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ONE–SIDED STAR PARTIAL ORDER PRESERVERS ON $B(H)$

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Abstract. Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. We classify (possibly non-additive) maps on $B(H)$, with H infinite dimensional, which preserve either the left-star or the right-star partial order in both directions. We also introduce natural, weaker versions of these partial orders and classify their preservers.

1. Introduction and statement of the main results

Let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. We denote by A^* the adjoint operator of $A \in B(H)$ and by $\mathrm{Im}A$ and $\mathrm{Ker}A$ the range and the kernel of $A \in B(H)$, respectively. Many partial orders can be defined on $B(H)$. One of the most used is the star partial order $\leqslant^*$ which was introduced by Drazin [6] and may be defined on $B(H)$ in the following way. We write

$$A \leqslant^* B \quad \text{when} \quad A^*A = A^*B \quad \text{and} \quad AA^* = BA^*, \quad A,B \in B(H).$$

If one of the two conditions defining the star order is omitted, then the remaining condition does not induce a partial order. However, it was shown in [4] that by adding conditions on the images of the considered operators we obtain the following two partial orders.

DEFINITION 1. The left-star partial order on $B(H)$ is a relation defined by

$$A \mathrel{*\!\leqslant} B \quad \text{when} \quad A^*A = A^*B \text{ and } \mathrm{Im}A \subseteq \mathrm{Im}B, \quad A,B \in B(H).$$

The right-star partial order on $B(H)$ is a relation defined by

$$A \leqslant\!* \; B \quad \text{when} \quad AA^* = BA^* \text{ and } \mathrm{Im}A^* \subseteq \mathrm{Im}B^*, \quad A,B \in B(H).$$

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It is interesting to find the form of the maps which preserve a relation, a quantity or some subsets. For example, let $\leqslant$ be any partial order on $B(H)$. We say the map Φ on $B(H)$ is a bi-preserver of $\leqslant$ (that is, Φ preserves $\leqslant$ in both directions) if

$$A \leqslant B \quad \text{if and only if} \quad \Phi(A) \leqslant \Phi(B), \quad A,B \in B(H).$$

Let $M_n(\mathbb{F})$, where $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$, be the set of all $n \times n$ real or complex matrices. Surjective bi-preservers of the star, or the left-star, or the right-star partial order on $M_n(\mathbb{F})$, $n \geqslant 3$, have already been characterized; see [10, 5] and also [8]. More precisely, in [5, Theorem 3] the following main result was proved.

PROPOSITION 2. *Let $n \geqslant 3$ be an integer. Then a surjection $\Phi\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is a bi-preserver of the left-star partial order if and only if there exist invertible $T,W \in M_n(\mathbb{F})$ such that Φ has the following form:*

$$\Phi(X) = T\left(\dot{X}\dot{X}^{\dagger} + (I-\dot{X}\dot{X}^{\dagger})\cdot T^{-1}T^{-*}\cdot\dot{X}\dot{X}^{\dagger}\cdot\left[\dot{X}\dot{X}^{\dagger}\cdot T^{-1}T^{-*}\cdot\dot{X}\dot{X}^{\dagger}\right]^{\dagger}\right)\dot{X}W.$$

Here the map $X \mapsto \dot{X}$ denotes either identity, or entrywise conjugation, or Moore-Penrose inverse, or entrywise-conjugated Moore-Penrose inverse on $M_n(\mathbb{F})$.

Results on star, or left-star, or right-star partial order preservers on $M_n(\mathbb{F})$ were extended to $B(H)$ or some subsets of $B(H)$ in [3, 4]. In [4] it is assumed that preservers of the left-star or the right-star partial orders on $B(H)$ with H infinite-dimensional are bijective and additive. It is the aim of this paper to further generalize this result by omitting additivity and injectivity.

Recall that the Moore-Penrose inverse of an operator $A \in B(H)$ is an operator, denoted by $A^{\dagger} \in B(H)$, which satisfies the four equations:

$$A^{\dagger}AA^{\dagger} = A^{\dagger}, \quad AA^{\dagger}A = A, \quad (A^{\dagger}A)^* = (A^{\dagger}A), \quad (AA^{\dagger})^* = (AA^{\dagger}).$$

Clearly, $(A^{\dagger})^{\dagger} = A$. By applying adjoint on all four equations we also see that $(A^{\dagger})^*$ is the Moore-Penrose inverse of A^*, that is,

$$(A^{\dagger})^* = (A^*)^{\dagger}.$$

Moreover, by the four equations which define the Moore-Penrose inverse, $AA^{\dagger}$ is a projection (i.e., a self-adjoint idempotent) onto $\operatorname{Im} A$, which must therefore be closed. Note also that $A \in B(H)$ has a Moore-Penrose inverse if and only if the range of A is closed (see, e.g. [12]). Since A^* has a Moore-Penrose inverse whenever A does, we see that $\operatorname{Im} A^*$ is closed whenever A has a Moore-Penrose inverse.

The Moore-Penrose inverse, when it exists, is unique. Namely, if B satisfies the same four equations, then

$$\begin{aligned} B &= BAB = B(AB)^* = BB^*A^* = BB^*A^*(A^*)^{\dagger}A^* = B(AB)^*(AA^{\dagger})^* \\ &= B(ABA)A^{\dagger} = BAA^{\dagger} = (BA)(A^{\dagger}A)A^{\dagger} = A^*B^*A^*(A^*)^{\dagger}A^{\dagger} \\ &= A^*(A^*)^{\dagger}A^{\dagger} = (A^{\dagger}A)A^{\dagger} = A^{\dagger}. \end{aligned}$$

Moreover it exists for all operators with closed range. In fact, if $A\colon H=\operatorname{Ker}A\oplus(\operatorname{Ker}A)^{\perp}\to H=(\operatorname{Im}A)^{\perp}\oplus(\operatorname{Im}A)$ is such an operator, then its Moore-Penrose inverse,

$$A^{\dagger}\colon H=(\operatorname{Im}A)^{\perp}\oplus(\operatorname{Im}A)\to H$$

is defined as zero on $(\operatorname{Im}A)^{\perp}$ and as the inverse, $\big(A|_{(\operatorname{Ker}A)^{\perp}}\big)^{-1}$ on $\operatorname{Im}A$ (see [9, Theorem 2.4, page 80]). It follows that $A^{\dagger}A$ is a projector onto $\operatorname{Im}A^{\dagger}=(\operatorname{Ker}A)^{\perp}=\overline{\operatorname{Im}A^{*}}=\operatorname{Im}A^{*}$.

In particular, for operators A,B with closed range,

$$\begin{aligned}\operatorname{Im}A^{\dagger}\subseteq\operatorname{Im}B^{\dagger}\Leftrightarrow\operatorname{Im}A^{*}\subseteq\operatorname{Im}B^{*}\Leftrightarrow(\operatorname{Ker}A)^{\perp}\subseteq(\operatorname{Ker}B)^{\perp}\\ \Leftrightarrow\operatorname{Ker}B\subseteq\operatorname{Ker}A\Leftrightarrow A(\operatorname{Ker}B)=0\Leftrightarrow A(I-B^{\dagger}B)=0\end{aligned}\tag{1}$$

where the last identify holds because $(I-B^{\dagger}B)$ is a projection onto $(\operatorname{Im}B^{*})^{\perp}=\operatorname{Ker}B$. Also, the following string of implications for a closed range operator T

$$\begin{aligned}T^{\dagger}X=0\Rightarrow TT^{\dagger}X=0\Rightarrow X^{*}(TT^{\dagger})=0\Rightarrow X^{*}TT^{\dagger}T=X^{*}T=0\Rightarrow T^{*}X=0\\ \Rightarrow X^{*}TT^{\dagger}=0\Rightarrow T^{\dagger}(TT^{\dagger})X=T^{\dagger}X=0\end{aligned}$$

proves that

$$T^{*}X=0\quad\text{if and only if}\quad T^{\dagger}X=0\tag{2}$$

(see also [1]). Hence, by its definition, and in view of (1)

$$A^{\dagger}\leqslant^{*} B^{\dagger}\Leftrightarrow(A^{\dagger})^{*}A^{\dagger}=(A^{\dagger})^{*}B^{\dagger}\quad\text{and}\quad A(I-B^{\dagger}B)=0.\tag{3}$$

By inserting $T=A^{\dagger}$ and $X=B^{\dagger}-A^{\dagger}$ into (2) we see that the first equality is equivalent to

$$AA^{\dagger}=(A^{\dagger})^{\dagger}A^{\dagger}=(A^{\dagger})^{\dagger}B^{\dagger}=AB^{\dagger}.\tag{4}$$

By multiplying it with $A^{\dagger}(\cdot)B$ and utilizing at the end also the second equality in (3) we get

$$A^{\dagger}AA^{\dagger}B=A^{\dagger}AB^{\dagger}B=A^{\dagger}A,$$

so $A^{\dagger}B=A^{\dagger}A$. By (2) this is equivalent to $A^{*}B=A^{*}A$. On the other hand, by multiplying (4) with $A^{\dagger}$ and taking the adjoints we get $(A^{\dagger})^{*}=(B^{\dagger})^{*}(A^{\dagger}A)^{*}=(B^{\dagger})^{*}(A^{\dagger}A)$. It follows that $\operatorname{Im}(A^{\dagger})^{*}\subseteq\operatorname{Im}(B^{\dagger})^{*}$ or equivalently, $\operatorname{Im}(A)\subseteq\operatorname{Im}(B)$. Hence, (3) implies $A\ {}^{*}\!\!\leqslant\ B$.

This shows that the Moore-Penrose inverse $X\mapsto X^{\dagger}$ is a well-defined map on the set of operators with closed range and it does preserve the ${}^{*}\!\!\leqslant$ order in both directions.

However, the general form of surjective bi-preservers of the left-star partial order on $B(H)$ cannot be of the same form as in Proposition 2, since an arbitrary operator in $B(H)$ does not necessarily have a closed range.

It is easy to check (see e.g., [3]) that the map $\Phi\colon B(H)\to B(H)$ defined by

$$\Phi(A)=UAT,\quad A\in B(H),\tag{5}$$

where $U \in B(H)$ is a unitary operator and $T \in B(H)$ is invertible, is a bi-preserver of the left-star partial order. We will show that such maps are the only possible surjective bi-preservers of the left-star partial order, with only one additional possibility that $U\colon H \to H$ may be an anti-unitary operator. Recall that, by its definition, an anti-unitary operator U is a conjugate-linear sujective isometry. Its adjoint, U^* is defined by $\langle Ux,y\rangle = \langle U^*y,x\rangle$, where $\langle\cdot,\cdot\rangle$ is a scalar product on H. Our main result therefore reads as follows.

THEOREM 3. *Let H be an infinite-dimensional complex Hilbert space. Then $\Phi\colon B(H) \to B(H)$ is a surjective bi-preserver of the left-star partial order $\overset{*}{\leqslant}$ if and only if*

$$\Phi(A) = UAT, \quad A \in B(H),$$

where U is a unitary (or anti-unitary) operator on H and T is an invertible bounded linear (respectively conjugate-linear) operator on H.

It is interesting to observe that for infinite-dimensional Hilbert spaces the structure of surjective left-star partial order bi-preservers is simpler than in finite dimensional spaces, see Proposition 2. In particular, this simpler structure shows yet again that the Moore-Penrose inverse cannot be extended to operators with non-closed range.

Observe that for $A,B \in B(H)$ the following holds (see, e.g., [4, Lemma 3])

$$A \overset{*}{\leqslant} B \quad \text{if and only if} \quad A^* \leqslant\!\!* \; B^*. \tag{6}$$

Let $\Phi\colon B(H) \to B(H)$ be a surjective bi-preserver of the right-star partial order. Applying Theorem 3 on the map $\Psi(X) = (\Phi(X^*))^*, X \in B(H)$, which by (6) is a bi-preserver of the left-star order, we obtain the next corollary.

COROLLARY 4. *Let H be an infinite-dimensional complex Hilbert space. Then $\Phi\colon B(H) \to B(H)$ is a surjective bi-preserver of the right-star partial order $\leqslant\!\!*$ if and only if*

$$\Phi(A) = TAU, \quad A \in B(H),$$

where U is a unitary (or anti-unitary) operator on H and T is an invertible bounded linear (respectively conjugate-linear) operator on H.

REMARK 5. Our results easily extend to classify converters from $\overset{*}{\leqslant}$ to $\leqslant\!\!*$ i.e., to surjective maps $\Psi\colon B(H) \to B(H)$, where H is infinite-dimensional, with the property $A \overset{*}{\leqslant} B$ if and only if $\Psi(A) \leqslant\!\!* \; \Psi(B)$. Namely, given any such Ψ the map $\Phi(X) = \Psi(X)^*$ preserves $\overset{*}{\leqslant}$ order.

Note that, unlike in finite-dimensional spaces, the images of operators on an infinite-dimensional Hilbert space H need not be closed. It is hence natural to consider also the weak counterparts to the left- and right- star partial orders where one compares the closures of images. They coincide with the classical ones on finite-dimensional spaces and are defined as follows:

DEFINITION 6. The weak left-star partial order on $B(H)$ is a relation defined by

$$A \mathrel{\ast\!\!\leqslant_w} B \quad \text{when} \quad A^*A = A^*B \text{ and } \overline{\operatorname{Im} A} \subseteq \overline{\operatorname{Im} B}, \quad A, B \in B(H).$$

The weak right-star partial order on $B(H)$ is a relation defined by

$$A \mathrel{\leqslant\!\!\ast_w} B \quad \text{when} \quad AA^* = BA^* \text{ and } \overline{\operatorname{Im} A^*} \subseteq \overline{\operatorname{Im} B^*}, \quad A, B \in B(H).$$

That these are actually partial orders is a straightforward consequence of the observation

$$A \mathrel{\ast\!\!\leqslant_w} B \iff A^* \mathrel{\leqslant\!\!\ast_w} B^* \tag{7}$$

and the following useful proposition.

PROPOSITION 7. *$A \mathrel{\ast\!\!\leqslant_w} B$ if and only if $A = PB$ for some projection P onto a closed subspace of $\overline{\operatorname{Im} B}$.*

Proof. $(\Rightarrow)$ Suppose $A \mathrel{\ast\!\!\leqslant_w} B$. Let P be the orthogonal projection onto $\overline{\operatorname{Im} A}$. Observe that $A^*(A-B) = 0$, so $\operatorname{Im}(A-B) \subseteq \operatorname{Ker} A^* = \overline{\operatorname{Im} A}^{\perp}$. Then

$$A = PA = PB + P(A-B) = PB.$$

$(\Leftarrow)$ Suppose $A = PB$ for some projection P onto a subspace of $\overline{\operatorname{Im} B}$. Then $\overline{\operatorname{Im} A} \subseteq \overline{\operatorname{Im} B}$ and

$$A^*A = B^*P^2B = B^*PB = A^*B. \quad \square$$

REMARK 8. If $A \mathrel{\ast\!\!\leqslant_w} B$, then actually $A = QB$ where Q is a projection onto $\overline{\operatorname{Im} A}$. This is seen by pre-multiplying the equation in Proposition 7 with Q.

We can now state our second main result.

THEOREM 9. *Let H be an infinite-dimensional complex Hilbert space. Then $\Phi\colon B(H) \to B(H)$ is a surjective bi-preserver of the weak left-star partial order $\ast\!\!\leqslant_w$ if and only if there exists an invertible positive definite $S \in B(H)$, a unitary (or anti-unitary) operator U on H, and an invertible bounded linear (respectively, conjugate-linear) operator T on H such that*

$$\Phi(A) = UP_{\overline{\operatorname{Im} SA}}S^{-1}AT, \quad A \in B(H),$$

where $P_{\overline{\operatorname{Im} SA}}$ is the orthogonal projection onto $\overline{\operatorname{Im} SA}$.

Similarly to Corollary 4 we can see that the following is true:

COROLLARY 10. *Let H be an infinite-dimensional complex Hilbert space. Then $\Phi\colon B(H) \to B(H)$ is a surjective bi-preserver of the weak right-star partial order $\leqslant\!\!\ast_w$ if and only if there exists an invertible positive definite $S \in B(H)$, a unitary (or anti-unitary) operator U on H, and an invertible bounded linear (respectively, conjugate-linear) operator T on H such that*

$$\Phi(A) = TAS^{-1}P_{\overline{\operatorname{Im} SA^*}}U, \quad A \in B(H),$$

where $P_{\overline{\operatorname{Im} SA^*}}$ is the orthogonal projection onto $\overline{\operatorname{Im} SA^*}$.

2. Preliminary results

We start with some notation and auxiliary results. Given a vector $w \in H$ we let w^* be a bounded linear functional on H given by $z \mapsto \langle z, w\rangle$. Denote by xw^* a rank-one operator given by $z \mapsto \langle z, w\rangle x$, where $w, x \in H$ are nonzero. Recall that every rank-one operator in $B(H)$ can be written in this form.

We will need in the sequel the following Propositions 11–16. Observe that Propositions 11–12 and 14–16 hold for both $*\!\leqslant$ and $*\!\leqslant_w$ orders, therefore we introduce a new notation $\mathscr{L}$ to denote either $*\!\leqslant$ or $*\!\leqslant_w$. Similarly, let $\mathscr{R}$ denote either $\leqslant\!*$ and $\leqslant\!*_w$.

PROPOSITION 11. *If $P \in B(H)$ is a projection and $A \mathscr{L} P$, then A is a projection and $AP = PA = A$.*

Proof. It suffices to show this when $\mathscr{L} = *\!\leqslant_w$ because if $A *\!\leqslant P$ then also $A *\!\leqslant_w P$. But for $*\!\leqslant_w$ this follows immediately from Remark 8. $\square$

PROPOSITION 12. *Let $A \in B(H)$ be nonzero. For every nonzero $x \in \operatorname{Im} A$ there exists a nonzero $y \in H$ such that $xy^* \mathscr{L} A$.*

Proof. Define $y = \frac{A^*x}{\|x\|^2}$. Since $x = Az \in \operatorname{Im} A$ for some $z \in H$, it follows that $y^*z = \frac{x^*x}{\|x\|^2} = 1$, so $y \neq 0$. The rest follows directly from the definition of $*\!\leqslant$ and $*\!\leqslant_w$. $\square$

Let us now show that a similar observation holds also for $\mathscr{R}$.

PROPOSITION 13. *Let $A \in B(H)$ be nonzero and suppose the range of A is closed. Let $y \in \operatorname{Im} A^*, y \neq 0$. Then there exists a nonzero $l \in H$ such that $yl^* \mathscr{R} A^*$.*

Proof. This was shown in [4] for the partial order $\leqslant\!*$. It holds also for $\leqslant\!*_w$ since $\leqslant\!*$ and $\leqslant\!*_w$ coincide when the range of A is closed. $\square$

We denote by $B_1(H)$ the set of all rank-one operators in $B(H)$. Let now xy^* and uv^* be two rank-one operators in $B(H)$. Let us define the following relation between operators in $B_1(H)$: we write $xy^* \sim uv^*$ if x and u are linearly dependent or y and v are linearly dependent. So, for two operators $A, B \in B_1(H)$ we write $A \sim B$ if $\operatorname{Im} A = \operatorname{Im} B$ or $\operatorname{Ker} A = \operatorname{Ker} B$.

PROPOSITION 14. *Let $A, B \in B(H)$, $A \neq B$, be rank-one operators in $B(H)$. Then $A \sim B$ if and only if there does not exist a rank-two operator $C \in B(H)$ such that $A \mathscr{L} C$ and $B \mathscr{L} C$.*

Proof. As in the proof of Proposition 13 this follows from [4]. $\square$

Let $x, y \in H$ be nonzero. Let us define the following sets of operators:

$$L_x = \{xv^* : v \in H \setminus \{0\}\} \quad \text{and} \quad R_y = \{zy^* : z \in H \setminus \{0\}\}.$$

Note that every operator in L_x and every operator in R_y is of rank-one.

PROPOSITION 15. *An operator A is invertible if and only if for every nonzero $x \in H$ and for every nonzero $y \in H$ there exist $B \in L_x$ and $C \in R_y$ such that $B \mathscr{L} A$ and $C \mathscr{L} A$.*

Proof. This was shown in [4] for the usual left-star partial order. For the weak left-star partial order the necessity follows from Proposition 12, Proposition 13, and equation (7). To prove sufficiency, first let $x \in H$ be nonzero. By hypothesis $xv^* \overset{*}{\leqslant}_w A$ for some nonzero $v \in H$, so by the definition of $\overset{*}{\leqslant}_w$ it follows that $x \in \overline{\operatorname{Im} A}$. Thus $\operatorname{Im} A$ is dense, so $\operatorname{Ker} A^* = 0$ and A^* is injective.

Now let $y \in H$ be nonzero, so there exists some nonzero z such that $zy^* \overset{*}{\leqslant}_w A$. By Remark 8, $zy^* = PA$ for the projection P whose range is $\mathbb{C}z$. It follows that $y \in \mathbb{C}A^*z$. Thus A^* is also surjective and the result follows. $\square$

The following result gives a characterization of rank-one operators in $B(H)$ that are dominated with respect to $\mathscr{L}$ by a given operator $B \in B(H)$ with $\operatorname{rank} B \geqslant 2$.

PROPOSITION 16. *Let* $\operatorname{rank} B \geqslant 2$.

1. *A rank-one $R \overset{*}{\leqslant} B$ if and only if $R = xx^*B$ for some vector $x \in \operatorname{Im} B$ with $\|x\| = 1$.*
2. *A rank-one $R \overset{*}{\leqslant}_w B$ if and only if $R = xx^*B$ for some vector $x \in \overline{\operatorname{Im} B}$ with $\|x\| = 1$.*

Proof. The first assertion may be proved in the same way as Lemma 6 in [5], and for the second assertion we can use Proposition 7 and Remark 8. $\square$

To streamline the proofs, we state and prove a common result for both the left-star partial order and its weaker version.

PROPOSITION 17. *Let H be an infinite-dimensional complex Hilbert space. Let $\Phi\colon B(H) \to B(H)$ be a surjective bi-preserver of either the left-star partial order $\overset{*}{\leqslant}$ or the weak left-star partial order $\overset{*}{\leqslant}_w$. Then Φ is bijective, preserves rank, and there exist a positive invertible operator $S \in B(H)$ and a unitary (or anti-unitary) operator U and an invertible bounded linear (respectively conjugate-linear) T on H such that*

$$U^*\Phi(xy^*)T^{-1} = \frac{Sxy^*S}{\|Sx\|^2}$$

for all rank-one operators xy^*.

Most of the arguments in the following proof hold at the same time for $\overset{*}{\leqslant}$ and for $\overset{*}{\leqslant}_w$; differences are noted whenever they occur. In particular, recall that $\overset{*}{\leqslant}$ and $\overset{*}{\leqslant}_w$ coincide on sets of operators acting on finite-dimensional spaces.

Proof. The proof will be divided into several steps. Recall that $\mathscr{L}$ denotes either $\preccurlyeq$ or $\preccurlyeq_w$. Let from now on H be an infinite-dimensional complex Hilbert space and $\Phi : B(H) \to B(H)$ as in Theorem 3, i.e., Φ is a surjective map such that for every pair $A,B \in B(H)$ we have

$$A \mathscr{L} B \quad \text{if and only if} \quad \Phi(A) \mathscr{L} \Phi(B).$$

Step 1. *First we show that Φ is injective and therefore bijective, and that $\Phi(0) = 0$.*

Indeed, if $\Phi(A) = \Phi(B)$, then $\Phi(A) \mathscr{L} \Phi(B) \mathscr{L} \Phi(A)$ and therefore we have $A \mathscr{L} B \mathscr{L} A$. So, $A = B$. Since $0 \mathscr{L} \Phi^{-1}(0)$, we have $\Phi(0) \mathscr{L} 0$ and thus $\Phi(0) = 0$.

Step 2. *Let $B \in B(H)$. Then* $\operatorname{rank} B = \infty$ *if and only if there exists an infinite chain $0 = A_0 \mathscr{L} A_1 \mathscr{L} \ldots \mathscr{L} B$ of pairwise distinct operators. Moreover,* $\operatorname{rank} B = r < \infty$ *if and only if there exists a chain*

$$0 = A_0 \mathscr{L} A_1 \mathscr{L} \ldots \mathscr{L} A_r = B$$

of $r+1$ pairwise distinct operators and no other such chain has larger length.

To see that the existence of the infinite chain implies $\operatorname{rank} B = \infty$, note that $\operatorname{Im} A_i \subseteq \overline{\operatorname{Im} B}$. So we are done if $\operatorname{rank} A_i = \infty$. However, if each A_i is of finite rank, then by Proposition 7 and Remark 8 (which hold also for $\preccurlyeq$ since the ranges of all operators A_i are closed) we obtain that $\operatorname{Im} A_i \subsetneq \operatorname{Im} A_{i+1}$ so again $\dim \operatorname{Im} B = \infty$. For the converse implication, take an orthonormal system $(x_n)_n \in \operatorname{Im} B$. By Proposition 16 we have $x_i x_i^* B \mathscr{L} B$ for each i. Also, one easily sees that $A_n = \sum_{i=1}^{n} x_i x_i^* B$ is a nested sequence of operators below B with respect to the order $\mathscr{L}$. One proceeds similarly when $\operatorname{rank} B < \infty$.

Step 3. Φ *preserves the rank of operators.*

Let $B \in B(H)$ with $\operatorname{rank} B = r < \infty$. By Step 2 there exists a chain $0 = A_0 \mathscr{L} A_1 \mathscr{L} \ldots \mathscr{L} A_r = B$ of $r+1$ pairwise distinct operators and no other such chain has larger length. Since Φ is injective and a bi-preserver of the order $\mathscr{L}$, it follows that $0 = \Phi(A_0) \mathscr{L} \Phi(A_1) \mathscr{L} \ldots \mathscr{L} \Phi(A_r) = \Phi(B)$ is a chain of $r+1$ pairwise distinct operators and no other such chain has larger length. Thus, again by Step 2, $\operatorname{rank} \Phi(B) = r$. Since Φ^{-1} has the same properties as Φ, we may conclude that for $B \in B(H)$, $\operatorname{rank} B = r < \infty$ if and only if $\operatorname{rank} \Phi(B) = r$.

Step 4. Φ *is a bi-preserver of the relation $\sim$.*

Indeed, it follows by Proposition 14 and Step 3 that for every pair $A,B \in B_1(H)$ we have $A \sim B$ if and only if $\Phi(A) \sim \Phi(B)$.

Step 5. *Action of Φ on the sets L_x, R_y.*

It is easy to see that for nonzero $x,y \in H$, L_x and R_y are the only maximal sets (with respect to the set inclusion) which consist of pairwise related rank-one operators via $\sim$. Since Φ is a bijective bi-preserver of the relation $\sim$, it follows that for every nonzero $x \in H$ there exists a nonzero $u \in H$ such that $\Phi(L_x) = L_u$, or there exists a nonzero $y \in H$ such that $\Phi(L_x) = R_y$. Similarly, for every nonzero $y \in H$ there exists a nonzero $x \in H$ such that $\Phi(R_y) = L_x$, or there exists a nonzero $v \in H$ such that $\Phi(R_y) = R_v$. The same holds for Φ^{-1}.

Step 6. Φ *preserves invertibility.*

Let now $A \in B(H)$ be an invertible operator and suppose $u \in H$ is nonzero. There exists a nonzero $x \in H$ such that $\Phi(L_x) = L_u$, or there exists a nonzero $y \in H$ such that $\Phi(R_y) = L_u$. Suppose $\Phi(L_x) = L_u$. Since A is invertible, it follows by Proposition 15 that there exists $B \in L_x$ such that $B \mathscr{L} A$. So, $\Phi(B) \mathscr{L} \Phi(A)$. Note that $\Phi(B) \in L_u$. Similarly, if $\Phi(R_y) = L_u$ there exists $C \in R_y$ such that $\Phi(C) \mathscr{L} \Phi(A)$ and $\Phi(C) \in L_u$. So, since Φ is surjective, we may find for every nonzero $u \in H$ an operator $D \in L_u$ such that $D \mathscr{L} \Phi(A)$. In the same way we prove that there exists an operator $E \in R_u$ such that $E \mathscr{L} \Phi(A)$. By Proposition 15 we may conclude that $\Phi(A)$ is an invertible operator. Since Φ^{-1} has the same properties as Φ it follows that $A \in B(H)$ is invertible if and only if $\Phi(A)$ is invertible.

Step 7. *Without loss of generality we may assume that* $\Phi(I) = I$.

Indeed, $\Phi(I)$, where I is the identity operator, is also invertible. By (5) we may replace the map Φ with the map $\Psi : B(H) \to B(H)$ which is defined in the following way: $\Psi(A) = \Phi(A)\Phi^{-1}(I)$. From now on we may and will assume that

$$\Phi(I) = I.$$

Step 8. Φ *leaves invariant the set* $\mathscr{P}(H)$ *of all projections in* $B(H)$.

By Definitions 1 and 6 it is clear that for every $P \in \mathscr{P}(H)$ we have $P \mathscr{L} I$. So, $\Phi(P) \mathscr{L} I$ and hence by Proposition 11, $\Phi(P)$ is also a projection. Since Φ is a bi-preserver of the left-star partial order, we may conclude that $\Phi(\mathscr{P}(H)) = \mathscr{P}(H)$.

Step 9. *Restriction of* Φ *on* $\mathscr{P}(H)$.

Let $P,Q \in \mathscr{P}(H)$. Proposition 11 yields that if $P \mathscr{L} Q$, then $PQ = QP = P$ and hence $P \leqslant Q$ where $\leqslant$ denotes the usual order on $\mathscr{P}(H)$ (i.e., $P \leqslant Q$ when $PQ = QP = P$). Also, directly by Definitions 1 and 6 it follows that if $PQ = QP = P$ for $P,Q \in \mathscr{P}(H)$, then $P \mathscr{L} Q$. The restriction of Φ to $\mathscr{P}(H)$ is a bijective map from $\mathscr{P}(H)$ to $\mathscr{P}(H)$ which preserves the usual order in both directions.

Step 10. *Action of* Φ *on* $\mathscr{P}(H)$.

We may identify closed subspaces in H with operators in $\mathscr{P}(H)$. So, the map Φ induces a lattice automorphism, i.e., a bijective map ω defined on the set of all closed subspaces in H, where $M \subseteq N$ if and only if $\omega(M) \subseteq \omega(N)$ for every pair of closed subspaces M,N in H. Recall that H is an infinite dimensional complex Hilbert space. By [7, Theorem 1] there exists a bicontinuous linear or conjugate-linear bijection $S\colon H \to H$ such that $\omega(M) = SM$ for every closed subspace M in H. Let from now on $P_M \in B(H)$ denote a projection with $\mathrm{Im} P_M = M$. It follows that

$$\Phi(P_M) = P_{S(M)}$$

for every $P_M \in \mathscr{P}(H)$.

Step 11. *Without loss of generality we may assume that the operator* S *(introduced in Step 10) is an invertible and a positive operator.*

Let the operator $S\colon H \to H$ be as in Step 10, i.e., a bicontinuous linear or conjugate-linear bijection. Suppose first S is linear and let $S = U|S|$ be its polar decomposition where U is a partial isometry and $|S| = \sqrt{S^*S}$, i.e., $|S|$ is a positive operator in $B(H)$.

Since S is invertible, $U \in B(H)$ is unitary. Step 10 implies that

$$\Phi(xx^*) = \frac{1}{\|Sx\|^2}(Sx)(Sx)^* = \frac{1}{\|Sx\|^2}Sxx^*S^*$$

for every $x \in H$ with $\|x\| = 1$. By replacing Φ with $U^*\Phi(\cdot)U$ we may by (5) without loss of generality assume that S is an invertible, positive operator in $B(H)$ (and thus self-adjoint).

Let now $S\colon H \to H$ be a bounded, conjugate-linear bijection. We will show that even in this case we may assume that $S \in B(H)$ is an invertible, positive (linear) operator. To show this let us recall some known facts about bounded conjugate-linear operators on Hilbert spaces (see for example [2]). A bounded conjugate-linear operator $T : H \to H$ has a unique conjugate-linear adjoint $T^* : H \to H$ defined with

$$\langle Tx, y\rangle = \langle T^*y, x\rangle$$

for all $x, y \in H$. As in the linear case, we say that T is self-adjoint when $T = T^*$, i.e., $\langle Tx, y\rangle = \langle Ty, x\rangle$ for every $x, y \in H$. Let A be a bounded conjugate-linear operator on a Hilbert space H and let $B \in B(H)$. Then both AB and B^*A^* are bounded conjugate-linear operators on H and since

$$\langle (AB)x, y\rangle = \langle A^*y, Bx\rangle = \langle B^*A^*y, x\rangle$$

we may by the uniqueness of the adjoint conclude that

$$(AB)^* = B^*A^*.$$

Similarly, if both A and B are bounded conjugate-linear operators on H, then AB, $B^*A^* \in B(H)$ and

$$\langle (AB)x, y\rangle = \langle A^*y, Bx\rangle = \overline{\langle Bx, A^*y\rangle} = \overline{\langle B^*A^*y, x\rangle} = \langle x, B^*A^*y\rangle$$

and therefore again $(AB)^* = B^*A^*$.

An example of a conjugate-linear operator on a complex Hilbert space H is the map J which, relative to a fixed orthonormal basis $(e_\lambda)_{\lambda\in\mathbb{N}\cup\Lambda}$ where Λ is the empty set in case H is a separable Hilbert space, is defined as follows: $J\colon x = \sum \alpha_\lambda e_\lambda \mapsto \sum \overline{\alpha_\lambda} e_\lambda$, $\alpha_\lambda \in \mathbb{C}$. Note that J is an involution, i.e., a conjugate-linear isometry from H onto H with $J^2 = I$, and that every involution is of this form (see [2]). Observe also $\langle Jx, y\rangle = \langle Jy, x\rangle$ for every $x, y \in H$, i.e., J is self-adjoint. Let $T : H \to H$ be a bounded conjugate-linear operator and let $J : H \to H$ be as above. Then $JT \in B(H)$ and

$$T^*T = T^*JJT = (J^*T)^*(JT) = (JT)^*(JT).$$

It follows that $|T| = |JT|$ is independent of J and hence well defined. If $JT = U|JT| = U|T|$ is the polar decomposition for JT, then $T = V|T|$ is the polar decomposition of T, where $V = JU$ is a conjugate-linear partial isometry. So, conjugate-linear operators have a well-defined polar decomposition with analogous properties to those of linear operators (see also [2]). Let $T=V|T|$ be the polar decomposition of a conjugate-linear bounded operator T. Suppose T is invertible. Observe that then V is anti-unitary, i.e., a conjugate-linear bounded operator on H with $V^*V=VV^*=I$. Also, $|T|=|JT|$ is a positive, invertible, bounded linear operator.

Let now U be an anti-unitary operator on H and $S:H\to H$ an invertible conjugate-linear bounded operator. Let $A,B\in B(H)$. Then $\operatorname{Im}A\subseteq\operatorname{Im}B$ if and only if $\operatorname{Im}UAS\subseteq\operatorname{Im}UBS$, and $\overline{\operatorname{Im}A}\subseteq\overline{\operatorname{Im}B}$ if and only if $\overline{\operatorname{Im}UAS}\subseteq\overline{\operatorname{Im}UBS}$ (for use with the usual and weak partial orders respectively). Also $A^*A=A^*B$ if and only if $(UAS)^*(UAS)=S^*A^*U^*UAS=S^*A^*AS=S^*A^*BS=S^*A^*U^*UBS=(UAS)^*(UBS)$, and therefore

$$A \mathrel{\mathscr{L}} B \quad \text{if and only if} \quad UAS \mathrel{\mathscr{L}} UBS. \tag{8}$$

Suppose $S\colon H\to H$ from Step 10 is a conjugate-linear, bijective, and bounded operator. Then we may write $S=U|S|$ where U is an anti-unitary operator on H and $|S|\in B(H)$ a positive, invertible operator. By again replacing Φ with $U^*\Phi(\cdot)U$, we may thus by (8) as in the linear case assume that S is a positive linear, bounded, and invertible operator on H.

From now on, let $S\in B(H)$ be an invertible and positive operator (and thus self-adjoint).

Step 12. *We show that $\Phi(P_MB(H)P_M)=P_{S(M)}B(H)P_{S(M)}$ where $P_MB(H)P_M=\{P_MAP_M : A\in B(H)\}$ and $P_M\in B(H)$ is a finite rank projection of rank $n\geqslant 2$.*

Since Φ^{-1} has the same properties as Φ, it is enough to show that

$$\Phi(P_MB(H)P_M)\subseteq P_{S(M)}B(H)P_{S(M)}.$$

First note that $A\in P_MB(H)P_M$ if and only if $\operatorname{Im}A\subseteq\operatorname{Im}P_M$ and $\operatorname{Ker}P_M\subseteq\operatorname{Ker}A$. Indeed, if $A\in P_MB(H)P_M$, then $A=P_MAP_M$ and therefore $\operatorname{Im}A\subseteq\operatorname{Im}P_M$ and $\operatorname{Ker}\,P_M\subseteq\operatorname{Ker}\,A$. Conversely, if $\operatorname{Im}A\subseteq\operatorname{Im}P_M$, then $A=P_MA$ and if $\operatorname{Ker}\,P_M\subseteq\operatorname{Ker}\,A$, then $\operatorname{Im}A^*\subseteq\operatorname{Im}P_M$ and therefore $A^*=P_MA^*$, i.e., $A=AP_M$. It follows that $A=P_MAP_M$ and so $A\in P_MB(H)P_M$.

First, let us show that for every rank-one operator $A\in P_MB(H)P_M$ it follows that $\Phi(A)\in P_{S(M)}B(H)P_{S(M)}$. Recall that

$$\Phi(xx^*)=\frac{1}{\|Sx\|^2}(Sx)(Sx)^*$$

for every $x\in H$ with $\|x\|=1$. Suppose $A=\alpha xy^*$ where $\|x\|=\|y\|=1$, $\alpha\in\mathbb{C}\setminus\{0\}$, and $A\in P_MB(H)P_M$. Then $x,y\in M$. Since $A\sim xx^*$ and $A\sim yy^*$, it follows by Step 4 that

$$\Phi(A)\sim\frac{1}{\|Sx\|^2}(Sx)(Sx)^* \text{ and } \Phi(A)\sim\frac{1}{\|Sy\|^2}(Sy)(Sy)^*.$$

If x and y are linearly independent, then by the bijectivity of S also Sx and Sy are linearly independent. It follows that $\Phi(A)=\lambda(Sx)(Sy)^*$ or $\Phi(A)=\mu(Sy)(Sx)^*,\lambda,\mu\in\mathbb{C}\setminus\{0\}$. In both cases $\Phi(A)\in P_{S(M)}B(H)P_{S(M)}$ and it is not a scalar multiple of a rank-one projection.

If $y \in \mathbb{C}x$, then $A \in \mathbb{C}xx^*$. By the previous argument applied on Φ^{-1} and since Φ preserves operators of rank-one we have that $\Phi(A)$ is a scalar multiple of a rank-one projection. Note that $A \sim xx^*$, so $\Phi(A) \sim \Phi(xx^*) \in \mathbb{C}(Sx)(Sx)^*$ and therefore $\Phi(A) \in P_{S(M)}B(H)P_{S(M)}$.

Second, let now $D \in P_M B(H) P_M$ be an operator of rank at least two. By Proposition 16, for each rank-one C such that $C \mathscr{L} D$, it follows $C = xx^*D$, $x \in \operatorname{Im} D = \overline{\operatorname{Im} D}$. This yields $C \in P_M B(H) P_M$ and hence

$$\Phi(C) \in P_{S(M)}B(H)P_{S(M)} \text{ for every rank one } C \mathscr{L} D. \tag{9}$$

So, $\operatorname{Im}\Phi(C) \subseteq \operatorname{Im} P_{S(M)}$. Since Φ is a bijective bi-preserver and maps the set of all rank-one operators onto itself (see also Proposition 12),

$$\operatorname{Im}\Phi(D) \subseteq \bigcup\{\operatorname{Im}\Phi(C) : C \in B_1(H) \text{ and } C \mathscr{L} D\} \subseteq \operatorname{Im} P_{S(M)}.$$

Recall that M, hence also $S(M)$, is a finite-dimensional subspace; since $S(M)$ contains the range of $\Phi(D)$, the range of $\Phi(D)$ is closed. In order to prove that $\Phi(D) \in P_{S(M)}B(H)P_{S(M)}$ it remains to show that $\operatorname{Im}\Phi(D)^* \subseteq \operatorname{Im} P_{S(M)}$.

For a nonzero $y \in \operatorname{Im}\Phi(D)^*$ there exists by Proposition 13, $l \in H$, $l \neq 0$, such that $\Phi(C)^* = yl^* \mathscr{R} \Phi(D)^*$ for some $C \in B_1(H)$. Note that $\operatorname{Im}\Phi(C)^* = \mathbb{C}y$, and since $y \in \operatorname{Im}\Phi(D)^*$ was arbitrary it follows

$$\begin{aligned}\operatorname{Im}\Phi(D)^* &\subseteq \bigcup\{\operatorname{Im}\Phi(C)^* : C \in B_1(H) \text{ and } \Phi(C)^* \mathscr{R} \Phi(D)^*\} \\ &= \bigcup\{\operatorname{Im}\Phi(C)^* : C \in B_1(H) \text{ and } C \mathscr{L} D\},\end{aligned}$$

where the last equality follows by (6) and (7).

Recall that $D \in P_M B(H) P_M$. For every $C \in B_1(H)$ where $C \mathscr{L} D$ we have $\Phi(C) \in P_{S(M)}B(H)P_{S(M)}$ by (9). It follows that $\operatorname{Im}\Phi(C)^* \subseteq \operatorname{Im} P_{S(M)}$. This implies that $\operatorname{Im}\Phi(D)^* \subseteq \operatorname{Im} P_{S(M)}$ and hence $\Phi(D) \in P_{S(M)}B(H)P_{S(M)}$.

Step 13. *Reduction of the problem to bijective bi-preservers on* $M_n(\mathbb{C})$.

Take any finite-dimensional subspace $M \subseteq H$ of dimension at least three and identify $P_M B(H) P_M$ and $P_{S(M)}B(H)P_{S(M)}$ with $M_n(\mathbb{C})$, $n = \dim M = \dim S(M)$. By Steps 9 and 10

$$\Phi(xx^*) = \frac{1}{\|Sx\|^2}(Sx)(Sx)^* = \frac{1}{\|Sx\|^2} Sxx^*S$$

for every unit vector $x \in H$. By [5, equations (9) and (10)] either

$$\Phi(xy^*) = \gamma_{xy^*} Sxy^*S$$

for every rank-one $xy^* \in P_M B(H) P_M$ or

$$\Phi(xy^*) = \gamma_{xy^*} Syx^*S$$

for every rank-one $xy^* \in P_M B(H) P_M$ where γ_{xy^*} is a nonzero scalar that depends on xy^*.

It easily follows that

$$\Phi(xy^*) = \gamma_{xy^*} Sxy^*S \tag{10}$$

for every rank-one $xy^* \in B(H)$ or

$$\Phi(xy^*) = \gamma_{xy^*} Syx^* S \tag{11}$$

for every rank-one $xy^* \in B(H)$. Note that in the former case we have $\gamma_{xy^*} = \frac{1}{\|Sx\|^2}$ by [5, Lemma 19].

Step 14. *We assume Φ is of the form (11) on the set of all rank-one operators in $B(H)$ and get a contradiction.*

Observe first that $xw^* = uv^*$ if and only if either one of x, w and one of u, v is zero or else $x \parallel u$ and $w \parallel v$ (i.e., are parallel). Suppose there exists an invertible, positive operator $S \in B(H)$ such that

$$\Phi(xy^*) = \frac{Syx^*S}{\lambda_{xy^*}}$$

for every nonzero $x, y \in H$ where λ_{xy^*} is some nonzero scalar that depends on xy^*. Fix an orthonormal basis $(e_\lambda)_{\lambda \in \mathbb{N} \cup \Lambda}$ where Λ is an empty set in case H is a separable Hilbert space. Define a linear operator B with $Be_n = \frac{1}{n} e_n$, $n \in \mathbb{N}$, and $Be_\lambda = e_\lambda$ for all $\lambda \in \Lambda$. Note that B is injective, bounded, and has a dense range. Let $C = \Phi(B)$ and let $x \in \operatorname{Im} B$ be a unit vector. By the assumption

$$\Phi(xx^*B) = \frac{SB^*xx^*S}{\alpha} \mathscr{L} C$$

where α is some nonzero scalar. Since Φ preserves rank, there exists by Proposition 16 a unit vector y such that $\frac{SB^*xx^*S}{\alpha} = yy^*C$. It follows that

$$y = \frac{SB^*x}{\|SB^*x\|} \mu \tag{12}$$

for some unimodular scalar μ which depends on x. Also,

$$y^*C = x^*S\delta \tag{13}$$

for some scalar δ. Equation (12) yields $y^*C = \frac{x^*BSC}{\|SB^*x\|} \overline{\mu}$ and thus by (13)

$$x^*S\delta = \frac{x^*BSC}{\|SB^*x\|} \overline{\mu}.$$

Since $x \in \operatorname{Im} B$, we may write $x = \frac{Bz}{\|Bz\|}$ for some nonzero $z \in H$. So,

$$z^*B^*S\delta_z = z^*B^*BSC\mu_z$$

where scalars δ_z and μ_z depend on z. This implies that B^*S and B^*BSC are locally linearly dependent. However since B^*S is of infinite rank, we may conclude (see e.g. [11, page 1869]) that they are linearly dependent, i.e., $B^*BSC = \lambda B^*S$ for some scalar λ. Note λ is nonzero because $\operatorname{Ker}(B^*BS) = \{0\}$ and $C = \Phi(B)$ is nonzero since $B \neq 0$. We have

$$C^*SB^*B = \overline{\lambda} SB.$$

Evaluate this identity on the vector e_n and use the fact that $B^*B = B^2$ to obtain $C^*\left(\frac{Se_n}{n^2}\right) = \frac{\overline{\lambda}Se_n}{n}$ and thus $C^*(Se_n) = n\overline{\lambda}Se_n$. We may (for each integer n) conclude that the operator C^* is unbounded which is a contradiction.

Step 15. *Conclusion of the proof.*

By the preceding steps, and by taking into account the assumptions from Steps 7 and 11, the result follows. $\square$

3. Proof of Theorem 3

Proof of Theorem 3. By our earlier remarks, sufficiency is clear. To prove necessity, suppose Φ is a surjective bi-preserver of $\preccurlyeq$. By Proposition 17 Φ is bijective and we may assume without loss of generality that Φ maps rank-one operators xy^* into

$$\Phi(xy^*) = \frac{Sxy^*S}{\|Sx\|^2}$$

for some invertible, positive operator $S \in B(H)$. It suffices to show that Φ is then the identity map.

Consider $A \in B(H)$ with a dense range and let $B = \Phi(A)$. If $x \in \operatorname{Im} A$ is a unit vector, then by Proposition 16 $xx^*A \preccurlyeq A$ (note that the range-kernel orthogonality $\operatorname{Ker} A^* = (\operatorname{Im} A)^\perp$ implies $x^*A \neq 0$) so applying Φ gives

$$\frac{Sxx^*AS}{\|Sx\|^2} \preccurlyeq B$$

which by Proposition 16 means that there exists a unit vector $z \in \operatorname{Im} B$ such that

$$\frac{Sxx^*AS}{\|Sx\|^2} = zz^*B. \tag{14}$$

It follows that $Sx = \delta_x z \in \operatorname{Im} B$ for every $x \in \operatorname{Im} A$ where δ_x is a nonzero scalar that depends on x, and in particular, $S \operatorname{Im} A \subseteq \operatorname{Im} B$. Conversely, if $z \in \operatorname{Im} B$ is a unit vector, then $0 \neq zz^*B \preccurlyeq B$, and since Φ^{-1} also preserves the order $\preccurlyeq$ and rank of operators, there exists a unit vector $x \in \operatorname{Im} A = \operatorname{Im} \Phi^{-1}(B)$ such that $\Phi(xx^*A) = zz^*B$, i.e., there exists $x \in \operatorname{Im} A$ such that $Sx = \delta_x z$. Hence

$$\operatorname{Im} B = S \operatorname{Im} A. \tag{15}$$

Let $x \in \operatorname{Im} A$ and $x = Aw \neq 0$. Insert for x in (14) $\frac{x}{\|x\|} = \frac{Aw}{\|x\|}$ to deduce

$$\gamma_w SAww^*A^*AS = zz^*B$$

where γ_w is some nonzero scalar. We infer that $z \in \mathbb{C}SAw$ and since $S^* = S$ we obtain

$$\mu_w SAww^*A^*AS = SAww^*A^*SB$$

for some nonzero scalar μ_w. Comparing both sides we get

$$\mu_w w^*A^*AS = w^*A^*SB$$

for all vectors w such that $Aw \neq 0$. Clearly this holds also if $Aw = 0$. Then A^*AS and A^*SB are locally linearly dependent and since A is not rank-one or zero we have that there exists $\lambda \in \mathbb{C}$ such that

$$\lambda A^*AS = A^*SB,$$

that is

$$A^*(\lambda A - SBS^{-1}) = 0.$$

Since $\operatorname{Im} A$ is dense, A^* is injective (by the range-kernel orthogonality) so $\lambda A = SBS^{-1}$ or equivalently,

$$B = \lambda S^{-1}AS.$$

It follows that $\operatorname{Im} B = S^{-1} \operatorname{Im} A = S \operatorname{Im} A$ where the last identity follows from (15). So,

$$S^2 \operatorname{Im} A = \operatorname{Im} A$$

whenever $\operatorname{Im} A$ is a dense space.

Suppose there exists a vector $x \in H$ such that x and $y = S^2x$ are linearly independent vectors. Fix an orthonormal basis $(e_\lambda)_{\lambda \in \mathbb{N}\cup\Lambda}$ where Λ is the empty set in case H is a separable Hilbert space and define a linear operator A with $Ae_n = \frac{1}{n}e_n$, $n \in \mathbb{N}$, and $Ae_\lambda = e_\lambda$ for all $\lambda \in \Lambda$. Recall that A is injective, bounded, and has a dense range. Let $\hat{y} = \sum \frac{1}{n} e_n$ and note that $\hat{y} \notin \operatorname{Im} A$. There exists a bounded linear bijection T on H which maps e_1 to x and $\hat{y}$ to y. Note that $x \in \operatorname{Im} TA$ and hence $y = S^2x \in S^2 \operatorname{Im} TA$, however $y \notin \operatorname{Im} TA$, a contradiction with $S^2 \operatorname{Im} TA = \operatorname{Im} TA$. It follows that S^2 and I are locally linearly dependent and since the positive operator S^2 is not rank-one or zero, there exists $\lambda > 0$ such that $S^2 = \lambda I$ and so its positive square root is $S = \sqrt{\lambda} I$. This implies that $\Phi(xy^*) = \frac{Sxy^*S}{\|Sx\|^2} = xy^*$ and so Φ is the identity map on operators of rank at most one. By applying [4, Lemma 13] we see that Φ is the identity.

Taking into account the assumption (see also Proposition 17) we may conclude that if H be an infinite-dimensional complex Hilbert space and $\Phi\colon B(H) \to B(H)$ a surjective bi-preserver of the left-star partial order, then

$$\Phi(A) = UAT, \quad A \in B(H),$$

where $U \in B(H)$ is a unitary operator and $T \in B(H)$ is an invertible operator, or U is an anti-unitary operator on H and T is an invertible conjugate-linear operator on H. $\square$

4. Proof of Theorem 9

Before proving Theorem 9, we first investigate a special class of maps that arise for bi-preservers of the weak left-star partial order.

LEMMA 18. *Fix an invertible positive $S \in B(H)$ and define $\psi : B(H) \to B(H)$ by*

$$\psi(A) = P_{\overline{\operatorname{Im} SA}} S^{-1}A, \quad A \in B(H).$$

Then $\operatorname{Ker} \psi(A) = \operatorname{Ker} A$ *and* $\overline{\operatorname{Im} \psi(A)} = \overline{\operatorname{Im} SA}$.

Proof. For the first assertion,

$$\begin{aligned}\operatorname{Ker}\psi(A) &= \{x : S^{-1}Ax \in \overline{\operatorname{Im} SA}^{\perp} = \operatorname{Ker}(SA)^*\}\\ &= \{x : A^*SS^{-1}Ax = 0\} = \{x : A^*Ax = 0\}\\ &= \operatorname{Ker} A.\end{aligned}$$

For the second assertion, it suffices to prove $\overline{\operatorname{Im}\psi(A)}^{\perp} = \overline{\operatorname{Im} SA}^{\perp}$. Note

$$\begin{aligned}\overline{\operatorname{Im}\psi(A)}^{\perp} = \operatorname{Ker}\psi(A)^* &= \{x : A^*S^{-1}P_{\overline{\operatorname{Im} SA}}x = 0\}\\ &= \overline{\operatorname{Im} SA}^{\perp} + \{x \in \overline{\operatorname{Im} SA} : A^*S^{-1}x = 0\}.\end{aligned}$$

But

$$\begin{aligned}\{x \in \overline{\operatorname{Im} SA} : A^*S^{-1}x = 0\} &= \{x = \lim SAx_n : A^*(\lim S^{-1}SAx_n) = 0\}\\ &\quad \text{write } y = S^{-1}x\\ &= \{Sy : y = \lim Ax_n, A^*y = 0\}\\ &= \{Sy : y \in \overline{\operatorname{Im} A} \cap \operatorname{Ker} A^* = \{0\}\} = \{0\},\end{aligned}$$

so the result follows. $\square$

Proof of Theorem 9. We begin by proving necessity. Suppose $\Phi : B(H) \to B(H)$ is a surjective bi-preserver of $\leqslant_w$. By Proposition 17 Φ is bijective, preserves rank, and we may assume without loss of generality that there exists an invertible positive definite $S \in B(H)$ such that

$$\Phi(xy^*) = \frac{Sxy^*S}{\|Sx\|^2}$$

for all nonzero $x \in H$. It suffices to show that, for all $B \in B(H)$,

$$\Phi(B) = P_{\overline{\operatorname{Im} SB}}S^{-1}BS$$

where $P_{\overline{\operatorname{Im} SB}}$ is the orthogonal projection onto $\overline{\operatorname{Im} SB}$.

Let $B \in B(H)$ and write $C = \Phi(B)$. By Proposition 7, and since Φ is a bijective rank-preserving bi-preserver of $\leqslant_w$, the following are equivalent.

(a) $R = xx^*B$ for some $x \in \overline{\operatorname{Im} B}$ with $\|x\| = 1$.

(b) $\operatorname{rank} R = 1$ and $R \leqslant B$.

(c) $\operatorname{rank}\Phi(R) = 1$ and $\Phi(R) \leqslant C$.

(d) $\Phi(R) = yy^*C$ for some $y \in \overline{\operatorname{Im} C}$ with $\|y\| = 1$.

Because (a) implies (d), for each unit vector $x \in \overline{\operatorname{Im} B}$ there exists a unit vector $y \in \overline{\operatorname{Im} C}$ such that

$$\frac{Sxx^*BS}{\|Sx\|^2} = yy^*C. \tag{16}$$

From this we conclude that $S\overline{\mathrm{Im}B} \subseteq \overline{\mathrm{Im}C}$. Conversely, because (d) implies (a), for each unit vector $y \in \overline{\mathrm{Im}C}$ there exist a unit vector $x \in \overline{\mathrm{Im}B}$ so that (16) holds, whence $\overline{\mathrm{Im}C} \subseteq S\overline{\mathrm{Im}B}$. Thus $\overline{\mathrm{Im}C} = S\overline{\mathrm{Im}B} = \overline{\mathrm{Im}SB}$.

Let x be a unit vector in $\overline{\mathrm{Im}B}$ and set $y = Sx/\|Sx\|$. By (16)

$$C^*Sx = SB^*x = SB^*S^{-1}(Sx).$$

Thus $C^* = SB^*S^{-1}$ when restricted to $\overline{\mathrm{Im}SB}$.

We also have $\mathrm{Ker}C^* = (\overline{\mathrm{Im}C})^\perp = (\overline{\mathrm{Im}SB})^\perp$. Thus for $x \in \overline{\mathrm{Im}SB}$, $y \in \overline{\mathrm{Im}SB}^\perp$ we have

$$C^*(x+y) = SB^*S^{-1}x = SB^*S^{-1}P_{\overline{\mathrm{Im}SB}}(x+y),$$

so $C^* = SB^*S^{-1}P_{\overline{\mathrm{Im}SB}}$ and the result follows.

To prove sufficiency, let S be an invertible positive operator in $B(H)$ and define $\psi : B(H) \to B(H)$ by

$$\psi(A) = P_{\overline{\mathrm{Im}SA}}S^{-1}A, \quad A \in B(H).$$

It suffices to prove that ψ is a surjective bi-preserver of $\overset{*}{\leqslant}_w$.

Let $A, B \in B(H)$. First suppose that $A \overset{*}{\leqslant}_w B$. By Lemma 18,

$$\overline{\mathrm{Im}\,\psi(A)} = \overline{\mathrm{Im}SA} = S(\overline{\mathrm{Im}A}) \subseteq S(\overline{\mathrm{Im}B}) = \overline{\mathrm{Im}SB} = \overline{\mathrm{Im}\,\psi(B)}.$$

We also have

$$\begin{aligned}
A \overset{*}{\leqslant}_w B &\implies A^*A = A^*B \qquad \text{take adjoint of both sides}\\
&\implies A^*A = B^*A \implies (A^* - B^*)S^{-1}SA = 0\\
&\implies (A^* - B^*)S^{-1}P_{\overline{\mathrm{Im}SA}} = 0\\
&\implies A^*S^{-1}P_{\overline{\mathrm{Im}SA}} = B^*S^{-1}P_{\overline{\mathrm{Im}SA}} = B^*S^{-1}P_{\overline{\mathrm{Im}SB}}P_{\overline{\mathrm{Im}SA}},
\end{aligned}$$

Taking adjoints of the last line gives $\psi(A) = P_{\overline{\mathrm{Im}SA}}\psi(B)$; by Proposition 7, $\psi(A) \overset{*}{\leqslant}_w \psi(B)$.

Conversely, suppose that $\psi(A) \overset{*}{\leqslant}_w \psi(B)$. By Lemma 18, $\overline{\mathrm{Im}SA} \subseteq \overline{\mathrm{Im}SB}$; applying S^{-1} gives $\overline{\mathrm{Im}A} \subseteq \overline{\mathrm{Im}B}$. By Proposition 7 and Lemma 18, $\psi(A) = P_{\overline{\mathrm{Im}SA}}\psi(B)$, so

$$\begin{aligned}
&P_{\overline{\mathrm{Im}SA}}S^{-1}A = P_{\overline{\mathrm{Im}SA}}S^{-1}B\\
&\implies P_{\overline{\mathrm{Im}SA}}S^{-1}(A-B) = 0\\
&\implies S^{-1}(A-B)x \in \overline{\mathrm{Im}SA}^\perp = \mathrm{Ker}\,(SA)^* \quad (\text{for all } x \in H)\\
&\implies A^*SS^{-1}(A-B) = 0\\
&\implies A^*A = A^*B.
\end{aligned}$$

Thus $A \overset{*}{\leqslant}_w B$.

Finally, to show surjectivity, fix $B \in B(H)$ and set $A = P_{\overline{\mathrm{Im}\, S^{-1}B}} SB$. By Lemma 18, $\overline{\mathrm{Im} A} = \overline{\mathrm{Im}\, S^{-1}B}$, so

$$\psi(A) = P_{\overline{\mathrm{Im}\, SA}} S^{-1} P_{\overline{\mathrm{Im}\, S^{-1}B}} SB = P_{\overline{\mathrm{Im}\, B}} S^{-1} P_{\overline{\mathrm{Im}\, S^{-1}B}} SB.$$

Let $x \in H$. Then $SBx = s + z$ for some $s \in \overline{\mathrm{Im}\, S^{-1}B}$ and some $z \in \overline{\mathrm{Im}\, S^{-1}B}^{\perp} = \mathrm{Ker}\, B^* S^{-1}$, and therefore $P_{\overline{\mathrm{Im}\, S^{-1}B}} SBx = s = SBx - z$. Thus

$$\begin{aligned}
\psi(A)x &= P_{\overline{\mathrm{Im}\, B}} S^{-1} P_{\overline{\mathrm{Im}\, S^{-1}B}} SBx \\
&= P_{\overline{\mathrm{Im}\, B}} S^{-1} (SBx - z) \\
&= Bx - P_{\overline{\mathrm{Im}\, B}} S^{-1} z = Bx
\end{aligned}$$

since $z \in \mathrm{Ker}\, B^* S^{-1}$ implies $S^{-1}z \in \mathrm{Ker}\, B^* = \overline{\mathrm{Im}\, B}^{\perp}$. Thus $\psi(A) = B$. $\square$

REMARK 19. Note the above proof shows that the inverse of the map $\psi(A) = P_{\overline{\mathrm{Im}\, SA}} S^{-1} A$ has the same form and is given by $\psi^{-1}(B) = P_{\overline{\mathrm{Im}\, S^{-1}B}} SB$.

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