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HARNACK TYPE INEQUALITIES FOR OPERATORS IN LOGARITHMIC SUBMAJORISATION

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Abstract. This paper studies the Harnack type logarithmic submajorisation and Fuglede–Kadison determinant inequalities for operators in a finite von Neumann algebra. In particular, the Harnack type determinant inequalities due to Lin–Zhang [17] and Yang–Zhang [28] are extended to the case of operators in a finite von Neumann algebra.

1. Introduction

The classical Harnack inequality, named after Carl Gustav Axel von Harnack, gives an estimate from above and an estimate from below for a positive harmonic function in a domain. Though the classical Harnack inequality is a direct consequence of the Poisson formula, variants and developed forms of the Harnack inequality have been demonstrated as an important tool in the general theory of harmonic functions and partial differential equations. There exist as yet extensive works on generalized Harnack inequalities in various forms, see [19, 27, 28] for a nice introduction about the inequality. The purpose of this paper is to investigate the Harnack type determinant inequality for operators and matrices.

With the help of Lagrange multiplier method, the following Harnack type determinant inequality was established by Tung [23], as a tool to study Harnack inequality: If $Z \in \mathbb{M}_n$ is a complex matrix with singular values r_k with $0 \leqslant r_k < 1$, $k = 1,2,\ldots,n$, then

$$\prod_{k=1}^{n} \frac{1-r_k}{1+r_k} \leqslant \frac{\det(\mathbb{I}-Z^*Z)}{|\det(\mathbb{I}-UZ)|^2} \leqslant \prod_{k=1}^{n} \frac{1+r_k}{1-r_k}, \quad U \in \mathbb{U}_n, \tag{1.1}$$

where $\mathbb{U}_n$ denotes the set of all $n \times n$ unitary matrices U. From these bounds Tung obtained upper and lower bounds of a Poisson kernel on $\mathbb{U}_n$ (see [23]), hence that the so-called Harnack's first and second theorems are established. Tung's work drew immediate attention of Hua and Marcus. Using majorisation theory and singular value (eigenvalue) inequalities of Weyl, Marcus [18] gave another proof of (1.1) and gave an equivalent form of (1.1). Almost at the same time, a proof of (1.1) was also given by Hua [11] based on the determinantal inequality he had previously obtained in [10].

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In the past decades, Tung's work has attracted attentions of mathematicians and been extended to various setting (see [15, 17, 19, 27, 28] and the references therein for more details). Among these outstanding works we will be interested in Lin–Zhang's and Yang–Zhang's work. Specifically, with $A = UZ$, (1.1) is equivalently rewritten in terms of eigenvalues ([15, 28]) as

$$\prod_{k=1}^{n} \frac{1-r_k}{1+r_k} \leqslant \prod_{k=1}^{n} \lambda_k((\mathbb{I}-A^*)^{-1}(\mathbb{I}-A^*A)(\mathbb{I}-A)^{-1}) \leqslant \prod_{k=1}^{n} \frac{1+r_k}{1-r_k}, \tag{1.2}$$

where $Z \in \mathbb{M}_n$ is a complex matrix with singular values r_k with $0 \leqslant r_k < 1, k = 1, 2, \dots, n$ and $U \in \mathbb{U}_n$. (1.2) leads to the study of inequalities of logarithmic submajorisation of eigenvalues and singular values. Following this line, an interesting generalization of (1.2) is presented by Yang–Zhang [28] and Jiang–Lin [15] as follows:

$$\prod_{k\in K} \lambda_k((\mathbb{I}-A^*)^{-1}(\mathbb{I}-A^*A)(\mathbb{I}-A)^{-1}) \leqslant \prod_{k\in K} \frac{1+r_k}{1-r_k}, \tag{1.3}$$

$$\prod_{i\in K} \lambda_{n-k+1}((\mathbb{I}-A^*)^{-1}(\mathbb{I}-A^*A)(\mathbb{I}-A)^{-1}) \geqslant \prod_{k\in K}(1-r_k^2) \prod_{i=1}^{|K|} \frac{1}{(1+r_i)^2}, \tag{1.4}$$

where K is a subset of $\{r_1, r_2, \dots, r_n\}$ and $|K|$ denotes the number of terms in K. The main theme of the paper is to continue with Jiang–Lin and Yang–Zhang's work and to show their results hold in the case of operators in finite von Neumann algebras.

We are concerned with the Harnack type logarithmic submajorisation inequality and Fuglede-Kadison determinant inequality for operators in a finite von Neumann algebra. The properties of the logarithmic submajorisation and Fuglede–Kadison determinant for operators in a finite von Neumann algebra was investigated by many authors, see for example [4, 2, 14]. Those properties are important, for example, in investigation of noncommutative Hardy spaces and invariant subspaces for operators in von Neumann algebras. By adapting the techniques in [28, 9, 21], we obtain some inequalities which is related to the Harnack type logarithmic submajorisation inequality and Fuglede–Kadison determinant inequality. In particular, we show that the inequalities (1.3) and (1.4) hold for operators in a finite von Neumann algebra. We will conclude this paper with a series of logarithmic submajorisation inequalities which is related to Cayley transform.

2. Preliminaries

2.1. Von Neumann algebras

Suppose that $\mathscr{H}$ is a separable Hilbert space over the field $\mathbb{C}$ and $\mathbb{I}$ is the identity operator in $\mathscr{H}$. We will denote by $\mathscr{B}(\mathscr{H})$ the $*$-algebra of all linear bounded operators in $\mathscr{H}$. Let $\mathscr{M}$ be a $*$-subalgebra of $\mathscr{B}(\mathscr{H})$ containing the identity operator $\mathbb{I}$. Then $\mathscr{M}$ is called a von Neumann algebra if $\mathscr{M}$ is weak* operator closed. Let $\mathscr{M}^+$ denote the positive part of $\mathscr{M}$. We recall that a weight on $\mathscr{M}$ is a map $\tau : \mathscr{M}^+ \to [0, \infty]$ satisfying

1. $\tau(x+y) = \tau(x) + \tau(y)$, for all $x, y \in \mathscr{M}^+$;

2. $\tau(\alpha x) = \alpha\tau(x)$ for all $x \in \mathscr{M}^+$ and $\alpha \in [0, \infty)$, with the convention $0 \cdot \infty = 0$.

The weight τ is called faithful if $\tau(x^*x) = 0$ implies $x = 0$, normal if $x_i \uparrow_i x$ in $\mathscr{M}^+$ implies that $0 \leqslant \tau(x_i) \uparrow_i \tau(x)$, tracial if $\tau(x^*x) = \tau(xx^*)$ for all $x \in \mathscr{M}$. Note that since (x_i) is bounded there is x in $\mathscr{M}^+$ such that, for any h in $\mathscr{H}$, $\langle x_i h, h\rangle \uparrow \langle xh, h\rangle$, which implies that x_i tends to x weak* and hence $x \in \mathscr{M}^+$. The operator x is obviously the least upper bound of (x_i), it is natural to denote it by $\sup_i x_i$. The self-adjoint part of $\mathscr{M}$, $\mathscr{M}^{sa}$, is a partially ordered vector space under the ordering $x \geqslant 0$ defined by $\langle x\xi, \xi\rangle \geqslant 0, \xi \in \mathscr{H}$. Recall that $x \in \mathscr{M}$ is contractive if $\|x\| \leqslant 1$ and strictly contractive if $\|x\| < 1$. Moreover, if x is strict contractive, then $\mathbb{I} - x^*x$ is invertible and $\mathbb{I} - x^*x \geqslant 0$.

It is also customary to say trace instead of tracial weight. A trace τ is called finite if $\tau(\mathbb{I}) < \infty$. A finite trace τ is extended uniquely to a positive linear functional on $\mathscr{M}$ which will also be denoted by τ. A positive linear functional τ on a von Neumann algebra is said to be a state if $\tau(\mathbb{I}) = 1$.

A von Neumann algebra $\mathscr{M}$ is called finite if the family formed of the finite normal traces separates the points of $\mathscr{M}$. Clearly this happens if $\mathscr{M}$ admits a single faithful normal finite trace. But a finite $\mathscr{M}$ may fail to have any faithful finite trace, for instance $\mathscr{M} = \ell^\infty(\mathbb{R})$ where $\mathbb{R}$ is equipped with counting measure. However, on a separable Hilbert space (i.e. if $\mathscr{M}$ is weak*-separable) the converse is also true i.e., $\mathscr{M}$ is finite if and only if it admits a faithful normal finite trace.

In what follows, we will keep all previous notations throughout the paper, and $\mathscr{M}$ will always denote a finite von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$, with a normal faithful finite tracial state τ, i.e., a normal faithful finite trace τ satisfies that $\tau(\mathbb{I}) = 1$. We refer to [24] for von Neumann algebras.

2.2. The eigenvalue function and generalized singular value function

DEFINITION 2.1. Let $x \in \mathscr{M}$ and $t > 0$. The "t-th singular number of x" $\mu_t(x)$ is defined by

$$\mu_t(x) = \inf\{\|xe\| : e \text{ is a projection in } \mathscr{M} \text{ with } \tau(e^\perp) \leqslant t\}.$$

We denote simply by $\mu(x)$ the function $t \to \mu_t(x)$. The generalized singular number function $t \to \mu_t(x)$ is decreasing right-continuous. For convenience to discuss the properties of $\mu_t(x)$ we define $\mu_t^\ell(x)$ by

$$\mu_t^\ell(x) = \inf\{\|xe\| : e \text{ is a projection in } \mathscr{M} \text{ with } \tau(e^\perp) < t\}.$$

Then $t \to \mu_t^\ell(x)$ is non-increasing and left-continuous. For $x \in \mathscr{M}$ and $t > 0$, we have $\mu_t(x) = \inf\{s : \tau(\mathbb{I} - e_s(|x|)) \leqslant t\}$ and $\mu_t^\ell(x) = \inf\{s : \tau(\mathbb{I} - e_s(|x|)) < t\}$, where the operators $e_s(|x|)$ are the spectral projection of $|x|$. Therefore, $\mu_t^\ell(x) = \mu_t(x)$ holds for almost every $t \in [0, 1]$ since the map $s \to \tau(\mathbb{I} - e_s(|x|))$ is non-increasing and continuous from the right (hence, it is almost everywhere continuous). See [6, 9, 25, 26] for basic properties and detailed information on $\mu_t(x)$ and $\mu_t^\ell(x)$.

If x is self-adjoint and $x = \int_{-\infty}^{\infty} t de_t(x) \in \mathscr{M}$ is the spectral resolution of x then for any Borel subset $B \subseteq (-\infty,\infty)$ we denote by $e_B(x)$ the corresponding spectral projection. However, we write $e_s(x) = e_{(-\infty,s]}(x)$. Given $x \in \mathscr{M}^{sa}$, the spectral scale $\lambda_t(x)$ on $(0,\tau(\mathbb{I}))$ is defined by

$$\lambda_t(x) = \inf\{s \in \mathbb{R} : \tau(\mathbb{I} - e_s(x)) \leqslant t\}.$$

Obviously, if $0 \leqslant x \in \mathscr{M}$ then $\lambda_t(x) = \mu_t(x)$ for $0 < t < 1$. The spectral scale $\lambda_t(x)$ is non-increasing and right-continuous. For the properties of $\lambda_t(\cdot)$, it is important to note that $\lambda_t(x + a\mathbb{I}) = \lambda_t(x) + a$ for every $x \in \mathscr{M}^{sa}$ and $a \in \mathbb{R}$. This property enables us to deduce estimations for $\lambda_t(x)$ from formulas on $\mu_t(x)$.

To achieve our main results, we state some properties of $\lambda(\cdot)$ and $\mu(\cdot)$ without proof (see [12, 9]).

PROPOSITION 2.2. (see [12, 9]) *Let $x,y \in \mathscr{M}$ and $v \in \mathscr{M}$. Then*

1. $\mu(|x|) = \mu(x) = \mu(x^*)$ *and* $\mu(\alpha x) = |\alpha|\mu_t(x)$, *for* $t > 0$ *and* $\alpha \in \mathbb{C}$.
2. *Let f be a bounded continuous increasing function on $[0,\infty)$ with $f(0) = 0$. Then $\mu(f(x)) = f(\mu(x))$ and $\tau(f(x)) = \int_0^{\tau(1)} f(\mu_t(x))dt$.*
3. $\mu_{s+t}(x+y) \leqslant \mu_t(x) + \mu_s(y),\ s,t > 0.$
4. *If $0 \leqslant x \leqslant y$, then $\mu_t(x) \leqslant \mu_t(y)$.*
5. $\mu_{t+s}(xy) \leqslant \mu_t(x)\mu_s(y),\ s,t > 0.$
6. *If x,y are self-adjoint, then $\lambda_{t+s}(x+y) \leqslant \lambda_t(x) + \lambda_s(y),\ t,s \geqslant 0,\ t+s \leqslant 1$.*
7. *If $0 \leqslant t \leqslant 1$ and x,y are self-adjoint, then $\lambda_t(x) \geqslant 0$ implies that $\lambda_t(v^*av) \leqslant \|v\|^2\lambda_t(x)$.*
8. *If x,y are self-adjoint and $x \leqslant y$, then $\lambda_t(x) \leqslant \lambda_t(y)$.*
9. *If x is self-adjoint, then $\lambda_t(f(x)) = f(\lambda_t(x)),\ t \in (0,\tau(\mathbb{I}))$, for every increasing continuous function f on $\mathbb{R}$.*

EXAMPLE 2.3. Let $\mathscr{H} = \mathbb{C}^n$ and let $\mathscr{M} = \mathscr{B}(\mathscr{H}) \cong \mathbb{M}_n(\mathbb{C})$ equipped with the normalized trace $\tau_n :\triangleq \frac{1}{n} tr_n$ where tr_n is the standard trace on $\mathbb{M}_n(\mathbb{C})$. If $x \in \mathscr{B}(\mathscr{H}) = \mathbb{M}_n(\mathbb{C})$ is self-adjoint, then x can be written as $x = \sum_{i=1}^n \alpha_j p_j$, where $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$ is the sequence of eigenvalues of x in which each is repeated according to its multiplicity and $\sum_{i=1}^n p_j = \mathbb{I}$. Therefore,

$$\lambda_t(x) = \sum_{j=1}^{n} \alpha_j \chi_{[\frac{j-1}{n},\frac{j}{n})}(t), t \in [0,1).$$

If $x \geqslant 0$, then $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n \geqslant 0$, $\lambda_t(x) = \mu_t(x)$ and $\mu_t^{\ell}(x) = \sum_{j=1}^n \alpha_j \chi_{(\frac{j-1}{n},\frac{j}{n}]}$.

If $x \in \mathbb{M}_n(\mathbb{C})$ is arbitrary, then $\mu_t(x) = \mu_t(|x|)$ and the eigenvalues of $|x|$ are usually called the singular values of x. It follows that

$$\mu_t(x) = \sum_{j=1}^{n} s_j \chi_{[\frac{j-1}{n}, \frac{j}{n})}(t)$$

and

$$\mu_t^{\ell}(x) = \sum_{j=1}^{n} s_j \chi_{(\frac{j-1}{n}, \frac{j}{n}]}(t),$$

where $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n \geqslant 0$ is the sequence of singular values of x, repeated according to multiplicity. It is clear that $\mu_{\frac{j-1}{n}}(x) = \mu^{\ell}_{\frac{j}{n}}(x)$.

Note that if $x \in \mathbb{M}_n(\mathbb{C})$ is self-adjoint, then x can also be written as $x = \sum_{i=1}^{m} \beta_j p_j$, where $\beta_1 > \beta_2 > \cdots > \beta_m (m \leqslant n)$. Then

$$\lambda_t(x) = \sum_{j=1}^{m} \beta_j \chi_{[d_{j-1}, d_j)}(t),$$

where $d_j = \sum_{i=1}^{j} \tau(p_i)$ for $j = 1, 2, \cdots, m$ and $d_0 = 0$. For each j, the length of the interval $[nd_{j-1}, nd_j)$ is $n\tau_n(p_j)$, which is the dimension of the eigenspace corresponding to β_j. See [13, 6] for more details of $\mu_t(\cdot)$ and $\lambda_t(\cdot)$ of operators and matrices (Note: the generalized singular values μ_k, as defined in [13], is denoted by $\mu^{\ell}_{\frac{k}{n}}$, in this paper; the generalized singular values $\mu_{\frac{k}{n}}$ and $\mu^{\ell}_{\frac{k}{n}}$ is noting but μ_{k+1} and μ_k, respectively, in [13])

EXAMPLE 2.4. Consider the algebra $\mathscr{M} = L^{\infty}([0,1])$ of all Lebesgue measurable essentially bounded functions on $[0,1]$. Algebra $\mathscr{M}$ can be seen as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathscr{H} = L^2([0,1])$, with the trace given by integration with respect to Lebesgue measure m. For a real measurable function $f \in L^{\infty}([0,1])$, the decreasing rearrangement f^* of the function f is given by

$$f^*(t) = \inf\{s \in \mathbb{R} : m(\{h \in [0,1] : f(h) > s\}) \leqslant t\}, \ \ 0 < t < 1.$$

Then $\mu_t(f) = |f|^*(t)$ and $\lambda_t(f) = f^*(t)$. Suppose that $f = \sum_1^n \alpha_i \chi_{B_i}$, where $B_i \subseteq [0,1]$ with $B_i \cap B_j = \emptyset$ whenever $i \neq j$, and $0 < \alpha_j \in \mathbb{R}$ $(j = 1, 2, \cdots, n)$ are such that $\alpha_i \neq \alpha_j$ whenever $i \neq j$. For the computation of $\mu_t(f)$, it may be assumed that $\alpha_1 > \alpha_2 > \cdots > \alpha_n$. Then

$$\lambda_t(f) = \sum_{j=1}^{n} \alpha_j \chi_{[d_{j-1}, d_j)}(t),$$

where $d_j = \sum_{i=1}^{j} m(B_i)$ for $j = 1, 2, \cdots, n$ and $d_0 = 0$. If $f \geqslant 0$, then $\alpha_1 > \alpha_2 > \cdots > \alpha_n \geqslant 0$, $\lambda_t(f) = \mu_t(f)$ and $\mu_t^{\ell}(f) = \sum_{j=1}^{n} \alpha_j \chi_{(d_{j-1}, d_j]}(t)$. See [6, 21] for more details.

2.3. Fuglede–Kadison determinant

Let $\mathscr{M}$ be a finite von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$, with a normal faithful finite tracial state τ. Recall that the Fuglede–Kadison determinant $\Delta = \Delta_\tau : \mathscr{M} \to \mathbb{R}^+$ is defined by $\Delta_\tau(x) = \tau(\log|x|)$ if $|x|$ is invertible; and otherwise, we define $\Delta_\tau(x) = \inf \Delta_\tau(|x| + \varepsilon\mathbb{I})$, the infimum takes over all scalars $\varepsilon > 0$. We define Fuglede–Kadison determinant-like function of x by

$$\Lambda_t(x) = \exp\{\int_0^t \log\mu_s(x)ds\}, \;\; t > 0.$$

Since $\tau(\mathbb{I}) = 1$, if $|x|$ is invertible, then

$$\Delta_\tau(x) = \Lambda_1(x) = \exp\{\int_0^1 \log\mu_s(x)ds\}.$$

We understanding that $\Delta(x) = 0$ if

$$\int_0^{\tau(\mathbb{I})} \log\mu_s(x)ds = -\infty.$$

Recall that x is said to be logarithmically submajorised by y(see [7, 14]), denoted by $x \prec\prec_{\log} y$ (or $\mu(x) \prec\prec_{\log} \mu(y)$), if $\Lambda_t(x) \leqslant \Lambda_t(y)$ for all $t > 0$.

We state for easy reference the following fact, obtained from [1, 4] for Fuglede-Kadison determinant which will be applied below.

PROPOSITION 2.5. *Let $x, y \in \mathscr{M}$. Then*

1. $\Delta_\tau(\mathbb{I}) = 1, \Delta_\tau(xy) = \Delta_\tau(x)\Delta_\tau(y)$,
2. $\Delta_\tau(x) = \Delta_\tau(x^*) = \Delta_\tau(|x|)$, $\Delta_\tau(|x|^\alpha) = (\Delta_\tau(|x|))^\alpha$, $\alpha \in \mathbb{R}^+$
3. $\Delta_\tau(x^{-1}) = (\Delta_\tau(x))^{-1}$, *if x is invertible in $\mathscr{M}$*
4. $\Delta_\tau(x) \leqslant \Delta_\tau(y)$, *if* $0 \leqslant x \leqslant y$
5. $\lim_{\varepsilon\to 0^+} \Delta_\tau(x + \varepsilon 1) = \Delta_\tau(x)$, *if* $0 \leqslant x$.
6. $\Delta_\tau(x) \leqslant \Delta_\tau(y)$, *if* $x \prec\prec_{\log} y$.

See [1, 4, 2] for basic properties and detailed information on Fuglede-Kadison determinant of $x \in \mathscr{M}$.

EXAMPLE 2.6. Let $\mathscr{H} = \mathbb{C}^n$ and let $\mathscr{M} = \mathscr{B}(\mathscr{H}) \cong \mathbb{M}_n(\mathbb{C})$ equipped with the normalized trace $\tau_n :\triangleq \frac{1}{n} tr_n$ where tr_n is the standard trace on $\mathbb{M}_n(\mathbb{C})$. If $x \in \mathscr{B}(\mathscr{H})$, then $\Delta_{\tau_n}(x) = (\det(|x|))^{\frac{1}{n}}$. See [13] for more information on determinant of matrices.

If $x, y \in \mathscr{M}$ and $0 < p < \infty$, then x is said to be p-submajorised by y, denoted by $x \prec\prec_p y$, if $\int_0^t \mu_s(x)^p ds \leqslant \int_0^t \mu_s(y)^p ds$ for all $t > 0$.

REMARK 2.7. Let $x,y \in \mathcal{M}^+$ be invertible. Then the following conditions are equivalent:

1. $\mathbb{I}+rx \prec\prec_{\log} \mathbb{I}+ry$, for all $r \in \mathbb{R}^+$;

2. $x \prec\prec_p y,\ \ 0<p<1$;

3. $x \prec\prec_{\log} y$;

4. $\int_0^t \varphi(\mu_s(x))ds \leqslant \int_0^t \varphi(\mu_s(y))ds$ for all $t>0$ and all nondecreasing functions φ on $[0,\infty)$ such that $\varphi(0)=0$ and $t \to \varphi(e^t)$ is convex.

Indeed, let ψ is a bounded positive measurable function on $[0,\infty)$ and

$$\pi_t(r)=\exp\{\int_0^t \log(1+r\psi(s))ds\}.$$

By [8, Lemma 3.2], we have

$$\int_0^t \psi(s)^p ds = \frac{p\sin(\pi p)}{\pi}\int_0^\infty \frac{\log \pi_t(r)}{r^{p+1}}dr,$$

which implies that (1)⇒(2) holds.

Note that if $(\int_0^t |\varphi(s)|^p \frac{ds}{t})^{\frac{1}{p}} < \infty$, $t>0$ for some $p>0$, then from [22, p. 71] we obtain

$$\exp\{\int_0^t \log|\varphi(s)|\frac{ds}{t}\} = \lim_{p\to 0}(\int_0^t |\varphi(s)|^p \frac{ds}{t})^{\frac{1}{p}},\ \ t>0,$$

which yields (2)⇒(3). (3)⇒(4) follows from the fact that $t \to \varphi(e^t)$ is convex and $\varphi(e^{\log\mu(x)}) = \varphi(\mu(x))$ (see [20, p.22, Theorem D.2]). It is easy to check that (4)⇒(1).

3. Unitary approximation and Logarithmic submajorisation

Our starting point is the following inequality for complex numbers:

$$||z|-1| \leqslant ||z|-v| \leqslant ||z|+1|,\ \ z,v \in \mathbb{C} \text{ with } |v|=1. \tag{3.1}$$

In this section, we will consider some Logarithmic submajorisation inequalities for operator version of (3.1). We start with a lemma which will be used in our proof.

LEMMA 3.1. *Let $x \in \mathcal{M}^+$. Then*

$$\lambda_s(-x) = -\mu^{\ell}_{1-s}(x),\ \ 0<s<1.$$

Proof. Let $x=\sum_{i=1}^n \alpha_i p_i$ with $\alpha_1>\alpha_2>\cdots>\alpha_n \geqslant 0$ and $p_ip_j=0, i\neq j$. Without loss of generality we can assume $\sum_{i=1}^n p_i = \mathbb{I}$. Indeed, if $\sum_{i=1}^n p_i \neq \mathbb{I}$ we write $p_{n+1}=\mathbb{I}-\sum_{i=1}^n p_i$. Replacing p_n by p_n+p_{n+1} in the equation $x=\sum_{i=1}^n \alpha_i p_i$ if $\alpha_n=0$, and replacing $x=\sum_{i=1}^{n}\alpha_i p_i$ by $x=\sum_{i=1}^{n+1}\alpha_i p_i$ if $\alpha_n\neq 0$. Set $d_i=\sum_{j=1}^{i}\tau(p_j)$, $1\leqslant i\leqslant n$ and $d_0=0$. Then $d_n=\sum_{j=1}^{n}\tau(p_j)=\tau(\mathbb{I})=1$,

$$\mu_s^{\ell}(x)=\sum_{i=1}^{n}\alpha_i\chi_{(d_{i-1},d_i]}(s),\ \ 0<s<1,$$

and

$$\lambda_s(-x)=\sum_{j=1}^{n}-\alpha_{n-j+1}\chi_{[1-d_j,1-d_{j-1})}(s),\ \ 0<s<1.$$

Thus,

$$\mu_{1-s}^{\ell}(x)=-\lambda_s(-x),\ \ 0<s<1,$$

For the general case, let $0\leqslant x\in\mathscr{M}$ and let $x=\int_0^{\|x\|}\lambda de_\lambda(x)$ be the spectral decomposition of x. Put

$$f_k(t)=\sum_{j=1}^{2^n}\frac{j\|x\|}{2^n}\chi_{[\frac{(j-1)\|x\|}{2^n},\frac{j\|x\|}{2^n})}(t).$$

Write $x_n=f_n(|x|)$. It follows that $\|x-x_n\|\leqslant\frac{\|x\|}{2^n}$ and $x_n\geqslant x_{n+1}\geqslant x$. The proof is completed by showing that

$$\lim_{n\to\infty}\mu_s^{\ell}(x_n)=\mu_s^{\ell}(x)\text{ and }\lim_{n\to\infty}\lambda_s(-x_n)=\lambda_s(-x).$$

For $\varepsilon>0$, we obtain

$$\mu_s^{\ell}(x_n)\leqslant\mu_{s-\varepsilon}^{\ell}(x)+\mu_{\varepsilon}^{\ell}(x_n-x)\leqslant\mu_{s-\varepsilon}^{\ell}(x)+\|x_n-x\|,\ \ 0<s<1$$

and

$$\lambda_{s+\varepsilon}(-x)-\|x_n-x\|\leqslant\lambda_{s+\varepsilon}(-x)-\lambda_{\varepsilon}(x_n-x)\leqslant\lambda_s(-x_n),\ \ 0<s<1.$$

Taking the $n\to\infty$ of the both side, we get

$$\limsup_{n\to\infty}\mu_s^{\ell}(x_n)\leqslant\mu_{s-\varepsilon}^{\ell}(x),\ \ 0<s<1$$

and

$$\liminf_{n\to\infty}\lambda_s(-x_n)\geqslant\lambda_{s+\varepsilon}(x),\ \ 0<s<1.$$

Since $\lambda(-x)$ is right-continuous and $\mu^{\ell}(x)$ is left-continuous on (0, 1), letting $\varepsilon\downarrow 0$, we obtain

$$\limsup_{n\to\infty}\mu_s^{\ell}(x_n)\leqslant\mu_s^{\ell}(x),\ \ \liminf_{n\to\infty}\lambda_s(-x_n)\geqslant\lambda_s(-x),\ \ 0<s<1.$$

On the other hand, since $x\leqslant x_n$, $-x\geqslant -x_n$, moreover,

$$\liminf_{n\to\infty}\mu_s^{\ell}(x_n)\geqslant\mu_s^{\ell}(x),\ \ \limsup_{n\to\infty}\lambda_s(-x_n)\leqslant\lambda_s(-x),\ \ 0<s<1.$$

Hence $\lim_{n\to\infty}\mu_s^{\ell}(x_n)=\mu_s^{\ell}(x)$, $\limsup_{n\to\infty}\lambda_s(-x_n)=\lambda_s(-x)$, $0<s<1$. This completes the proof. $\square$

PROPOSITION 3.2. *Let $x \in \mathscr{M}$. Then*

$$-\mu^{\ell}_{1-s}(x) \leqslant \lambda_s(Rex) \leqslant \mu_s(x), \quad -\mu^{\ell}_{1-s}(x) \leqslant \lambda_s(Imx) \leqslant \mu_s(x), \quad 0<s<1.$$

Proof. The proof is adapted from [15, Lemma 2.1]. For any $t>0$, we have

$$t^2x^*x+\frac{1}{t^2}\mathbb{I}-(x^*+x)=\Big(tx-\frac{1}{t}\mathbb{I}\Big)^*\Big(tx-\frac{1}{t}\mathbb{I}\Big)\geqslant 0.$$

which tell us that

$$t^2x^*x+\frac{1}{t^2}\mathbb{I}\geqslant 2Rex\geqslant -\Big(t^2x^*x+\frac{1}{t^2}\mathbb{I}\Big). \tag{3.2}$$

Proposition 2.2(8) now yields

$$\lambda_s\Big(t^2x^*x+\frac{1}{t^2}\mathbb{I}\Big)\geqslant \lambda_s(2Rex)\geqslant \lambda_s\Big(-\Big(t^2x^*x+\frac{1}{t^2}\mathbb{I}\Big)\Big), \quad 0<s<1 \tag{3.3}$$

for all $s>0$. An easy calculation shows that $\mu_s(y+\mathbb{I})=\mu_s(y)+1$ and $\mu^{\ell}_s(y+\mathbb{I})=\mu^{\ell}_s(y)+1$ for $y\geqslant 0$ and $0<s<1$. Combining this with (3.3) and Proposition 2.2 we can assert that

$$\begin{aligned} t^2\mu_s(x)^2+\frac{1}{t^2}1 &= \mu_s\Big(t^2x^*x+\frac{1}{t^2}\mathbb{I}\Big) \\ &= \lambda_s\Big(t^2x^*x+\frac{1}{t^2}\mathbb{I}\Big) \\ &\geqslant \lambda_s(2Rex), \quad 0<s<1. \end{aligned}$$

If it was true that $\mu_s(x)=0$ for some $s>0$, there would be $2\lambda_t(Rex)=\lambda_t(2Rex)\leqslant 0$ by take $t\to\infty$. Otherwise, we take $t=\frac{1}{\mu_s(x)^{\frac{1}{2}}}$, it follows that $\lambda_s(Rex)\leqslant\mu_s(x),\ 0<s<1$.

On the other hand, combining (3.3) with Proposition 2.2 and Lemma 3.1 yields

$$\begin{aligned} \lambda_s(2Rex) &\geqslant \lambda_s\Big(-\Big(t^2x^*x+\frac{1}{t^2}\mathbb{I}\Big)\Big) \\ &= -\mu^{\ell}_{1-s}\Big(t^2x^*x+\frac{1}{t^2}\mathbb{I}\Big) \\ &= -t^2\mu^{\ell}_{1-s}(x^*x)-\frac{1}{t^2}, \quad 0<s<1. \end{aligned}$$

We now apply the above argument again, with $\mu(x)$ replaced by $\mu^{\ell}(x)$, to obtain $\lambda_s(Rex)\geqslant -\mu^{\ell}_{1-s}(x),\ 0<s<1$. Finally, since $Re(-ix)=Imx$, from what has already been proved we see that $-\mu^{\ell}_{1-s}(x)=-\mu^{\ell}_{1-s}(-ix)\leqslant\lambda_s(Imx)\leqslant\mu_s(-ix)=\mu_s(x),\ 0<s<1$. $\square$

REMARK 3.3.

1. Let $x\in\mathscr{M}$. From the proof of Lemma 3.1 we have

$$\lambda_s(Rex)\leqslant\mu_s(x), \quad \lambda_s(Imx)\leqslant\mu_s(x), \quad 0<s<1.$$

2. Let $x \in \mathscr{M}$. It follows from inequality (3.2) that

$$t^2x^*x + \frac{1}{t^2}\mathbb{I} \geqslant 2Rex \geqslant -\Big(t^2x^*x + \frac{1}{t^2}\mathbb{I}\Big).$$

Moreover, [14, Lemma 4.2] means that $\mu(2Rex) \prec\prec_{\log} \mu(t^2x^*x + \frac{1}{t^2}\mathbb{I})$ for all $t > 0$, with $-\infty$ allowed for values. Moreover, we have

$$\Delta_\tau(2Rex) \leqslant \Delta_\tau\Big(t^2x^*x + \frac{1}{t^2}\mathbb{I}\Big), \ \ t > 0.$$

COROLLARY 3.4. *Let $x, y \in \mathscr{M}$ and let $\alpha \in \mathbb{R}$. If $x^* = x$, then*

$$\lambda_s(iRey - iy) \leqslant \mu_s(y - \alpha x), \ \ 0 < s < 1$$

and

$$\lambda_s(y - iImy) \leqslant \mu_s(y - i\alpha x), \ \ 0 < s < 1.$$

Proof. The results follow from Remark 3.3(1) along with the fact that $-i(y - Rey) = Imy = Im(y - \alpha x)$ and $y - iImy = Rey = Re(y - i\alpha x)$. □

PROPOSITION 3.5. *Let $0 \leqslant x \in \mathscr{M}$ such that $\|x\| > 1$.*

1. *If $u \in \mathscr{M}$ is an unitary operator, then*

$$\mu(x - Reu) \prec\prec_{\log} \mu(x + \mathbb{I}),$$

which implies that

$$\Delta_\tau(x - Reu) \leqslant \Delta_\tau(x + \mathbb{I}).$$

2. *If $u \in \mathscr{M}$ is an unitary operator and $\tau(|x - \mathbb{I}|) = \tau(|x - u|)$, then*

$$\Delta_\tau(x - u) \leqslant \Delta_\tau(x - \mathbb{I}).$$

Proof. (1). From $-\mathbb{I} \leqslant -Reu \leqslant \mathbb{I}$, we deduce that $-(x + \mathbb{I}) \leqslant x - \mathbb{I} \leqslant x - Reu \leqslant x + \mathbb{I}$. Then we conclude from [14, Lemma 4.2] that

$$\mu(x - Reu) \prec\prec_{\log} \mu(x + \mathbb{I}).$$

Hence we see that

$$\Delta_\tau(x - Reu) \leqslant \Delta_\tau(x + \mathbb{I}).$$

(2). Note that [5, Corollary 2.6] leads to

$$\mu(x - \mathbb{I}) \prec\prec \mu(x - u).$$

Since $\tau(|x - \mathbb{I}|) = \tau(|x - u|)$, [3, Theorem 3.3] shows that $\tau(|x - u|^p) \leqslant \tau(|x - \mathbb{I}|^p)$, $0 < p < 1$, i.e.

$$\int_0^1 \mu_t(x - u)^p \leqslant \int_0^1 \mu_t(x - \mathbb{I})^p dt, \ \ 0 < p < 1.$$

Hence, from $\int_0^1 |f(s)|^p ds)^{\frac{1}{p}} < \infty$ and [22, p.74] we obtain

$$exp\{\int_0^1 \log|f(s)|ds\} = \lim_{p\to 0}(\int_0^1 |f(s)|^p ds)^{\frac{1}{p}},$$

which force $\Delta_\tau(x-u) \leqslant \Delta_\tau(x-\mathbb{I})$. $\square$

LEMMA 3.6. *Let $0 \leqslant x \in \mathscr{M}$ be invertible. Then*

$$\mu_t^\ell(x^{-1}) = \mu_{1-t}(x)^{-1}, \ \ 0<t<1.$$

Proof. Without loss of generality, we may assume that $\mathscr{M}$ has no minimal projections (otherwise we consider the von Neumann algebra $\mathscr{M}\otimes L^\infty([0,1])$). Since $0 \leqslant x \in \mathscr{M}$ is invertible, $\mu_s(x) > 0$ for all $0<s<1$, and so $\mu_{1-t}(x)^{-1}$ is well defined for $0<t<1$. First we assume that $x = \sum_{i=1}^n \alpha_i p_i$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_n > 0$ and $\sum_{i=1}^n p_i = 1, p_i p_j = 0, i \neq j$. Thus $x^{-1} = \sum_{i=1}^n \frac{1}{\alpha_i} p_i$. Let $d_i = \sum_{j=1}^i \tau(p_j), 1 \leqslant i \leqslant n$. Then $d_n = \tau(\mathbb{I}) = 1$,

$$\mu_t(x) = \alpha_1 \chi_{(0,d_1)}(t) + \sum_{i=2}^n \alpha_i \chi_{[d_{i-1},d_i)}(t), \ \ 0<t<1,$$

and

$$\mu_t^\ell(x^{-1}) = \sum_{i=2}^n \frac{1}{\alpha_i}\chi_{(1-d_i,1-d_{i-1}]}(t) + \frac{1}{\alpha_1}\chi_{(1-d_1,1)}(t), \ \ 0<t<1.$$

Therefore,

$$\mu_t^\ell(x^{-1}) = \frac{1}{\mu_{1-t}(x)}, \ \ 0<t<1.$$

For the general case, let $0 \leqslant x \in \mathscr{M}$. Since $x^{-1} \in \mathscr{M}$, there exists $\delta > 0$ such that $x = \int_\delta^{\|x\|} \lambda de_\lambda(x)$ is the spectral decomposition of x. Put

$$f_k(t) = \sum_{j=1}^{2^n} \Big(\delta + \frac{(j-1)a}{2^n}\Big)\chi_{[\delta+\frac{(j-1)a}{2^n},\delta+\frac{ja}{2^n})}(t),$$

where $a = \|x\| - \delta > 0$. Obviously, $0 \leqslant f_k(t) \leqslant f_{k+1}(t) \leqslant t$. Set

$$x_n = f_n(|x|) = \sum_{j=1}^{2^n} \Big(\delta + \frac{(j-1)a}{2^n}\Big) e_{[\delta+\frac{(j-1)a}{2^n},\delta+\frac{ja}{2^n})}(x).$$

Then

$$x_n^{-1} = f_n(|x|) = \sum_{j=1}^{2^n} \Big(\delta + \frac{(j-1)a}{2^n}\Big)^{-1} e_{[\delta+\frac{(j-1)a}{2^n},\delta+\frac{ja}{2^n})}(x).$$

It follows that $\|x - x_n\| \leqslant \frac{a}{2^n}$ and

$$\|x^{-1} - x_n^{-1}\| \leqslant \Big(\delta + \frac{(j-1)a}{2^n}\Big)^{-1} - \Big(\delta + \frac{ja}{2^n}\Big)^{-1} \leqslant \frac{1}{\delta^2}\frac{a}{2^n}.$$

Hence we infer from [9, Lemma 3.4] that $\mu_t(x) = \lim_{n\to\infty}\mu_t(x_n)$. On the other hand, picking up a small $\varepsilon > 0$, we obtain

$$\mu_t^\ell(x_n^{-1}) \leqslant \mu_{t-\varepsilon}^\ell(x^{-1}) + \mu_\varepsilon^\ell(x^{-1} - x_n^{-1}) \leqslant \mu_{t-\varepsilon}^\ell(x^{-1}) + \|x^{-1} - x_n^{-1}\|.$$

Letting $\varepsilon \downarrow 0$ we get

$$\mu_t^\ell(x_n^{-1}) \leqslant \mu_t^\ell(x^{-1}) + \|x^{-1} - x_n^{-1}\|.$$

In consequence, $\limsup_{n\to\infty}\mu_t^\ell(x_n^{-1}) \leqslant \mu_t^\ell(x^{-1})$. Therefore, $x^{-1} \leqslant x_n^{-1}$ tells us that

$$\liminf_{n\to\infty}\mu_t^\ell(x_n^{-1}) \geqslant \mu_t^\ell(x^{-1}).$$

Hence $\mu_t^\ell(x) = \lim_{n\to\infty}\mu_t^\ell(x_n)$. This completes the proof. □

EXAMPLE 3.7. Let $\mathscr{H} = \mathbb{C}^n$ and let $\mathscr{M} = \mathscr{B}(\mathscr{H}) \cong \mathbb{M}_n(\mathbb{C})$ equipped with the normalized trace $\tau_n :\triangleq \frac{1}{n}tr_n$ where tr_n is the standard trace on $\mathbb{M}_n(\mathbb{C})$. If $x \in \mathbb{M}_n(\mathbb{C})$ is positive and invertible, then x can be written as $x = \sum_{i=1}^n \alpha_j p_j$, where $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n > 0$ is the sequence of eigenvalues of x in which each is repeated according to its multiplicity and $\sum_{i=1}^n p_j = \mathbb{I}$. The proof of Lemma 3.6 tells us that

$$\mu_t^\ell(x^{-1}) = \mu_{\tau(1-t}(x)^{-1},\ \ 0 < t < 1.$$

In particular,

$$\mu_{\frac{k}{n}}^\ell(x^{-1}) = (\alpha_{n+1-k})^{-1} = \mu_{1-\frac{k}{n}}(x)^{-1},\ \ k = 2,\ldots,n.$$

We conclude this section with a series of inequalities of generalized singular value function.

LEMMA 3.8. *Let $x, y \in \mathscr{M}$.*

1. *If $x^* = x$, then $\lambda_t(x) \leqslant \mu_t(x)$.*
2. *If $s, t > 0$ such that $s + t < 1$, then $1 \leqslant \mu_t(x) + \mu_s(\mathbb{I} - x)$ and $1 \leqslant \mu_t^\ell(x) + \mu_s^\ell(\mathbb{I} - x)$.*
3. *For any $t > 0$ we have $1 \leqslant \mu_t(x) + \mu_{1-t}^\ell(\mathbb{I} - x)$, $1 \leqslant \mu_t^\ell(x) + \mu_{1-t}^\ell(\mathbb{I} - x)$ and $1 \leqslant \mu_t^\ell(x) + \mu_{1-t}(\mathbb{I} - x)$.*
4. *For any $t > 0$ we have $1 \leqslant \mu_t(x) + \mu_{1-t}^\ell(x \pm i\mathbb{I})$, $1 \leqslant \mu_t^\ell(x) + \mu_{1-t}^\ell(x \pm i\mathbb{I})$ and $1 \leqslant \mu_t^\ell(x) + \mu_{1-t}(x \pm i\mathbb{I})$.*
5. *If $0 \leqslant x \in \mathscr{M}$ and $\|x\| \leqslant 1$, then*

$$\mu_t(1 - x) = 1 - \mu_{1-t}^\ell(x),\ \ \mu_t^\ell(1 - x) = 1 - \mu_{1-t}(x).$$

Proof. (1). Since $-|x| \leqslant x \leqslant |x|$, $\lambda_t(x) \leqslant \lambda_t(|x|) = \mu_x(x)$. (2)-(4) follow from the fact $\mu_{s+t}(x+y) \leqslant \mu_t(x) + \mu_s(y)$ and $\mu_{s+t}^\ell(x+y) \leqslant \mu_t^\ell(x) + \mu_s^\ell(y)$. (5). This follows by the same method as in Lemma 3.6. □

4. Harnack type inequality for operator

In this section Harnack type inequalities for operators in Logarithmic submajorisation are stated and proved. We will extend the results of Yang–Zhang [28] and Lin–Zhang [17] to the case of finite von Neumann algebra. We start with a lemma which follows by the same method as in [28, Proposition 2].

LEMMA 4.1. *Let $x \in \mathcal{M}$. If $\mathbb{I}-x$ is invertible, then*

$$\begin{aligned}(\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1} =& 2Re((\mathbb{I}-x)^{-1})-\mathbb{I}\\ =& 2Re\Big((\mathbb{I}-x)^{-1}-\frac{1}{2}\mathbb{I}\Big)\\ =& Re((\mathbb{I}+x)(\mathbb{I}-x)^{-1}) = S^*S,\end{aligned}$$

where $S=(\mathbb{I}-x^*x)^{\frac{1}{2}}(\mathbb{I}-x)^{-1}$. Moreover, if $x \in \mathcal{M}$ with $\|x\|<1$, then $\mathbb{I}-x$ is invertible, which implies that the equalities above are true.

THEOREM 4.2. *Let $x \in \mathcal{M}$ with $\|x\|<1$. Then*

$$\mu_t((\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1}) \leqslant \frac{1+\mu_t(x)}{1-\mu_t(x)}, \quad 0<t<1. \tag{4.1}$$

Moreover, for any subset $K \subseteq [0,1]$ we have

$$\begin{aligned}\int_K \log \mu_t((\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1})dt \leqslant& \int_K \log\frac{1+\mu_t(x)}{1-\mu_t(x)}dt\\ \leqslant& \int_0^1 \log\frac{1+\mu_t(x)}{1-\mu_t(x)}dt.\end{aligned}$$

In particular,

$$\frac{\Delta_\tau(\mathbb{I}-x^*x)}{\Delta_\tau(\mathbb{I}-x)^2} \leqslant \exp\int_0^1 \log\frac{1+\mu_t(x)}{1-\mu_t(x)}dt.$$

Proof. We conclude from the definition of $\mu_t(\cdot)$ and $\lambda_t(\cdot)$ that

$$\begin{aligned}\mu_t((\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1}) =& \lambda_t((\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1})\\ =& \lambda_t(2Re((\mathbb{I}-x)^{-1})-\mathbb{I}) \ \text{(Lemma 4.1)}\\ =& \lambda_t(2Re((\mathbb{I}-x)^{-1}))-1 \ \text{(Proposition 2.2(9))}\\ \leqslant& \mu_t(2(\mathbb{I}-x)^{-1})-1 \ \text{(Remark 3.3)}\\ =& \frac{2}{\mu^\ell_{1-s}(\mathbb{I}-x)}-1 \ \text{(Lemma 3.6)}\\ \leqslant& \frac{2}{1-\mu_t(x)}-1 \ \text{(Lemma 3.8)}\\ =& \frac{1+\mu_t(x)}{1-\mu_t(x)}, \quad 0<t<1.\end{aligned}$$

Furthermore, since $\frac{1+\mu_t(x)}{1-\mu_t(x)} \geqslant 1$, (4.1) means that

$$\int_K \log \mu_t((\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1})dt \leqslant \int_K \log \frac{1+\mu_t(x)}{1-\mu_t(x)}dt \leqslant \int_0^1 \log \frac{1+\mu_t(x)}{1-\mu_t(x)}dt.$$

Finally, by (4.1) and Proposition 2.5(1)-(3), we have

$$\begin{aligned}\frac{\Delta_\tau(\mathbb{I}-x^*x)}{\Delta_\tau(\mathbb{I}-x)^2} &= \Delta_\tau((\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1}) \\ &= \exp \int_0^1 \log \mu_t((\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1})dt \\ &\leqslant \exp \int_0^1 \log \frac{1+\mu_t(x)}{1-\mu_t(x)}dt. \quad \square\end{aligned}$$

To achieve one of our main results, we state for easy reference the following fact, which will be applied below.

LEMMA 4.3. ([21, Theorem 2]) *Let $x,y \in \mathscr{M}$ be invertible. If K is a Borel subset of $[0,1]$ with $m(K)=t$ ($m(K)$ denotes the Lebesgue measure of K), then*

$$\int_K \log \mu_s(xy)ds \leqslant \int_0^t \log \mu_s(x)ds + \int_K \log \mu_s(y)ds.$$

LEMMA 4.4. *Let $x,y \in \mathscr{M}$ be invertible. If K is a Borel subset of $[0,1]$ with $m(K)=t$ ($m(K)$ denotes the Lebesgue measure of K), then*

$$\int_K \log \mu_s(x)ds + \int_0^t \log \mu_{1-s}(y)ds \leqslant \int_K \log \mu_s(xy)ds.$$

Proof. Let K^c denote the set $\{t \in [0,1] : t \notin K\}$. Then $m(K^c) = 1-t$. We conclude from Lemma 4.3 that

$$\int_{K^c} \log \mu_s(xy)ds \leqslant \int_{K^c} \log \mu_s(x) + \int_0^{1-t} \log \mu_s(y)ds. \tag{4.2}$$

Note that $x,y \in \mathscr{M}$ are invertible. By Proposition 2.5(1) and (3) we have $\Delta(x) \neq 0$, $\Delta(y) \neq 0$ and

$$-\infty < \int_0^1 \log(\mu_s(x))ds + \int_0^1 \log \mu_s(y)ds = \int_0^1 \log \mu_s(xy)ds < \infty \tag{4.3}$$

Subtracting (4.2) from (4.3) yields

$$\int_K \log \mu_s(x)ds + \int_{1-t}^1 \log \mu_s(y)ds \leqslant \int_K \log \mu_s(xy)ds,$$

i.e.,

$$\int_K \log \mu_s(x)ds + \int_0^t \log \mu_{1-s}(y)ds \leqslant \int_K \log \mu_s(xy)ds. \quad \square$$

REMARK 4.5.

1. Let $x, y \in \mathscr{M}$ and let K be a Borel subset of $[0,1]$ with $m(K) = t$ (here $m(K)$ denotes the Lebesgue measure of K). Then

$$\int_K \log \mu_s(x)ds + \int_0^t \log \mu_{1-s}(y)ds \leqslant \int_K \log \mu_s(xy)ds.$$

Indeed, if x, y are invertible, then it follows from Lemma 4.4. We write $x = u|x|$ and $y = v|y|$ for unitary operators $u, v \in \mathscr{M}$. Then $z = u|x||y^*|v^*$ and $\mu_t(x) = \mu_t(|x|)$, $\mu_t(y) = \mu_t(|y^*|)$, $\mu_t(z) = \mu_t(|x||y^*|)$. Thus, we may without loss of generality assume $x \geqslant 0, y \geqslant 0$ and let

$$z(\varepsilon_1, \varepsilon_2) = (x + \varepsilon_1 \mathbb{I})(y + \varepsilon_2 \mathbb{I}).$$

Note that $\mu_s(x + \varepsilon_1 \mathbb{I}) = \mu_s(x) + \varepsilon_1$ and $\mu_s(y + \varepsilon_2 \mathbb{I}) = \mu_s(y) + \varepsilon_2$. From Lemma 4.4 we see that

$$\begin{aligned} &\int_K \log(\mu_s(x) + \varepsilon_1)ds + \int_0^t \log(\mu_{1-s}(y) + \varepsilon_2)ds \\ &\leqslant \int_K \log \mu_s(z(\varepsilon_1, \varepsilon_2))ds. \end{aligned} \tag{4.4}$$

Moreover, for any projection operators $e \in \mathscr{M}$, we have

$$\|z(\varepsilon_1, \varepsilon_2)e\|^2 = \|e(y + \varepsilon_2 \mathbb{I})(x^2 + 2\varepsilon_1 x + \varepsilon_1^2 \mathbb{I})(y + \varepsilon_2 \mathbb{I})e\|,$$

which implies that $\mu_s(z(\varepsilon_1, \varepsilon_2))$ is decreasing in ε_1. Similarly, $\mu_s(z(\varepsilon_1, \varepsilon_2))$ is decreasing in ε_2. Letting $\varepsilon_i \to 0$ and using the monotone convergence theorem in (4.4), we obtain the desired inequality.

2. Let $x, y \in \mathscr{M}$ and let K be a Borel subset of $[0,1]$ with $m(K) = t$ ($m(K)$ denotes the Lebesgue measure of K). Combining Lemma 4.3 with Lemma 4.4 we can assert that

$$\begin{aligned} \int_K \log \mu_s(x)ds + \int_0^t \log \mu_{1-s}(y)ds &\leqslant \int_K \log \mu_s(xy)ds \\ &\leqslant \int_K \log \mu_s(x) + \int_0^t \log \mu_s(y)ds. \end{aligned}$$

In particular, if $K = [0,t]$, then

$$\begin{aligned} \int_0^t \log \mu_s(x)ds + \int_0^t \log \mu_{1-s}(y)ds &\leqslant \int_0^t \log \mu_s(xy)ds \\ &\leqslant \int_0^t \log \mu_s(x) + \int_0^t \log \mu_s(y)ds. \end{aligned}$$

THEOREM 4.6. *Let $x \in \mathscr{M}$ with $\|x\| < 1$. If K is a Borel subset of $[0,1]$ with $m(K) = t$ ($m(K)$ denotes the Lebesgue measure of K), then*

$$\begin{aligned}&\int_K \log \mu_s((\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1})ds \\ &\geqslant \int_0^t 2\log\frac{1}{1+\mu_s(x)}ds + \int_K \log(1-\mu_{1-s}(x)^2)ds, t>0.\end{aligned}$$

Proof. For convenience, we write $A := (\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1}$. Since $\|x\| < 1$, A is invertible, hence that $\Delta(A) > 0$. Therefore, $\int_0^1 \log\mu_s(A)ds > -\infty$. Using Lemma 4.4 twice, we have

$$\int_K \log\mu_s(A)ds \geqslant \int_0^t 2\log\mu_{1-s}((\mathbb{I}-x)^{-1})ds + \int_K \log\mu_s(\mathbb{I}-x^*x)ds.$$

It follows from Lemma 3.8(3)-(5) and Lemma 3.6 that

$$\begin{aligned}\int_K \log\mu_s(A)ds &\geqslant \int_0^t 2\log\mu_{1-s}((\mathbb{I}-x)^{-1})ds + \int_K \log\mu_s(\mathbb{I}-x^*x)ds \\ &= \int_0^t 2\log\frac{1}{\mu_s^\ell(\mathbb{I}-x)}ds + \int_K \log\mu_s(\mathbb{I}-x^*x)ds \\ &\geqslant \int_0^t 2\log\frac{1}{1+\mu_s^\ell(x)}ds + \int_K \log(1-\mu_{1-s}^\ell(x)^2)ds \\ &= \int_0^t 2\log\frac{1}{1+\mu_s(x)}ds + \int_K \log(1-\mu_{1-s}(x)^2)ds,\end{aligned}$$

because $\mu_s^\ell(x) = \mu_s(x)$ holds for almost every $t \in [0,1]$. □

COROLLARY 4.7. *Let $x \in \mathscr{M}$ with $\|x\| < 1$. Then*

$$\int_0^t \log\mu_{1-s}((\mathbb{I}-x^*)^{-1}(\mathbb{I}-x^*x)(\mathbb{I}-x)^{-1})ds \geqslant \int_0^t \log\frac{1-\mu_s(x)}{1+\mu_s(x)}ds,\ t>0.$$

In particular,

$$\frac{\Delta_\tau(\mathbb{I}-x^*x)}{\Delta_\tau(\mathbb{I}-x)^2} \geqslant \exp\int_0^1 \log\frac{1-\mu_s(x)}{1+\mu_s(x)}ds.$$

Proof. Replacing K by $[1-t,1]$, in Theorem 4.6 we have

$$\begin{aligned}\int_0^t \log\mu_{1-s}(A)ds &= \int_{1-t}^1 \log\mu_s(A)ds \\ &\geqslant \int_0^t 2\log\frac{1}{1+\mu_s(x)}ds + \int_{1-t}^1 \log(1-\mu_{1-s}(x)^2)ds \\ &= \int_0^t 2\log\frac{1}{1+\mu_s(x)}ds + \int_0^t \log(1-\mu_s(x)^2)ds \\ &= \int_0^t \log\frac{1-\mu_s(x)}{1+\mu_s(x)}ds.\end{aligned}$$

Therefore, letting $t \to 1$ yields

$$\int_0^1 \log \mu_s(A)ds = \int_0^1 \log \mu_{1-s}(A)ds \geqslant \int_0^1 \log \frac{1+\mu_s(x)}{1-\mu_s(x)}ds.$$

This completes the proof. $\square$

THEOREM 4.8. *Let $0 \leqslant x_i \in \mathscr{M}$ with $\|x_i\| < 1$, $i = 1,2,\cdots,n$. Then for any unitary operator $u \in \mathscr{M}$ and positive scalars ω_i, $i = 1,2,\cdots,n$, $\sum_i^n \omega_i = 1$, we have*

$$\prod_{i=1}^n \left[\exp \int_0^1 \log \frac{1-\mu_t(x_i)}{1+\mu_t(x_i)}dt\right]^{\omega_i} \leqslant \frac{\Delta_\tau(\mathbb{I}-W^2)}{\Delta_\tau(\mathbb{I}-uW)^2} \leqslant \prod_{i=1}^n \left[\exp \int_0^1 \log \frac{1+\mu_t(x_i)}{1-\mu_t(x_i)}dt\right]^{\omega_i},$$

where $W = \sum_{i=1}^n \omega_i x_i$.

Proof. An easy calculation shows that $1-W^2$ and $1-uW$ are invertible and $W \geqslant 0$ with $\|W\| < 1$. Theorem 4.2 and Corollary 4.7 tell us that

$$\exp \int_0^1 \log \frac{1-\mu_t(x)}{1+\mu_t(x)}dt \leqslant \frac{\Delta(\mathbb{I}-x^*x)}{\Delta(\mathbb{I}-x)^2} \leqslant \exp \int_0^1 \log \frac{1+\mu_t(x)}{1-\mu_t(x)}dt. \tag{4.5}$$

Note that [9, Theorem 4.4] tells us that

$$\int_0^t \mu_s(W)ds \leqslant \int_0^t \sum_{i=1}^n \omega_i \mu_s(x_i)ds.$$

The rest of the proof run as [17, Theorem 5]. For the convenience of the reader, we add a proof. Indeed, the convexity and the monotonicity of the function $f(t) = \log \frac{1+t}{1-t}, 0 \leqslant t < 1$ mean that

$$\int_0^t f(\mu_s(W))ds \leqslant \int_0^t f(\sum_{i=1}^n \omega_i \mu_s(x_i))ds.$$

On the other hand, by Lewent's inequality([17, 16]), we obtain

$$\frac{1+\sum_{i=1}^n \omega_i \mu_s(x_i)}{1-\sum_{i=1}^n \omega_i \mu_s(x_i)} \leqslant \prod_{i=1}^n \left(\frac{1+\mu_s(x_i)}{1-\mu_s(x_i)}\right)^{\omega_i}.$$

Thus

$$\int_0^t f(\mu_s(W))ds \leqslant \int_0^t \log \prod_{i=1}^n \left(\frac{1+\mu_s(x_i)}{1-\mu_s(x_i)}\right)^{\omega_i} ds = \sum_{i=1}^n \omega_i \int_0^t \log \left(\frac{1+\mu_s(x_i)}{1-\mu_s(x_i)}\right)ds.$$

It follows that

$$\exp\left\{\int_0^t \log\left(\frac{1+\mu_s(W)}{1-\mu_s(W)}\right)ds\right\} \leqslant \prod_{i=1}^n \left[\exp \int_0^1 \log \frac{1+\mu_t(x_i)}{1-\mu_t(x_i)}dt\right]^{\omega_i}. \tag{4.6}$$

Moreover, the inequalities in (4.6) reverse by taking reciprocals, which implies

$$\exp\left\{\int_0^t \log\left(\frac{1-\mu_s(W)}{1+\mu_s(W)}\right)ds\right\} \geqslant \prod_{i=1}^n \left[\exp \int_0^1 \log \frac{1-\mu_t(x_i)}{1+\mu_t(x_i)}dt\right]^{\omega_i}. \tag{4.7}$$

Combining (4.5) with (4.6) and (4.7) yields

$$\prod_{i=1}^{n}\left[\exp\int_0^1 \log\frac{1-\mu_t(x_i)}{1+\mu_t(x_i)}dt\right]^{\omega_i} \leqslant \frac{\Delta_\tau(\mathbb{I}-W^2)}{\Delta_\tau(\mathbb{I}-uW)^2} \leqslant \prod_{i=1}^{n}\left[\exp\int_0^1 \log\frac{1+\mu_t(x_i)}{1-\mu_t(x_i)}dt\right]^{\omega_i}. \quad \square$$

5. Cayley transform with logarithmic submajorisation

In this section, we will consider some logarithmic submajorisation inequalities related to Cayley transform. We will extend some results of Yang–Zhang [28] to the case of finite von Neumann algebra.

Let $x \in \mathscr{M}$. If $x+i\mathbb{I}$ is invertible, we call $\mathscr{C}(x) = (x-i\mathbb{I})(x+i\mathbb{I})^{-1}$ the Cayley transform of x.

THEOREM 5.1. *Let $x, y \in \mathscr{M}$ with $\|x\| < 1$, $\|y\| < 1$ and let $\mathscr{C}(x)$ and $\mathscr{C}(y)$ be the Cayley transforms of x and y, respectively. If K is a Borel subset of $[0,1]$ with $m(K) = t$ ($m(K)$ denotes the Lebesgue measure of K), then*

$$\begin{aligned}
&\int_K \log(1-\mu_{1-s}(x))ds - \int_0^t \log(1+\mu_s(x))ds \\
&\leqslant \int_K \log\mu_s(\mathscr{C}(x))ds \\
&\leqslant \int_K \log(1+\mu_s(x)) - \int_0^t \log(1-\mu_s(x))ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_K \log\mu_s(\mathscr{C}(x)-\mathscr{C}(y))ds \\
&\leqslant \int_K \log 2\mu_s(x-y)ds - \int_0^t \log[(1-\mu_s(x))(1-\mu_s(y))]ds.
\end{aligned}$$

Proof. Let us first compute the upper bounds. Remark 4.5 shows that

$$\begin{aligned}
\int_K \log\mu_s(\mathscr{C}(x))ds &= \int_K \log\mu_s((x-i\mathbb{I})(x+i\mathbb{I})^{-1})ds \\
&\leqslant \int_K \log\mu_s(x-i\mathbb{I})ds + \int_0^t \log\mu_s((x+i\mathbb{I})^{-1})ds.
\end{aligned}$$

Together with Lemma 3.8 this gives

$$\begin{aligned}
\int_0^t \log\mu_s((x+i\mathbb{I})^{-1})ds &\leqslant \int_0^t \log[\mu_{1-s}^{\ell}(x+i\mathbb{I})]^{-1}ds \\
&\leqslant \int_0^t \log(1-\mu_s(x))^{-1}ds \\
&= -\int_0^t \log(1-\mu_s(x))ds.
\end{aligned}$$

Thus

$$\int_K \log \mu_s(\mathscr{C}(x))ds \leqslant \int_K \log(1+\mu_s(x)) - \int_0^t \log(1-\mu_s(x))ds$$

The lower bound follows easily by using Remark 4.5. Indeed, from Remark 4.5 we obtain

$$\begin{aligned}
\int_K \log \mu_s(\mathscr{C}(x))ds &\geqslant \int_0^t \log \mu_{1-s}(x-i\mathbb{I})ds + \int_K \log \mu_s((x+i\mathbb{I})^{-1})ds \\
&\geqslant \int_0^t \log(1-\mu_s^\ell(x))ds - \int_K \log \mu_{1-s}^\ell(x+i\mathbb{I})ds \\
&\geqslant \int_0^t \log(1-\mu_s^\ell(x))ds - \int_K \log(1+\mu_s^\ell(x))ds \\
&= \int_0^t \log(1-\mu_s(x))ds - \int_K \log(1+\mu_s(x))ds.
\end{aligned}$$

For the second part, an easy calculation shows that $\mathscr{C}(x) = 1-2i(x+i\mathbb{I})^{-1}$ and

$$\mathscr{C}(x)-\mathscr{C}(y) = 2i(y+i\mathbb{I})^{-1}(x-y)(x+i\mathbb{I})^{-1}.$$

Hence, Remark 4.5 implies that

$$\begin{aligned}
\int_K \log \mu_s(\mathscr{C}(x)-\mathscr{C}(y))ds &= \int_K \log 2\mu_s((y+i\mathbb{I})^{-1}(x-y)(x+i\mathbb{I})^{-1})ds \\
&\leqslant \int_0^t \log \mu_s((y+i\mathbb{I})^{-1})ds + \int_K \log 2\mu_s(x-y)ds \\
&\quad + \int_0^t \log \mu_s((x+i\mathbb{I})^{-1})ds \\
&\leqslant \int_0^t \log[\mu_{1-s}^\ell(y+i\mathbb{I})]^{-1}ds + \int_K \log 2\mu_s(x-y)ds \\
&\quad + \int_0^t \log[\mu_{1-s}^\ell(x+i\mathbb{I})]^{-1}ds \\
&\leqslant \int_0^t \log[1-\mu_s(y)]^{-1}ds + \int_K \log 2\mu_s(x-y)ds \\
&\quad + \int_0^t \log[1-\mu_s(x)]^{-1}ds \\
&= \int_K \log 2\mu_s(x-y)ds \\
&\quad - \int_0^t \log[(1-\mu_s(x))(1-\mu_s(y))]ds. \qquad \square
\end{aligned}$$

If we replace K by $[0,1]$, in Theorem 5.1 we have the following corollary.

COROLLARY 5.2. *Let $x,y \in \mathscr{M}$ with $\|x\|<1$, $\|y\|<1$ and let $\mathscr{C}(x)$ and $\mathscr{C}(y)$ be the Cayley transforms of x and y, respectively. Then*

$$\int_0^1 \log \frac{1-\mu_{1-s}(x)}{1+\mu_s(x)}ds \leqslant \int_0^1 \log \mu_s(\mathscr{C}(x))ds \leqslant \int_0^1 \log \frac{1+\mu_s(x)}{1-\mu_s(x)}ds$$

and

$$\int_0^1 \log \mu_s(\mathscr{C}(x)-\mathscr{C}(y))ds \leqslant \int_0^1 \log \frac{2\mu_s(x-y)}{(1-\mu_s(x))(1-\mu_s(y))} ds.$$

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