha\alpha} \in P_0 \cap Q$, then $SOL(q,A)$ is connected for every q.*

THEOREM 2.9. ([3], Theorem 2) *Suppose $A \in E_c \cap Q_0$. Then Lemke's algorithm terminates at a solution of $LCP(q,A)$ or determines that $FEA(q,A) = \emptyset$.*

THEOREM 2.10. ([14], Proposition 2) *Suppose that A is pseudomonotone on R^n_+. Then A is a P_0 matrix.*

THEOREM 2.11. ([17], Theorem 3) *Suppose that $A \in E_c$. Then $A \in E_0^f$.*

THEOREM 2.12. ([30], Theorem 4.3) *Any 2×2 P_0-matrix with positive diagonal is sufficient.*

THEOREM 2.13. ([8], Corollary 3.9.19) [12] *L-matrices are Q_0-matrices.*

THEOREM 2.14. ([7], Theorem 2 and Theorem 2') *Let $A \in R^{n\times n}$ where $n \geqslant 2$. Then A is sufficient if and only if A and each of its principal pivot transforms are sufficient of order 2.*

THEOREM 2.15. ([22], Theorem 6.1) *Suppose $A \in E_0$. If $A \in R_0$ then $A \in Q$.*

THEOREM 2.16. ([13], Equation 6) $Q_b = Q \cap R_0$.

3. Some properties of E_0^s-matrices

We begin by the definition of semimonotone star (E_0^s) matrix.

DEFINITION 3.1. A semimonotone matrix A is said to be a semimonotone star (E_0^s) matrix if $x^T Ax = 0$, $Ax \geqslant 0$, $x \geqslant 0 \implies A^T x \leqslant 0$.

REMARK 3.1. Note that $E_0 \cap R_0 \subseteq E_0^s$.

EXAMPLE 3.1. Consider the matrix $A = \begin{bmatrix} 0 & -5 \\ 2 & 0 \end{bmatrix}$. Now $x^T Ax = -3x_1x_2$. Consider $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_1,\ x_2 \geqslant 0$. Hence we consider the following cases.

Case I: For $x_1 = x_2 = 0,\ x = 0,\ Ax = 0,\ x^T Ax = 0$ implies $A^T x = 0$.
Case II: For $x_1 > 0,\ x_2 = 0,\ x \geqslant 0,\ Ax \geqslant 0,\ x^T Ax = 0$ implies $A^T x \leqslant 0$.
Case III: For $x_1 = 0,\ x_2 > 0,\ x \geqslant 0$. However $Ax \ngeqslant 0$.
Case IV: For $x_1 > 0,\ x_2 > 0,\ x > 0$. However $x^T Ax \neq 0$.
Hence $A \in E_0^s$.

The following result shows that E_0^s-matrices are invariant under principal rearrangement and scaling operations.

THEOREM 3.1. *If $A \in E_0^s$-matrix and $P \in R^{n\times n}$ is any permutation matrix if and only if $PAP^T \in E_0^s$.*

Proof. Let $A \in E_0^s$ and $P \in R^{n\times n}$ be any permutation matrix. Then PAP^T is an E_0-matrix by Theorem 4.3 of [29]. Consider $x \geqslant 0$, $(PAP^T)x \geqslant 0$ and $x^T(PAP^T)x = 0$. Let $y = P^T x$. Note that $x^T PAP^T x = y^T Ay = 0$, $AP^T x = Ay \geqslant 0$. This implies $A^T y = A^T P^T x \leqslant 0$. It follows that $(PAP^T)^T x \leqslant 0$, since P is a permutation matrix. It follows that PAP^T is an E_0^s-matrix. The converse of the above theorem follows from the fact that $P^T P = I$ and therefore invertible with $P^{-1} = P^T$. □

THEOREM 3.2. *Suppose A is a E_0^s-matrix. Let $D \in R^{n\times n}$ be a positive diagonal matrix. Then $A \in E_0^s$ if and only if $DAD^T \in E_0^s$.*

Proof. Consider $A \in E_0^s$ and let $D \in R^{n\times n}$ be a positive diagonal matrix. Then DAD^T is an E_0-matrix [29]. Consider $x \geqslant 0$, $(DAD^T)x \geqslant 0$ and $x^T(DAD^T)x = 0$. Let $y = D^T x$. Note that $x^T DAD^T x = y^T Ay = 0$, $AD^T x = Ay \geqslant 0 \Rightarrow A^T y = A^T D^T x \leqslant 0$. It follows that $(DAD^T)^T x \leqslant 0$, since D is a positive diagonal matrix. Thus $DAD^T \in E_0^s$. The converse follows from the fact that D^{-1} is a positive diagonal matrix and $A = D^{-1}(DAD^T)(D^{-1})^T$. □

The following example shows that $A \in E_0^s$-matrix does not imply $(A + A^T) \in E_0^s$-matrix.

EXAMPLE 3.2. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. Clearly $A \in E_0^s$, since $x^T Ax = 0$, $Ax \geqslant 0$, $x \geqslant 0$ implies $A^T x \leqslant 0$.

It is easy to show that $A + A^T = \begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not an E_0^s-matrix.

We show that PPT of E_0^s-matrix need not be an E_0^s-matrix.

EXAMPLE 3.3. We consider the matrix as in Example 3.2. Note that $A \in E_0^s$ and it is easy to show that $A^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -2 \\ 2 & 1 & 2 \end{bmatrix}$ is not a E_0^s-matrix. Therefore any PPT of E_0^s-matrix need not be E_0^s-matrix.

Note that a matrix is in E_0 if and only if its transpose is in E_0. We show that $A \in E_0^s$-matrix does not imply $A^T \in E_0^s$-matrix in general.

EXAMPLE 3.4. Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$. Note that $A \in E_0^s$. Consider $B = A^T = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$. Now $x^T Bx = 3x_1x_2$. Consider $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, where x_1, x_2, $x_3 \geqslant 0$. Now for $x_1 > 0$, $x_2 = 0$, $x_3 = 0$ implies $x \geqslant 0$, $Bx \geqslant 0$, $x^T Bx = 0$. But $B^T x \nleqslant 0$. Therefore $B = A^T$ is not an E_0^s-matrix.

Now we show a condition under which A^T satisfies E_0^s-property.

THEOREM 3.3. *Suppose that A is pseudomonotone on R^n_+ and $0 \neq x \geqslant 0$, $A^T x = 0$ has no solution. Then A^T satisfies E_0^s-property.*

Proof. Since A is pseudomonotone on R^n_+, then A is a P_0 matrix by Theorem 2.10. Hence $A \in E_0$. We have to show that A^T satisfies the following property.

$$0 \neq x \geqslant 0, \;\; A^T x \geqslant 0, \text{ and } x^T A^T x = 0 \implies Ax \leqslant 0.$$

As $0 \neq x \geqslant 0$, $A^T x = 0$ has no solution, therefore $(A^T x)_i > 0$ for some index i. We consider the vector e_i which has one at the ith position and zeros elsewhere. Now consider $y = e_i + \lambda e_j$, where $i \neq j$ and $\lambda \geqslant 0$. Then, for any small $\delta > 0$, we get

$$(x - \delta y)^T A(\delta y) = \delta[(A^T x)_i + \lambda (A^T x)_j - \delta y^T Ay] \geqslant 0.$$

By pseudomonotonicity, $(x - \delta y)^T Ax \geqslant 0$. Thus $y^T Ax \leqslant 0$. This gives $(Ax)_i + \lambda (Ax)_j \leqslant 0$. As λ is arbitrary, $(Ax)_i \leqslant 0$ and $(Ax)_j \leqslant 0$. Hence $Ax \leqslant 0$. □

REMARK 3.2. If A is pseudomonotone on R^n_+ and $A^T \in R_0$. Then it can be easily verified that A^T satisfies E_0^s-property.

COROLLARY 3.1. *Suppose that A is pseudomonotone on R^n_+ and satisfies one of the following conditions:*

(i) *A is invertible*

(ii) *A is normal i.e. $AA^T = A^T A$.*

Then $A^T \in E_0^s$.

Proof. To prove the result, we consider following cases.

(i) If A is invertible then the system $0 \neq x \geqslant 0$, $A^T x = 0$ has no solution. Hence the proof follows from the Theorem 3.3.

(ii) Since $x^T AA^T x = x^T A^T Ax$, so if $A^T x = 0$ then $Ax = 0$. Again if $A^T x \neq 0$, then for at least one i, $(A^T x)_i > 0$. Therefore the proof follows from the Theorem 3.3. □

We say that E_0^s is not a complete class which can be illustrated with the following example.

EXAMPLE 3.5. Consider the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ -4 & -5 & 0 \end{bmatrix}$. Note that $A \in E_0^s$. Consider $\alpha = \{1,2\}$. Then $A_{\alpha\alpha} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$. Now $x^T A_{\alpha\alpha} x = 3x_1x_2$. Suppose $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_1,\ x_2 \geqslant 0$. Now for $x_1 > 0,\ x_2 = 0$ implies $x \geqslant 0,\ A_{\alpha\alpha}x \geqslant 0,\ x^T A_{\alpha\alpha} x = 0$. But $A_{\alpha\alpha}^T x \nleq 0$. Therefore $A_{\alpha\alpha}$ is not an E_0^s-matrix.

DEFINITION 3.2. A matrix A is said to be a completely semimonotone star ($\bar{E}_0^s$) matrices if all its principal submatrices are semimonotone star matrix.

THEOREM 3.4. *Let $A \in PSBD \cap E_0$ with $rank(A) \geqslant 2$. Further, suppose A is not a skew-symmetric matrix. Then $A \in E_0^s$-matrix.*

Proof. Let A be a PSBD as well as E_0-matrix with $\operatorname{rank}(A) \geqslant 2$. By Theorem 2.2, we have the following three cases.

Case I: A is a PSD matrix. This implies $A \in E_0^s$.

Case II: $A \in C_0^*$. This implies $A \in E_0^s$.

Case III: $(A + A^T) \leqslant 0$. For $x \geqslant 0$, $Ax \geqslant 0$ implies $(A + A^T)x \leqslant 0$. Hence $A^T x \leqslant -Ax \leqslant 0$. Therefore, A is an E_0^s-matrix. $\square$

REMARK 3.3. Note that $C_0^* \subseteq E_0^s$.

EXAMPLE 3.6. Consider the matrix $A = \begin{bmatrix} 0 & 3 \\ -1 & 0 \end{bmatrix}$. As A is a P_0-matrix, A is an E_0-matrix. Also it is easy to show that A is a PSBD matrix with $\operatorname{rank}(A) \geqslant 2$. Hence by Theorem 3.4, $A \in E_0^s$.

4. PPT based matrix classes under E_0^s-property

We consider some PPT based matrix classes with E_0^s-property in the context of linear complementarity problem to show that these classes are processable by Lemke's algorithm under certain conditions. Aganagic and Cottle [2] showed that Lemke's algorithm processes LCP(q,A) if $A \in P_0 \cap Q_0$. Verifying whether a matrix class belongs to $P_0 \cap Q_0$ or not is difficult. We show that the class identified in this paper is a subclass of $P_0 \cap Q_0$. The identification of this matrix class motivates the study of further application in matrix theory.

DEFINITION 4.1. A matrix $A \in E_0^s$ is said to be $\tilde{E}_0^s$-matrix if for $x \in \operatorname{SOL}(0,A)$, $(A^T x)_i \neq 0 \implies (Ax)_i \neq 0\ \forall\, i$.

REMARK 4.1. Note that, a matrix A is said to be a completely $\tilde{E}_0^s$-matrix if every principal submatrix of A is $\tilde{E}_0^s$.

EXAMPLE 4.1. Consider $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & -4 & 0 \end{bmatrix}$. Note that, $A \notin C_0^*$. For $k > 0$ and $x = \begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix}$, $x \geqslant 0$, $Ax \geqslant 0$, $x^T Ax = 0$ implies $A^T x \leqslant 0$. Hence $A \in E_0^s$.

Now $A^T x = \begin{bmatrix} -2k \\ -4k \\ 0 \end{bmatrix}$ and $Ax = \begin{bmatrix} k \\ 2k \\ 0 \end{bmatrix}$. Therefore $\forall\, i$, $(A^T x)_i \neq 0 \implies (Ax)_i \neq 0$. Hence $A \in \tilde{E}_0^s$.

REMARK 4.2. It is easy to show that $C_0^+ \subseteq \tilde{E}_0^s$.

Note that not every E_0^s-matrix is an $\tilde{E}_0^s$-matrix. We consider the following example from the paper [16].

EXAMPLE 4.2. Consider $A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Note that $A \in P_0$. Hence $A \in E_0$. The only nonzero vectors in $\mathrm{SOL}(0,A)$ are of the form $x = \begin{bmatrix} k \\ k \\ 0 \end{bmatrix}$ for $k > 0$. Now for such x, $A^T x \leqslant 0$ holds. Hence $A \in E_0^s$. Now $A^T x = \begin{bmatrix} 0 \\ 0 \\ -2k \end{bmatrix}$ and $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Note that $(A^T x)_3 \neq 0$ but $(Ax)_3 = 0$. Hence $A \notin \tilde{E}_0^s$.

THEOREM 4.1. *Let $A \in \tilde{E}_0^s$ $(n \geqslant 3)$. Assume that each PPT of A is either almost E or completely $\tilde{E}_0^s$. Then $A \in P_0$.*

Proof. Since $A \in \tilde{E}_0^s$, then for $x \in \mathrm{SOL}(0,A)$ implies $A^T x \leqslant 0$. Note that, $(A^T x)_i < 0$ implies that $(Ax)_i > 0$ for any i. Again by definition $(A^T x)_i \neq 0$ which implies $(Ax)_i \neq 0$ $\forall\, i$. Now by taking $D_2 = I$, where I represents the identity matrix. Then $D_2 x = Ix \neq 0$. So $(D_1 A + A^T I)x = 0$ by taking,

$$D_{ii} = \begin{cases} \dfrac{-(A^T x)_i}{(Ax)_i}, & (Ax)_i \neq 0, \\ 0, & (Ax)_i = 0, \end{cases}$$

where D_{ii} denotes the ith diagonal of D_1. So $A \in E_0^s \cap L_2$. Therefore $A \in Q_0$ by Theorem 2.13. Note that every legitimate PPT of A is either almost E or completely $\tilde{E}_0^s$. Suppose M is a PPT of A so that $M \in$ almost E. Then all principal submatrices of M upto $n-1$ order are $\bar{Q}$. Hence $M \in \bar{Q}_0$. The rest of the proof follows from the Theorem 3.6 of Das [10]. Since the PPT of A is almost E, it follows that all proper principal submatrices are P_0. Now to complete the proof, we need to show that

$\det A \geqslant 0$. Suppose not. Then $\det A < 0$. This implies that A is an almost P_0-matrix. By Theorem 2.4, $A^{-1} \in N_0$. If $A^{-1} \in$ almost E then this contradicts that the diagonal entries are positive. Therefore $\det A \geqslant 0$. It follows that $A \in P_0$.

Now suppose M is a PPT of A so that M is completely $\tilde{E}_0^s$. Then $M \in \bar{Q}_0$. As $A \in \tilde{E}_0^s$, it follows that $A \in E_0^f$. Therefore $M \in E_0^f \cap \bar{Q}_0$. By Corollary 3.6 of [21], $M \in P_0$. Hence $A \in P_0$. $\square$

COROLLARY 4.1. *Let $A \in R^{n\times n} \cap \tilde{E}_0^s$. Assume that every legitimate PPT of A is either almost E or completely $\tilde{E}_0^s$. Then LCP(q,A) is processable by Lemke's algorithm.*

Earlier Das [10] proposed *exact order 2* C_0^f-matrices in connection with PPT based matrix classes. We define *exact order k* C_0^f-matrices. For detail explanation and examples, see [10].

DEFINITION 4.2. A is said to be an *exact order k* C_0^f-matrix if its PPTs are either exact order k C_0 or E_0 and there exists at least one PPT M of A for some $\alpha \subset \{1,2,\cdots,n\}$ that is exact order k C_0.

We prove the following theorem.

THEOREM 4.2. *Let $A \in \tilde{E}_0^s \cap$ exact order k C_0^f $(n \geqslant k+2)$. Assume that each PPT of A is either exact order k C_0 or E_0 with at least k positive diagonal entries. Then LCP(q,A) is processable by Lemke's algorithm.*

Proof. We show that $A \in P_0$. Suppose M is a PPT of A so that M is exact order k C_0. By Theorem 2.1, all the principal submatrices of order $(n-k)$ of M are PSD. Let $M^{(n-k+1)}$ be the principal submatrix of M of order $(n-k+1)$. It is enough to show that $\det M^{(n-k+1)} \geqslant 0$. Suppose not. Then $\det M^{(n-k+1)} < 0$. We consider $B = M^{(n-k+1)}$ is an almost P_0-matrix. Therefore $B^{-1} \in N_0$ and there exists a nonempty subset $\alpha \subset \{1,2,\ldots,n-k+1\}$ satisfying [10]

$$B^{-1}_{\alpha\alpha} \leqslant 0,\ B^{-1}_{\bar{\alpha}\bar{\alpha}} \leqslant 0,\ B^{-1}_{\alpha\bar{\alpha}} \geqslant 0 \text{ and } B^{-1}_{\bar{\alpha}\alpha} \geqslant 0. \tag{4.1}$$

By definition $B^{-1} \in E_0$ with at least k positive diagonal entry. This contradicts Equation 4.1. Therefore $\det M^{(n-k+1)} \geqslant 0$. Now by the same argument as above, we show that $\det M \geqslant 0$. Therefore it follows that $A \in P_0$. Hence $A \in P_0 \cap \tilde{E}_0^s$. So LCP$(q,A)$ is processable by Lemke's Algorithm. $\square$

We establish the condition under which a matrix A is sufficient whenever it satisfies $(++)$-property.

THEOREM 4.3. *Suppose $A \in E_0$ satisfies $(++)$-property. If each legitimate PPT of A is either almost C_0 or completely $\tilde{E}_0^s$ with full rank second order principal submatrices, then A is sufficient.*

Proof. As $A \in E_0$ with $(++)$-property. Hence $A \in P_0$ by Theorem 2.5. Suppose M is a PPT of A. We consider the following cases.

Case I: If M is an almost C_0-matrix, then by Theorem 2.1, M is a PSD matrix of order $(n-1)$. Hence M is a PSD matrix of order 2 also. So by Theorem 2.14, M is a sufficient matrix of order $(n-1)$.

Case II: If M is completely $\tilde{E}_0^s$ then sign pattern of all 2×2 principal submatrices of M will be in the following subcases:

Subcase I: If the sign patterns are $\begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & - \\ + & 0 \end{bmatrix}$ then these two patterns are sufficient.

Subcase II: If the sign patterns are $\begin{bmatrix} + & + \\ - & + \end{bmatrix}$ or $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$ or $\begin{bmatrix} + & + \\ + & + \end{bmatrix}$ or $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ then by Theorem 2.12, these patterns are sufficient.

Subcase III: If the sign patterns are $\begin{bmatrix} 0 & + \\ - & + \end{bmatrix}$ or $\begin{bmatrix} 0 & - \\ + & + \end{bmatrix}$ or $\begin{bmatrix} + & - \\ + & 0 \end{bmatrix}$ or $\begin{bmatrix} + & + \\ - & 0 \end{bmatrix}$ then these patterns are sufficient.

Then for every PPT of A of order 2 is sufficient. By Theorem 2.14, A is sufficient. $\square$

5. Properties of SOL(q,A) under E_0^s-property

We show that solution set of LCP(q,A) is connected if $A \in E_0^s$ with the following structure $A = \begin{bmatrix} A_{\alpha\alpha} & + \\ - & 0 \end{bmatrix}$, where $A_{\alpha\alpha} \in R^{(n-1)\times(n-1)}$.

THEOREM 5.1. *Let $A \in R^{n\times n}$ with $A = \begin{bmatrix} A_{\alpha\alpha} & + \\ - & 0 \end{bmatrix}$ and $A_{\alpha\alpha} \in P_0$. Then $A \in \tilde{E}_0^s$-matrix.*

Proof. First we show that $A = \begin{bmatrix} A_{\alpha\alpha} & + \\ - & 0 \end{bmatrix}$ with $A_{\alpha\alpha} \in P_0$ is E_0-matrix. We consider the vector $(u_\alpha, v) \in R^n_+$ where $\alpha = \{1,2,\cdots,(n-1)\}$. We assume $u_\alpha \neq 0$. Now as $A_{\alpha\alpha} \in P_0$, we can write $A_{\alpha\alpha} \in E_0$. By the semimonotonicity of $A_{\alpha\alpha}$ $\exists$ an index i such that $(u_\alpha)_i > 0$ and $(A_{\alpha\alpha}u_\alpha)_i \geqslant 0$. If we let $A = \begin{bmatrix} A_{\alpha\alpha} & P \\ N & 0 \end{bmatrix}$, for such an index i, $(A_{\alpha\alpha}u_\alpha + vP)_i \geqslant 0$, where P and N denotes positive and negative values. Hence $A \in E_0$. We consider the following two cases:

Case I: First we take $x = [x_\alpha, 0]^T$, where $\alpha = \{1,2,\cdots,(n-1)\}$. Then suppose $x^T A x = 0$, $x \geqslant 0$, but in this case $Ax \ngeq 0$.

Case II: Take $x = [x_\alpha, x_{\bar{\alpha}}]^T$, where $x_\alpha, x_{\bar{\alpha}} \geqslant 0$. Then suppose for this x, $x^T A x = 0$, but $Ax \ngeq 0$.

So the vector x for which $x^T A x = 0$, $Ax \geqslant 0$, $x \geqslant 0$, are the zero vector and $[0,0,\cdots,c]^T, c > 0$ and for both cases $A^T x \leqslant 0$.

Hence A is E_0^s-matrix. Now it is easy to show that for $x = [0,0,\cdots,c]^T$, $(A^T x)_i \neq 0 \implies (Ax)_i \neq 0$ for each i. Hence $A \in \tilde{E}_0^s$. $\square$

REMARK 5.1. Suppose $A \in R^{n\times n}$ with $A = \begin{bmatrix} A_{\alpha\alpha} & + \\ - & 0 \end{bmatrix}$ and $A_{\alpha\alpha} \in P_0 \cap Q$. Then A is a connected matrix (E_c) from the Theorem 2.8 of [27].

REMARK 5.2. Suppose $A \in R^{n\times n}$ with $A = \begin{bmatrix} A_{\alpha\alpha} & + \\ - & 0 \end{bmatrix}$ and $A_{\alpha\alpha} \in P_0 \cap Q$. Now as $A \in E_c$ so $A \in E_c \cap Q_0$ and by Theorem 2.9, Lemke's algorithm processes LCP(q,A).

THEOREM 5.2. *Suppose that* $A \in R^{n\times n}$ *with* $A = \begin{bmatrix} A_{\alpha\alpha} & + \\ - & 0 \end{bmatrix}$ *and* $A_{\alpha\alpha} \in P_0 \cap Q$. *Then* $A \in P_0$.

Proof. Since $A \in R^{n\times n}$ with $A = \begin{bmatrix} A_{\alpha\alpha} & + \\ - & 0 \end{bmatrix}$ and $A_{\alpha\alpha} \in P_0 \cap Q$ then by Remark 5.1, $A \in E_c$. Again by Theorem 2.11, $A \in E_0^f$. As $A \in \tilde{E}_0^s$ by Theorem 5.1, $A \in L$ by Theorem 4.1. By applying degree theory, $A \in P_0$ in view of Corollary 3.1 of [20]. $\square$

REMARK 5.3. Jones and Gowda [17] raised the following open problem: Is it true that $P_0 \cap Q_0 = E_c \cap Q_0$? Cao and Ferris [3] showed that $P_0 \cap Q_0 = E_c \cap Q_0$ is true for second order matrices. We settle the above open problem partially by considering a subclass of $P_0 \cap \tilde{E}_0^s = E_c \cap Q_0$. Note that $P_0 \cap \tilde{E}_0^s \subseteq P_0 \cap Q_0$.

In general, SOL(q,A) is not bounded for every $q \in$ *int pos*$[-A,I]$ and $A \in \tilde{E}_0^s$. *int pos*$[-A,I]$ denotes the relative interior of *pos*$[-A,I]$. A vector $q \in$ *int pos*$[-A,I]$ if and only if $\exists\ z \geqslant 0$ and $w > 0$ such that $w = q + Az$. For details see [8]. Here we establish the following results.

THEOREM 5.3. *Let* $A \in \tilde{E}_0^s$ *and suppose* $SOL(q,A)$ *is not bounded for all* $q \in$ *int pos*$[-A,I]$. *Suppose* $r \in K(A)$ *and* $\exists$ *vectors* z *and* $z^\lambda = \hat{z} + \lambda z$ *such that* $z \in SOL(0,A) \setminus \{0\}$, $z^\lambda \in SOL(q,A)$ *for any suitable* $\hat{z}$ *and* $\forall\, \lambda \geqslant 0$ *with* $w \in SOL(r,A)$. *Then* $(z^\lambda - w)_\alpha (A(z^\lambda - w))_\alpha < 0\ \forall\ \alpha = \{i : z_i \neq 0\}$.

Proof. Suppose $A \in \tilde{E}_0^s$ and SOL(q,A) is not bounded for all $q \in$ *int pos*$[-A,I]$. Note that $A \in E_0^s \cap L_2$ as shown in Theorem 4.1 and $q \in$ *int pos*$[-A,I]$ and there exist vectors z and $z^\lambda = \hat{z} + \lambda z$ such that $z \in$ SOL$(0,A) \setminus \{0\}$ and $z^\lambda \in$ SOL(q,A) $\forall\, \lambda \geqslant 0$. We select an $r \in K(A)$ such a way that $\alpha = \{i : z_i \neq 0\}$. Then $r_i - q_i < 0$ Now for sufficiently large λ, $(z^\lambda - w)_\alpha > 0$ and $w \in$ SOL(r,A). We write

$$(A(z^\lambda - w))_\alpha = -q_\alpha - (Aw)_\alpha \leqslant -q_\alpha + r_\alpha < 0.$$

This implies

$$(z^\lambda - w)_\alpha (A(z^\lambda - w))_\alpha < 0. \quad \square$$

However, strict inequality of $(z^\lambda - w)_\alpha (A(z^\lambda - w))_\alpha < 0$ does not hold in case of $\alpha \neq \{i : z_i \neq 0\}$. For details see [1].

THEOREM 5.4. *Let $A \in \tilde{E}_0^s$ and suppose $SOL(q,A)$ is not bounded for all $q \in$ int pos$[-A,I]$. Suppose $r \in K(A)$ and $\exists$ vectors z and $z^\lambda = \hat{z} + \lambda z$ such that $z \in SOL(0,A) \setminus \{0\}$, $z^\lambda \in SOL(q,A)$ for any suitable $\hat{z}$ and $\forall\, \lambda \geqslant 0$ with $w \in SOL(r,A)$. Then $(z^\lambda - w)_\alpha (A(z^\lambda - w))_\alpha \leqslant 0 \;\forall\, \alpha = \{i : \hat{z}_i \geqslant 0, z_i = 0\}$.*

Proof. The first part of the proof follows from the proof of Theorem 5.3. Now we select an $r \in K(A)$ and consider $\alpha = \{i : z_i \neq 0\}$. We select an $r \in K(A)$ and consider $\alpha = \{i : z_i = 0\}$. Then $r_i - q_i \geqslant 0$. Now for sufficiently large λ, $(z^\lambda - w)_\alpha > 0$ and $w \in$ SOL(r,A). Now we consider following two cases.

Case I: Let $\alpha = \{i : \hat{z}_i > 0, z_i = 0\}$. Then $r_i - q_i = 0$. We write

$$\begin{aligned}(z^\lambda - w)_i (A(z^\lambda - w))_i &= (z^\lambda - w)_i ((Az^\lambda)_i - (Aw)_i + q_i - r_i) \\ &= z_i^\lambda ((Az^\lambda)_i + q_i) - w_i((Az^\lambda)_i + q_i) \\ &\quad + z_i^\lambda(-(Aw)_i - r_i) - w_i(-(Aw)_i - r_i) \\ &\leqslant 0.\end{aligned}$$

Case II: Let $\alpha = \{i : z_i = \hat{z}_i = 0\}$. Then $r_i - q_i > 0$. We write

$$\begin{aligned}(z^\lambda - w)_i (A(z^\lambda - w))_i &= -w_i((Az^\lambda)_i - (Aw)_i) \\ &\leqslant -w_i((Az^\lambda)_i - (Aw)_i + q_i - r_i) \\ &= -w_i(Az^\lambda + q)_i + w_i(Aw + r)_i \\ &= -w_i(Az^\lambda + q)_i \\ &\leqslant 0. \quad \square\end{aligned}$$

Now we show the condition for which SOL(q,A) is compact where $A \in \tilde{E}_0^s$. To establish the result we use game theoretic approach and Ville's theorem of alternative.

THEOREM 5.5. *Suppose $A \in \tilde{E}_0^s$ with $v(A) > 0$. Then $SOL(q,A)$ is compact.*

Proof. By theorem 4.1, $\tilde{E}_0^s \subseteq E_0 \cap L_2$. This implies $A \in Q_0$. Since $v(A) > 0$, $A \in E_0^s \cap Q$. Now to establish $A \in R_0$ it is enough to show that LCP$(0,A)$ has only trivial solution. Suppose not, then LCP$(0,A)$ has nontrivial solution, i.e. say, $0 \neq x \in$ SOL$(0,A)$ then $0 \neq x \geqslant 0$, $Ax \geqslant 0$ and $x^T Ax = 0$. Since $A \in E_0^s$, we can write $A^T x \leqslant 0$. Now $A^T x \leqslant 0$, $0 \neq x \geqslant 0$ has a solution. According to Ville's theorem of alternative, there does not exist $x > 0$ such that $Ax > 0$. However, $Ax > 0$, $x > 0$ has a solution since $A \in Q$. See [[8], Page no. 184]. This is a contradiction. Hence LCP$(0,A)$ has only trivial solution. Therefore $A \in Q \cap R_0$. Now by Theorem 2.16, $A \in Q_b$. Hence SOL(q,A) is nonempty and compact. $\square$

We illustrate the result with the help of an example.

EXAMPLE 5.1. Consider the matrix $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 1 \\ -2 & -2 & 1 \end{bmatrix}$. Now $x^T Ax = 3x_1x_2 + x_3^2 - x_3(x_1 + x_2)$. Now we consider the following four cases.

Case I: For $x_1 = 0,\ x_2 = k,\ x_3 = 0,$ where $k > 0$. Here $x \geqslant 0,\ x^T Ax = 0$ holds but in this case $Ax \ngeqslant 0$.

Case II: For $x_1 = k,\ x_2 = 0,\ x_3 = 0,$ where $k > 0$. Here $x \geqslant 0,\ x^T Ax = 0$ holds but in this case $Ax \ngeqslant 0$.

Case III: $x_1 = 0,\ x_2 = k,\ x_3 = k,$ where $k > 0$. Here $x \geqslant 0,\ x^T Ax = 0$ holds but in this case $Ax \ngeqslant 0$.

Case IV: $x_1 = k,\ x_2 = 0,\ x_3 = k,$ where $k > 0$. Here $x \geqslant 0,\ x^T Ax = 0$ holds but in this case $Ax \ngeqslant 0$.

Hence zero vector is the only vector for which $x \geqslant 0,\ Ax \geqslant 0,\ x^T Ax = 0$ implies $A^T x \leqslant 0$ holds. So $A \in E_0^s$-matrix. Also it is clear that $A \in \tilde{E}_0^s$. Here we get that LCP$(0,A)$ has unique solution. Hence $A \in R_0$.

The following result shows that the solution set of LCP(q,A) is stable when $A \in \tilde{E}_0^s$.

THEOREM 5.6. *Suppose $A \in \tilde{E}_0^s$ with $v(A) > 0$, if the LCP(q,A) has unique solution x^*, then LCP(q,A) is stable at x^*.*

Proof. As $A \in \tilde{E}_0^s$ with $v(A) > 0$, then by Theorem 5.5, $A \in R_0$. Again as shown in the Theorem 2.6, $A \in \operatorname{int}(Q) \cap R_0$. So by Theorem 2.7, if the LCP(q,A) has unique solution x^*, then LCP(q,A) is stable at x^*. □

6. Iterative algorithm to process LCP(q,A)

Todd and Ye [28] proposed a projective algorithm to solve linear programming problem considering a suitable merit function. Using the same merit function Pang [26] proposed an iterative descent type algorithm with a fixed value of the parameter κ to process LCP(q,A) where A is a row sufficient matrix. Kojima et al. [18] proposed an interior point method to process P_0-matrices using similar type of merit function. Here we propose a modified version of interior point algorithm by using a dynamic κ for each iterations in line with Pang [26] for finding solution of LCP(q,A) given that $A \in \tilde{E}_0^s$. Note that $\tilde{E}_0^s$ contains P_0-matrices as well as non P_0-matrices. We prove that the search directions generated by algorithm are descent and show that the proposed algorithm converges to the solution under some defined conditions.

ALGORITHM. Let $z > 0,\ w = q + Az > 0,$ and $\psi : R_{++}^n \times R_{++}^n \to R$ such that $\psi(z,w) = \kappa^k \log(z^T w) - \sum_{i=1}^n \log\,(z_i w_i) \geqslant 0$. Further suppose $\rho^k = \min_i \{z_i^k w_i^k\}$ and $\kappa^k > \max\,(n, \frac{z^T w}{\rho^k})$ for k-th iteration.

Step 1: Set $k = 0$. Let $\beta \in (0,1)$ and $\sigma \in (0, \frac{1}{2})$ following line search step and z^0 be a strictly feasible point of LCP(q,A) and $w^0 = q + Az^0 > 0$.

$$\nabla_z \psi_k = \nabla_z \psi(z^k, w^k), \qquad \nabla_w \psi_k = \nabla_w \psi(z^k, w^k)$$

and

$$Z^k = diag(z^k), \qquad W^k = diag(w^k).$$

Step 2: Now to find the search direction, consider the following problem

$$\text{minimize} \qquad (\nabla_z \psi_k)^T d_z + (\nabla_w \psi_k)^T d_w$$

$$\text{subject to} \qquad d_w = A d_z, \qquad \|(Z^k)^{-1} d_z\|^2 + \|(W^k)^{-1} d_w\|^2 \leqslant \beta^2.$$

Step 3: Find the smallest $m_k \geqslant 0$ such that

$$\psi(z^k + 2^{-m_k} d_z^k, w^k + 2^{-m_k} d_w^k) - \psi(z^k, w^k) \leqslant \sigma 2^{-m_k} [(\nabla_z \psi_k)^T d_z^k + (\nabla_w \psi_k)^T d_w^k].$$

Step 4: Set

$$(z^{k+1}, w^{k+1}) = (z^k, w^k) + 2^{-m_k} (d_z^k, d_w^k).$$

Step 5: If $(z^{k+1})^T w^{k+1} \leqslant \varepsilon$, where ε is a very small positive quantity, stop else $k = k+1$.

REMARK 6.1. The algorithm is based on the existence of a strictly feasible point. As $A \in \tilde{E}_0^s$ implies $A \in Q_0$ in view of Theorem 4.1 then existence of a strictly feasible points for such a matrix will eventually lead to the solution of LCP(q, A).

Now we prove the following lemma for E_0-matrices.

LEMMA 6.1. *Suppose* $A \in E_0$, $z > 0$, $w = q + Az > 0$, *and* $\psi : R^n_{++} \times R^n_{++} \to R$ *such that* $\psi(z, w) = \kappa^k \log(z^T w) - \sum_{i=1}^n \log (z_i w_i)$. *Further suppose* $\rho^k = \min_i \{z_i^k w_i^k\}$ *and* $\kappa^k > \max (n, \frac{z^T w}{\rho^k})$ *for each* k*th iteration. Then the search direction* (d_z^k, d_w^k) *generated by the algorithm is descent direction.*

Proof. Consider $r^k = \nabla_z \psi_k + A^T \nabla_w \psi_k$ and first we show that $r^k \neq 0$ for kth iteration. Consider the merit function $z > 0$, $w = q + Az > 0$ and $\psi : R^n_{++} \times R^n_{++} \to R$ such that $\psi(z, w) = \kappa^k \log(z^T w) - \sum_{i=1}^n \log (z_i w_i) \geqslant 0$. Note that

$$\begin{aligned}\big(\nabla_z \psi(z, w)\big)_i &= \tfrac{\kappa^k}{z^T w} v_i - \tfrac{1}{z_i w_i} w_i \\ &= w_i \big[\tfrac{\kappa^k}{z^T w} - \tfrac{1}{z_i w_i}\big].\end{aligned}$$

Similarly we show

$$\big(\nabla_w \psi(z, w)\big)_i = z_i \big[\tfrac{\kappa^k}{z^T w} - \tfrac{1}{z_i w_i}\big].$$

Again for kth iteration $\kappa^k > \max (n, \frac{z^T w}{\rho^k})$ where $\rho^k = \min_i \{z_i^k w_i^k\}$. By the construction of κ^k, it implies

$$z_i\left(\frac{\kappa^k}{z^T w} - \frac{1}{z_i w_i}\right) > 0.$$

For details, see (page no. 462, [8]). Therefore $\left(\nabla_w \psi(z,w)\right)_i > 0 \ \forall\, i$. In a similar way we can show that $\left(\nabla_z \psi(z,w)\right)_i > 0 \ \forall\, i$. Now $A \in E_0$. So $A^T \in E_0$. By the definition of semimonotonicity for $\left(\nabla_w \psi(z,w)\right) > 0 \ \exists$ a j such that $(A^T \nabla_w \psi(z,w))_j \geqslant 0$. Therefore $(\nabla_z \psi(z,w))_j + (A^T \nabla_v \psi(z,w))_j \neq 0$ for at least one j. Hence $\nabla_z \psi(z,w) + A^T \nabla_v \psi(z,w) \neq 0$. Again $A^k = (Z^k)^{-2} + A^T (W^k)^{-2} A$ is positive definite as

$$\begin{aligned} x^T A^T (W)^{-2} A x &= (Ax)^T (W)^{-2} A x \\ &= (y)^T (W)^{-2} y \end{aligned}$$

and $(y)^T (W)^{-2} y \geqslant 0, \ \forall\, y \in R^n, \ A^T (W)^{-2} A$ is positive semidefinite.

So $\tau_k = \dfrac{\sqrt{(r^k)^T (A^k)^{-1} r^k}}{\beta}$ is positive. We have $d_z^k = -\frac{(A^k)^{-1} r^k}{\tau_k}$, $d_w^k = A d_z^k$ from the algorithm. Now we show that $(\nabla_z \psi_k)^T d_z^k + (\nabla_w \psi_k)^T d_w^k < 0$. We derive

$$\begin{aligned} (\nabla_z \psi_k)^T d_z^k + (\nabla_w \psi_k)^T d_w^k &= \left[\nabla_z \psi_k + A^T \nabla_w \psi_k\right]^T d_w^k \\ &= -\tfrac{1}{\tau_k}\left(\sqrt{(r^k)^T (A^k)^{-1} r^k}\right)^2 \\ &= -\tau_k \beta^2 < 0. \end{aligned}$$

We consider

$$\psi(z^k + 2^{-m_k} d_z^k, w^k + 2^{-m_k} d_w^k) - \psi(z^k, w^k) \leqslant \sigma 2^{-m_k} [(\nabla_z \psi_k)^T d_z^k + (\nabla_w \psi_k)^T d_w^k].$$

Since $0 < \beta, \sigma < 1$, we say $\psi(z^k + 2^{-m_k} d_z^k, w^k + 2^{-m_k} d_w^k) - \psi(z^k, w^k) < 0$. Hence (d_z^k, d_w^k) is descent direction in this algorithm. □

REMARK 6.2. Note that the Lemma 6.1 is true for $\tilde{E}_0^s$-matrices as $\tilde{E}_0^s \subseteq E_0$.

REMARK 6.3. We consider dynamic κ to extend the applicability of the algorithm proposed by Pang [26]. By choosing different values of κ instead of a fixed value in each iterations, we extend the use of Lemma 6.1 for E_0-matrices.

We prove the following theorem to show that the proposed algorithm converges to the solution under some defined condition.

THEOREM 6.1. *If $A \in \tilde{E}_0^s$ and $LCP(q,A)$ has a strictly feasible solution, then every accumulation point of $\{z^k\}$ is the solution of $LCP(q,A)$.*

Proof. The proof follows from the Theorem 4 of [26]. □

7. Numerical illustration

A numerical example is considered to show the performance of the proposed algorithm.

EXAMPLE 7.1. We consider the following example of LCP(q,A), where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & -5 & 0 \end{pmatrix} \text{ and } q = \begin{pmatrix} -4 \\ -7 \\ 10 \end{pmatrix}.$$

It is easy to show that $A \in \tilde{E}_0^s$. We apply proposed algorithm to find solution of the given problem. According to Theorem 6.1 algorithm converges to solution with $z^0, w^0 > 0$. To start with we initialize $\beta = 0.5,\ \gamma = 0.5,\ \sigma = 0.2,$ and $\varepsilon = 0.00001$. We set $z^0 = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ and obtain $w^0 = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$.

Iteration (k)	z^k	w^k	d_z^k	d_w^k	$\psi(z^k, w^k)$
1	$\begin{pmatrix} 1.05 \\ 1.09 \\ 4.76 \end{pmatrix}$	$\begin{pmatrix} 1.85 \\ 4.62 \\ 2.42 \end{pmatrix}$	$\begin{pmatrix} 0.106 \\ 0.189 \\ -0.487 \end{pmatrix}$	$\begin{pmatrix} -0.298 \\ -0.761 \\ -1.155 \end{pmatrix}$	29.3308
2	$\begin{pmatrix} 1.1 \\ 1.17 \\ 4.53 \end{pmatrix}$	$\begin{pmatrix} 1.7 \\ 4.25 \\ 1.94 \end{pmatrix}$	$\begin{pmatrix} 0.0853 \\ 0.1607 \\ -0.4551 \end{pmatrix}$	$\begin{pmatrix} -0.294 \\ -0.74 \\ -0.974 \end{pmatrix}$	23.2919
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
50	$\begin{pmatrix} 1.07 \\ 1.57 \\ 2.43 \end{pmatrix}$	$\begin{pmatrix} 0.00608 \\ 0.00389 \\ 0.00281 \end{pmatrix}$	$\begin{pmatrix} 0.00047 \\ -0.00017 \\ -0.00154 \end{pmatrix}$	$\begin{pmatrix} -0.00171 \\ -0.00215 \\ -0.00009 \end{pmatrix}$	2.4617
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
96	$\begin{pmatrix} 1.07 \\ 1.57 \\ 2.43 \end{pmatrix}$	$\begin{pmatrix} 0.00001 \\ 0.000000 \\ 0.00000 \end{pmatrix}$	$\begin{pmatrix} -0.000001 \\ -0.00000 \\ -0.000003 \end{pmatrix}$	$\begin{pmatrix} -0.00000 \\ -0.00000 \\ -0.00000 \end{pmatrix}$	1.1684
97	$\begin{pmatrix} 1.07 \\ 1.57 \\ 2.43 \end{pmatrix}$	$\begin{pmatrix} 0.00001 \\ 0.000009 \\ 0.000005 \end{pmatrix}$	$\begin{pmatrix} 0.000002 \\ 0.000000 \\ -0.000000 \end{pmatrix}$	$\begin{pmatrix} -0.000000 \\ -0.000000 \\ -0.000000 \end{pmatrix}$	1.1684
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
100	$\begin{pmatrix} 1.07 \\ 1.57 \\ 2.43 \end{pmatrix}$	$\begin{pmatrix} 0.00000 \\ 0.00000 \\ 0.00000 \end{pmatrix}$	$\begin{pmatrix} 0.000000 \\ -0.000000 \\ -0.000000 \end{pmatrix}$	$\begin{pmatrix} -0.000000 \\ -0.000000 \\ 0.00000 \end{pmatrix}$	1.0565

Table 1: *Summary of computation for the proposed algorithm*

Table 1 summarizes the computations for the first 2 iterations, 50th iteration and 96th, 97th iterations and 100th iteration. At the 100th iteration, sequence $\{z^k\}$ and

$\{w^k\}$ produced by the proposed algorithm give the solution of the given LCP(q,A) i.e. $z^* = \begin{pmatrix} 1.0714 \\ 1.5714 \\ 2.4285 \end{pmatrix}$ and $w^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

8. Concluding remark

In this article, we showed that LCP(q,A) is processable by Lemke's algorithm and the solution set of LCP(q,A) is bounded if $A \in \tilde{E}_0^s \cap P_0$, a subclass of $E_0^s \cap P_0$. It can be shown that non-negative matrices with zero diagonal with at least one $a_{ij} > 0$ with $i \neq j$ is not a $\tilde{E}_0^s$-matrix. Whether a matrix class belongs to $P_0 \cap Q_0$ or not is difficult to verify. We find some conditions under which $\tilde{E}_0^s$-matrix will belong $P_0 \cap Q_0$ which will motivate further study and applications in matrix theory. Finally we propose an iterative and descent type interior point method to compute solution of LCP(q,A).

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(Received May 28, 2020)

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