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SPHERICAL SYMMETRY OF SOME UNITARY INVARIANTS FOR COMMUTING TUPLES

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Abstract. We discuss spherical and Euclidean analogues of joint spectral radius, joint operator norm and joint numerical radius associated with commuting d-tuples of Hilbert space operators. In particular, we deduce their invariance under the action of the group $\mathscr{U}(d)$ of $d\times d$ unitary matrices. Unlike spectral and numerical radii, the analogues of joint operator norm differ in dimension $d>1$. The joint hyponormality ensures that these analogues of joint operator norm agree in all dimensions. However, the separate hyponormality fails to ensure so.

1. Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{C}$ denote the complex plane. Let $\mathscr{H}$ be a complex Hilbert space and $\mathscr{B}(\mathscr{H})$ stand for the C^*-Algebra of all bounded linear operators on $\mathscr{H}$ with identity $I_{\mathscr{H}}$ (or I if no confusion arises). For $A,B\in\mathscr{B}(\mathscr{H})$, the *cross-commutator* of A and B is given by $[A,B]:=AB-BA$. Let $\mathscr{B}(\mathscr{H})^{(d)}$ denote the set of d-tuples $\mathbf{T}=(T_1,\ldots,T_d)$ of operators in $\mathscr{B}(\mathscr{H})$. When the operators $T_1,\ldots,T_d$ are pairwise commuting, that is $[T_i,T_j]=0$ for all $i,j\in\{1,\ldots,d\}$, we say that $\mathbf{T}$ is a *commuting d-tuple*. A commuting d-tuple $\mathbf{T}$ is said to be *doubly commuting* if $[T_i^*,T_j]=0$ for all $1\leqslant i\neq j\leqslant d$. We say that a commuting d-tuple $\mathbf{T}=(T_1,\ldots,T_d)\in\mathscr{B}(\mathscr{H})^{(d)}$ is *jointly hyponormal* if the operator matrix $(\!([T_j^*,T_i])\!)_{1\leqslant i,j\leqslant d}$ is a positive operator on the d-fold inflation $\oplus_{i=1}^d\mathscr{H}$ of $\mathscr{H}$ (see [2, Definition 1]). For an account on various related notions of hyponormality in several variables, one may refer to [11].

The Taylor spectrum and approximate point spectrum of a commuting d-tuple $\mathbf{T}=(T_1,\ldots,T_d)\in\mathscr{B}(\mathscr{H})^{(d)}$ are denoted by $\sigma_T(\mathbf{T})$ and $\sigma_\pi(\mathbf{T})$, respectively (the reader is referred to [17, 8, 10] for the definitions and basic properties). Recall that $\sigma_T(\mathbf{T})$ is a nonempty compact subset of $\mathbb{C}^d$. The *joint spectral radius* of $\mathbf{T}$ is defined by

$$r(\mathbf{T})=\max\{\|\lambda\|_2,\ \lambda=(\lambda_1,\ldots,\lambda_d)\in\sigma_T(\mathbf{T})\},$$

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where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{C}^d$. We invoke the following analogue of the Gelfand-Beurling spectral radius formula for commuting tuples of Hilbert space operators (see [5, Lemma 5], [15, Theorem 1] and [9, Theorem 1]):

$$r(\mathbf{T}) = \inf_{n\in\mathbb{N}^*} \Big\| \sum_{\substack{|\alpha|=n,\\ \alpha\in\mathbb{N}^d}} \frac{n!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} \Big\|^{\frac{1}{2n}} = \lim_{n\to\infty} \Big\| \sum_{\substack{|\alpha|=n,\\ \alpha\in\mathbb{N}^d}} \frac{n!}{\alpha!} \mathbf{T}^{*\alpha}\mathbf{T}^{\alpha} \Big\|^{\frac{1}{2n}}. \tag{1.1}$$

Here for the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, we used the notations $|\alpha| := \sum_{j=1}^d |\alpha_j|$, $\alpha! := \prod_{k=1}^d \alpha_k!$ and $\mathbf{T}^\alpha := \prod_{k=1}^d T_k^{\alpha_k}$. The *spherical norm* of $\mathbf{T} \in \mathscr{B}(\mathscr{H})^{(d)}$ is given by

$$\|\mathbf{T}\| := \sup\left\{ \left(\sum_{k=1}^d \|T_k x\|^2 \right)^{\frac{1}{2}} ;\, x \in \mathscr{H},\ \|x\| = 1 \right\}.$$

Note that $\|\mathbf{T}\| = \Big\| \sum_{k=1}^d T_k^* T_k \Big\|^{\frac{1}{2}}$. It may be now deduced from (1.1) that $r(\mathbf{T}) \leqslant \|\mathbf{T}\|$.

We are also interested in the *Euclidean* analogue of joint spectral radius and joint norm of a d-tuple $\mathbf{T}$ (see [16, Page 26]):

$$r_e(\mathbf{T}) := \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} r(\lambda_1 T_1 + \ldots + \lambda_d T_d),$$
$$\|\mathbf{T}\|_e := \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \|\lambda_1 T_1 + \ldots + \lambda_d T_d\|,$$

where $\mathbb{B}_d$ denotes the open unit ball of $\mathbb{C}^d$ with respect to the Euclidean norm $\|\cdot\|_2$. It turns out that $\|\mathbf{T}\|$ and $\|\mathbf{T}\|_e$ are always equivalent on $\mathscr{B}(\mathscr{H})^{(d)}$ (see [16, Theorem 1.18] and [13, Proposition 2.1]):

$$\frac{1}{\sqrt{d}}\|\mathbf{T}\| \leqslant \|\mathbf{T}\|_e \leqslant \|\mathbf{T}\|, \quad \mathbf{T} \in \mathscr{B}(\mathscr{H})^{(d)}. \tag{1.2}$$

Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^{(d)}$. Following [8], we define the *joint* (*or spherical*) *numerical radius* of $\mathbf{T}$ as

$$\omega(\mathbf{T}) = \sup\left\{ \left(\sum_{k=1}^d |\langle T_k x, x\rangle|^2 \right)^{\frac{1}{2}} ;\, x \in \mathscr{H},\ \|x\| = 1 \right\}.$$

It was shown in [16, Proof of Theorem 1.19] that $\omega(\mathbf{T})$ coincides with the *Euclidean numerical radius* $\omega_e(\mathbf{T})$ given by

$$\omega_e(\mathbf{T}) := \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \omega(\lambda_1 T_1 + \ldots + \lambda_d T_d).$$

We mention that the usage of notations in [16] differs from this. For any commuting d-tuple $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^{(d)}$, we have

$$r(\mathbf{T}) \leqslant \max\left\{ \frac{1}{2\sqrt{d}}\|\mathbf{T}\|, r(\mathbf{T}) \right\} \leqslant \omega(\mathbf{T}) \leqslant \|\mathbf{T}\|_e \leqslant \|\mathbf{T}\|. \tag{1.3}$$

(see [4, Theorem 2.4] and [3, Theorem 2.2]). We also mention that the second last inequality above follows from the fact that $\omega_e(\mathbf{T}) = \omega(\mathbf{T})$. The inequalities in (1.3) can simultaneously be strict even if $d = 1$. Indeed, let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ be an operator on $\mathbb{C}^2$. One can verify that $r(T) = 1$, $\|T\|_e = \|T\| = \frac{\sqrt{5}+1}{2}$ and $\omega(T) = \frac{3}{2}$. It is clear that the invariants $r(\cdot), \|\cdot\|, \omega(\cdot)$ (spherical) and $r_e(\cdot), \|\cdot\|_e, \omega_e(\cdot)$ (Euclidean) are all unitary invariants.

One of the main results of this note shows that all these invariants have spherical symmetry, and except the joint norm, their spherical and Euclidean analogues coincide (see Theorem 2.1). This disparity is addressed in the rest of this note. In particular, it it shown that we have equality of spherical and Euclidean analogues of all the three invariants for jointly hyponormal tuples (see Corollary 2.1). Finally, we characterize all commuting d-tuples for which joint norm and joint spectral radius coincide (see Theorem 2.2).

2. Spherical symmetry

Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathscr{B}(\mathscr{H})^{(d)}$ and let $\mathscr{U}(d)$ denote the group of complex $d \times d$ unitary matrices. For $U = (u_{jk})_{1 \leqslant j,k \leqslant d} \in \mathscr{U}(d)$, the d-tuple $\mathbf{T}_U$ is given by

$$(\mathbf{T}_U)_j = \sum_{k=1}^{d} u_{jk} T_k, \quad 1 \leqslant j \leqslant d.$$

If $\mathbf{T}$ is commuting, then so is $\mathbf{T}_U$ for every $U \in \mathscr{U}(d)$. The unitary group $\mathscr{U}(d)$ acts faithfully on $\mathscr{B}(\mathscr{H})^{(d)}$ via the group action

$$(U, \mathbf{T}) \to \mathbf{T}_U, \quad U \in \mathscr{U}(d),\ \mathbf{T} \in \mathscr{B}(\mathscr{H})^{(d)}.$$

Following [7], we say that a commuting d-tuple $\mathbf{T}$ is *spherical* if the orbit $\{\mathbf{T}_U : U \in \mathscr{U}(d)\}$ of $\mathbf{T}$ is single-ton up to the unitary equivalence. Interestingly some of the unitary invariants, seen as a function on $\mathscr{B}(\mathscr{H})^{(d)}$, are constant on the orbit of a possibly non-spherical d-tuple.

THEOREM 2.1. *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathscr{B}(\mathscr{H})^{(d)}$ be a commuting d-tuple. Then, for every $U \in \mathscr{U}(d)$, the following hold:*

(i) $r(\mathbf{T}_U) = r(\mathbf{T}) = r_e(\mathbf{T}) = r_e(\mathbf{T}_U)$.

(ii) $\omega(\mathbf{T}_U) = \omega(\mathbf{T}) = \omega_e(\mathbf{T}) = \omega_e(\mathbf{T}_U)$.

(iii) $\|\mathbf{T}_U\| = \|\mathbf{T}\|$ *and* $\|\mathbf{T}\|_e = \|\mathbf{T}_U\|_e$.

REMARK 2.1. There are several remarks.

(1) Geometrically, $\mathscr{U}$-invariance in (i) $(resp.\ (ii))$ says that although Taylor spectrum (resp. joint numerical range) need not be invariant under "rotation" by a unitary, its supremum is unchanged under the same effect.

(2) The equality $r(\mathbf{T}) = r_e(\mathbf{T})$ and $\omega(\mathbf{T}) = \omega_e(\mathbf{T})$ was first observed in [3, Theorem 2.1] and [16, Proof of Theorem 1.19], respectively. It is worth mentioning that (ii) holds even for non-commuting d-tuples.

(3) In general, $\|T\| \neq \|T\|_e$, so there is a disparity in (iii) and (i)–(ii) (see Example 2.1 below for details).

Proof.
(i) We need the following elementary facts:

$$\sup_{(\lambda_1,\dots,\lambda_d)\in\mathbb{B}_d} \Big|\sum_{j=1}^{d} \lambda_j z_j\Big| = \|z\|_2, \quad z \in \mathbb{C}^d. \tag{2.1}$$

For two nonempty sets A, B and a bounded function $f : A \times B \to [0,\infty)$,

$$\sup_{a\in A}\sup_{b\in B} f(a,b) = \sup_{b\in B}\sup_{a\in A} f(a,b). \tag{2.2}$$

We contend that

$$r_e(\mathbf{T}_U) = r(\mathbf{T}), \quad U \in \mathscr{U}(d). \tag{2.3}$$

For $\lambda = (\lambda_1,\dots,\lambda_d) \in \mathbb{C}^d$ and $U \in \mathscr{U}(d)$, consider $S_{U,\lambda} := \sum_{k=1}^{d} \lambda_k (\mathbf{T}_U)_k$. By the spectral mapping property of the Taylor spectrum (see [10, Corollary 3.7] and [17, Lemma 3.1]), we have $\sigma_T(\mathbf{T}_U) = U\sigma_T(\mathbf{T})$, and hence

$$r(S_{U,\lambda}) = \sup \sigma_T(S_{U,\lambda}) = \sup_{z\in\sigma_T(\mathrm{T})} \Big|\sum_{j=1}^{d} \lambda_j (Uz)_j\Big|, \quad \lambda = (\lambda_1,\dots,\lambda_d) \in \mathbb{C}^d.$$

Taking supremum on both sides over unit ball, we may infer from (2.1) and (2.2) that

$$\begin{aligned} r_e(\mathbf{T}_U) &= \sup_{(\lambda_1,\dots,\lambda_d)\in\mathbb{B}_d}\ \sup_{z\in\sigma_T(\mathbf{T})} \Big|\sum_{j=1}^{d} \lambda_j (Uz)_j\Big| \\ &= \sup_{z\in\sigma_T(\mathbf{T})} \|Uz\|_2 \\ &= \sup_{z\in\sigma_T(\mathbf{T})} \|z\|_2 \\ &= r(\mathbf{T}). \end{aligned}$$

Thus the claim stands verified. Letting U to be the identity matrix in (2.3), we obtain $r_e(\mathbf{T}) = r(\mathbf{T})$, and hence applying this fact to the commuting d-tuple $\mathbf{T}_U$, we get $r_e(\mathbf{T}_U) = r(\mathbf{T}_U)$. Another application of (2.3) completes the proof of (i).

(ii) For $U \in \mathscr{U}(d)$ and $\lambda = (\lambda_1,\dots,\lambda_d) \in \mathbb{C}^d$, note that

$$\sum_{j=1}^{d} \lambda_j (\mathbf{T}_U)_j = \sum_{k=1}^{d} \Big(\sum_{j=1}^{d} \lambda_j u_{jk}\Big) T_k.$$

Since $\omega_e(\mathbf{S}^*) = \omega_e(\mathbf{S})$ for any d-tuple $\mathbf{S}$, by the spherical symmetry of $\mathbb{B}_d$,

$$\begin{aligned}\omega_e(\mathbf{T}_U) &= \omega_e(\mathbf{T}_U^*)\\ &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d} \omega\left(\sum_{k=1}^d\Big(\sum_{j=1}^d \overline{\lambda}_j\overline{u}_{jk}\Big)T_k^*\right)\\ &= \sup_{\lambda\in\mathbb{B}_d}\omega\Big(\sum_{k=1}^d (U^*\overline{\lambda})_k T_k^*\Big)\\ &= \omega_e(\mathbf{T}),\end{aligned}$$

where $\overline{\lambda}$ denotes the d-tuple $(\overline{\lambda}_1,\ldots,\overline{\lambda}_d)$ in $\mathbb{C}^d$. Since $\omega(\cdot) = \omega_e(\cdot)$ (see Remark 2.1), we obtain (ii).

(iii) As in the preceding paragraph, one may see that $\|\mathbf{T}_U\|_e = \|\mathbf{T}\|_e$ for any $U \in \mathscr{U}(d)$. Indeed,

$$\begin{aligned}\|\mathbf{T}_U\|_e &= \|\mathbf{T}_U^*\|_e\\ &= \sup_{(\lambda_1,\ldots,\lambda_d)\in\mathbb{B}_d}\Big\|\sum_{k=1}^d\Big(\sum_{j=1}^d \overline{\lambda}_j\overline{u}_{jk}\Big)T_k^*\Big\|\\ &= \sup_{\lambda\in\mathbb{B}_d}\Big\|\sum_{k=1}^d (U^*\overline{\lambda})_k T_k^*\Big\|\\ &= \|\mathbf{T}\|_e.\end{aligned}$$

Finally, the equality $\|\mathbf{T}_U\| = \|\mathbf{T}\|$ is immediate from the identity

$$\sum_{j=1}^d (\mathbf{T}_U^*)_j(\mathbf{T}_U)_j = \sum_{j=1}^d T_j^*T_j, \quad U\in\mathscr{U}(d)$$

(cf. [12, Equation (11)]). □

The following example illustrates that there is disparity in (iii) and (i)-(ii) of Theorem 2.1.

EXAMPLE 2.1. For an arbitrary commuting d-tuple $\mathbf{T}$, we need not have $\|\mathbf{T}\|_e = \|\mathbf{T}\|$. Indeed, $\|\mathbf{T}^*\|_e = \|\mathbf{T}\|_e$ is always true, while $\|\mathbf{T}^*\| = \|\mathbf{T}\|$ fails in general. To see this, we consider the *Drury-Arveson d-shift* $\mathbf{M}_z$, that is, the d-tuple of operators $\mathbf{M}_{z_1},\ldots,\mathbf{M}_{z_d}$ of multiplication by the coordinate functions $z_1,\ldots,z_d$ on the reproducing kernel Hilbert space associated with the positive definite kernel κ given by

$$\kappa(z,w) := \frac{1}{1-\langle z,w\rangle}, \quad z,w\in\mathbb{B}_d.$$

By [1, Remark 3.2], $\sum_{j=1}^d \mathbf{M}_{z_j}\mathbf{M}_{z_j}^* \leqslant I$. Further, it is not difficult to see that $\sum_{j=1}^d \mathbf{M}_{z_j}^*\mathbf{M}_{z_j} \geqslant I$, where equality holds if and only if $d=1$. Putting all these facts together, we may conclude that $\|\mathbf{M}_z^*\| = \|\mathbf{M}_z\|$ if and only if $d=1$. Moreover, by [1, Corollary, Page

192], $\mathbf{M}_{z_j}$ is a hyponormal operator for every $j = 1, \ldots, d$. By (1.1), the Taylor spectrum of $\mathbf{M}_z$ is contained closed unit ball in $\mathbb{C}^d$, and hence it may be deduced from [6, Lemma 3.10] that $\mathbf{M}_z$ is jointly hyponormal if and only if $d = 1$.

The last example together with Theorem 2.1 motivates us to the following question.

QUESTION 2.1. *What are all commuting d-tuples $\mathbf{T}$ for which $\|\mathbf{T}_U\|_e = \|\mathbf{T}_U\|$ for every $U \in \mathscr{U}(d)$?*

In view of Theorem 2.1(iii), the question above is equivalent to characterizing commuting tuples $\mathbf{T}$ for which $\|\mathbf{T}\|_e = \|\mathbf{T}\|$. One obvious necessary condition for this is that $\|\mathbf{T}^*\| = \|\mathbf{T}\|$. This fails even for commuting d-tuples of hyponormal operators (see Example 2.1). It turns out however that the answer to above question is affirmative for jointly hyponormal tuples. Indeed, we have the following general fact.

PROPOSITION 2.1. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^{(d)}$ be a commuting d-tuple. If $r(\mathbf{T}) = \|\mathbf{T}\|$, then $\|\mathbf{T}_U\|_e = \|\mathbf{T}_U\|$ for every $U \in \mathscr{U}(d)$.*

Proof. Assume that $r(\mathbf{T}) = \|\mathbf{T}\|$. We already recorded that $\|\mathbf{T}\|_e \leqslant \|\mathbf{T}\|$ (see (1.2)). We check that $\|\mathbf{T}\|_e \geqslant \|\mathbf{T}\|$. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$, consider the operator $S_\lambda := \sum_{k=1}^d \lambda_k T_k$. Recall the fact that norm of bounded linear operator is at least its spectral radius. Combining this with $r(\mathbf{T}) = \|\mathbf{T}\|$ and Theorem 2.1(i), we obtain

$$\|\mathbf{T}\|_e = \sup_{\lambda \in \mathbb{B}_d} \|S_\lambda\| \geqslant \sup_{\lambda \in \mathbb{B}_d} r(S_\lambda) = r_e(\mathbf{T}) = r(\mathbf{T}) = \|\mathbf{T}\|.$$

We may now apply Theorem 2.1(iii). □

There exists a commuting d-tuple $\mathbf{T}$ for which $\|\mathbf{T}\| = \|\mathbf{T}\|_e$ but $r(\mathbf{T}) \neq \|\mathbf{T}\|$. Indeed, if A is a nonzero nilpotent operator and $\mathbf{T} = (A, \ldots, A)$ is a commuting d-tuple, then $\|\mathbf{T}\| = \|\mathbf{T}\|_e = \sqrt{d}\,\|A\| \neq 0$, but $r(\mathbf{T}) = \sqrt{d}\, r(A) = 0$. Moreover, the equality $r(\mathbf{T}) = \omega(\mathbf{T})$ does not imply in general $r(\mathbf{T}) = \|\mathbf{T}\|_e$, even if $d = 1$. Indeed, let

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

One can verify that $r(T) = 1$, $\|T\| = 2$ and $\omega(T) = 1$. Thus, $r(T) = \omega(T)$ however, $r(T) \neq \|T\|$.

The following result generalizes [13, Theorem 2.1] (in view of the fact that any doubly commuting tuple of hyponormal operators is jointly hyponormal):

COROLLARY 2.1. *Let $\mathbf{T} = (T_1, \ldots, T_d) \in \mathscr{B}(\mathscr{H})^{(d)}$ be a jointly hyponormal d-tuple. Then, for every $U \in \mathscr{U}(d)$,*

$$\|\mathbf{T}_U\| = \|\mathbf{T}\| = \|\mathbf{T}\|_e = \|\mathbf{T}_U\|_e.$$

Proof. By [6, Lemma 3.10], $r(\mathbf{T}) = \|\mathbf{T}\|$. Now apply Proposition 2.1. $\square$

Proposition 2.1 raises the problem of characterizing all commuting tuples $\mathbf{T}$ for which $r(\mathbf{T}) = \|\mathbf{T}\|$. The following theorem answers this plus little more. In particular, the assertion (i) below generalizes [14, Problem 218].

THEOREM 2.2. *Let $\mathbf{T} = (T_1, \dots, T_d) \in \mathscr{B}(\mathscr{H})^{(d)}$ be a commuting d-tuple. The following statements are true:*

(i) $r(\mathbf{T}) = \|\mathbf{T}\|$ if and only if $\omega(\mathbf{T}) = \|\mathbf{T}\|$.

(ii) $r(\mathbf{T}) = \|\mathbf{T}\|_e$ if and only if $\omega(\mathbf{T}) = \|\mathbf{T}\|_e$.

Proof. Assume that $r(\mathbf{T}) = \|\mathbf{T}\|$. Then, by (1.3), we obtain $\omega(\mathbf{T}) = \|\mathbf{T}\|$. Conversely, assume that $\omega(\mathbf{T}) = \|\mathbf{T}\|$. Since $r(\mathbf{T}) \leqslant \|\mathbf{T}\|$ is always true, in order to get the desired result, it suffices to prove that $r(\mathbf{T}) \geqslant \|\mathbf{T}\|$. To see this, let $JtW(\mathbf{T})$ denote the bounded subset of $\mathbb{C}^d$ given by

$$JtW(\mathbf{T}) := \{(\langle T_1 x, x\rangle, \dots, \langle T_d x, x\rangle)\, ; x \in \mathscr{H},\ \|x\| = 1\}.$$

Note that

$$\|\mathbf{T}\| = \omega(\mathbf{T}) = \sup\{\|\lambda\|_2;\ \lambda \in JtW(\mathbf{T})\} = \sup\left\{\|\lambda\|_2;\ \lambda \in \overline{JtW(\mathbf{T})}\right\}.$$

Since $\overline{JtW(\mathbf{T})}$ is compact, there exist $\lambda = (\lambda_1, \dots, \lambda_d) \in \overline{JtW(\mathbf{T})}$ and a sequence $\{x_n\}_{n \geqslant 0}$ of unit vectors in $\mathscr{H}$ such that

$$\left.\begin{aligned} \|\lambda\|_2 &= \|\mathbf{T}\|, \\ \lambda_k = \lim_{n \to +\infty} \langle T_k x_n, x_n\rangle&, \quad k \in \{1, \dots, d\} \end{aligned}\right\}. \tag{2.4}$$

If $\Re(z)$ denotes the real part of the complex number z, then

$$\begin{aligned} \sum_{k=1}^{d} \|(T_k - \lambda_k I)x_n\|^2 &= \sum_{k=1}^{d} \left(\|T_k x_n\|^2 + |\lambda_k|^2 - 2\Re\left(\overline{\lambda}_k \langle T_k x_n, x_n\rangle\right)\right) \\ &\leqslant \|\mathbf{T}\|^2 + \|\lambda\|_2^2 - 2\sum_{k=1}^{d} \Re\left(\overline{\lambda}_k \langle T_k x_n, x_n\rangle\right) \\ &\xrightarrow{n \to +\infty} 0, \end{aligned}$$

where we employed (2.4) in the last step. It follows that

$$\lim_{n \to \infty} \|(T_k - \lambda_k)x_n\| = 0, \quad k \in \{1, \dots, d\},$$

which in turn yields that $\lambda \in \sigma_\pi(\mathbf{T})$. Since $\sigma_\pi(\mathbf{T}) \subseteq \sigma_T(\mathbf{T})$ (see [10, Page 22]), we must have $\|\mathbf{T}\| \leqslant r(\mathbf{T})$. This yields (i).

The conclusion in (ii) may be obtained by imitating the argument of (i). However, for the sake of completeness, we provide a direct argument. To see this, we notice first that by (1.3), $r(\mathbf{T}) = \|\mathbf{T}\|_e$ implies $\omega(\mathbf{T}) = \|\mathbf{T}\|_e$. Conversely, suppose that $\omega(\mathbf{T}) = \|\mathbf{T}\|_e$. By (1.3), we have $r(\mathbf{T}) \leqslant \|\mathbf{T}\|_e$. So, in order to get the desired result, it suffices to prove that $r(\mathbf{T}) \geqslant \|\mathbf{T}\|_e$. For $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$, we let $S_\lambda := \sum_{k=1}^d \lambda_k T_k$. By (i) (the case of $d = 1$), we have

$$r(S_\lambda) = \|S_\lambda\| \Longleftrightarrow \omega(S_\lambda) = \|S_\lambda\|, \quad \lambda \in \mathbb{C}^d. \tag{2.5}$$

On the other hand, since $\omega(\mathbf{T}) = \omega_e(\mathbf{T})$ (see [16, Proof of Theorem 1.19]), by the continuity of the numerical radius and the operator norm there exists $\mu \in \overline{\mathbb{B}}_d$ such that

$$\omega(S_\mu) = \omega(\mathbf{T}) = \|\mathbf{T}\|_e \geqslant \|S_\mu\|.$$

Since $\omega(S_\mu) \leqslant \|S_\mu\|$,

$$\|\mathbf{T}\|_e = \omega(S_\mu) = \|S_\mu\|. \tag{2.6}$$

By (2.5), $r(S_\mu) = \|S_\mu\|$. So, by Theorem 2.1(i) and (2.6), we get

$$r(\mathbf{T}) = r_e(\mathbf{T}) \geqslant r(S_\mu) = \|S_\mu\| = \|\mathbf{T}\|_e.$$

This completes the proof. $\square$

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