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ON ERGODIC THEOREM FOR A FAMILY OF OPERATORS

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Abstract. In this paper, we obtain a generalization of the uniform ergodic theorem to the family of bounded linear operators on a Banach space. We present some elementary results in this setting and we show that Lin's theorem can be extended from the case of a bounded linear operator to the case of a family of bounded linear operators acting on a Banach space.

1. Introduction

Let T be a bounded linear operator on a complex Banach space $\mathscr{X}$. The uniform ergodicity for T was already developed in different directions (see, e.g. [2, 3, 4, 5, 7, 8, 9, 12]). For example, in [3], it was shown that if $\frac{1}{n}\|T^n\| \longrightarrow 0$ as $n \to \infty$, then T is uniformly ergodic if and only if $(I-T)^2\mathscr{X}$ is closed. Hence $(I-T)^k\mathscr{X}$ is closed for each integer $k \geqslant 1$. In [8] M. Lin has established the following theorem which characterizes the uniform ergodicity for an operator acting on a Banach space.

THEOREM 1. *Let T be a bounded linear operator on a Banach space $\mathscr{X}$ satisfying $\frac{1}{n}\|T^n\| \longrightarrow 0$ as $n \to \infty$. Then the following conditions are equivalent:*

(1) There exists a bounded linear operator P such that

$$\left\| \frac{1}{n}\sum_{k=0}^{n-1} T^k - P \right\| \longrightarrow 0 \text{ as } n \to \infty.$$

(2) $(I-T)\mathscr{X}$ is closed and $\mathscr{X} = \{x \in \mathscr{X} : Tx = x\} \oplus (I-T)\mathscr{X}$.

(3) $(I-T)^2\mathscr{X}$ is closed.

(4) $(I-T)\mathscr{X}$ is closed.

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In this paper we introduce the notion of the uniform ergodicity for a family of bounded linear operators from the Banach algebra $C_b((0,1],\ \mathscr{B}(\mathscr{X}))$ (respectively from $\mathscr{B}_\infty$), see below for the definitions. We give relations between these two definitions, see Proposition 1 below, and we extend the equivalent properties of Theorem 1 for a family of bounded linear operators acting on a Banach space.

Krishna and Johnson have analyzed completeness of a collection of bounded linear operators between normed spaces in [6]. We are motivated by the papers [10, 11] of S. Macovei which contain some interesting properties of families of bounded linear operators acting on a Banach space.

2. Preliminaries

Let $\mathscr{X}$ be an infinite-dimensional complex Banach space and $\mathscr{B}(\mathscr{X})$ the Banach algebra of all bounded linear operators on $\mathscr{X}$. We denote by I the identity operator on $\mathscr{X}$.

Let $T \in \mathscr{B}(\mathscr{X})$, we denote the Cesàro mean by

$$M_n(T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k.$$

Recall that T is uniformly ergodic if there exists $P \in \mathscr{B}(\mathscr{X})$ such that

$$\|M_n(T) - P\| \longrightarrow 0 \text{ as } n \to \infty.$$

In [10], Macovei showed that the set

$$C_b((0,1],\mathscr{B}(\mathscr{X})) = \Big\{ \{T_h\}_{h\in(0,1]} \subset \mathscr{B}(\mathscr{X}) : \{T_h\}_{h\in(0,1]} \text{ is a bounded family, i.e. } \sup_{h\in(0,1]} \|T_h\| < \infty \Big\},$$

is a Banach algebra non-commutative with norm

$$\|\{T_h\}\| = \sup_{h\in(0,1]} \|T_h\|.$$

And

$$C_0((0,1],\mathscr{B}(\mathscr{X})) = \left\{ \{T_h\}_{h\in(0,1]} \subset C_b((0,1],\mathscr{B}(\mathscr{X})) : \lim_{h\to 0} \|T_h\| = 0 \right\}$$

is a closed bilateral ideal of $C_b((0,1],\mathscr{B}(\mathscr{X}))$. The quotient algebra

$$C_b((0,1],\mathscr{B}(\mathscr{X}))/C_0((0,1],\ \mathscr{B}(\mathscr{X})),$$

which will be denoted $\mathscr{B}_\infty$, is also a Banach algebra with quotient norm

$$\left\|\{\dot{T_h}\}\right\| = \inf_{\{U_h\}_{h\in(0,1]} \in C_0((0,1],\mathscr{B}(\mathscr{X}))} \|\{T_h\} + \{U_h\}\| = \inf_{\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}} \|\{S_h\}\| \leqslant \|\{S_h\}\|,$$

for any $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$. On the other hand we have

$$\limsup_{h\to 0} \|\{S_h\}\| \leqslant \left\|\{\dot{T_h}\}\right\|$$

for any $\{S_h\}_{h\in(0,1]} \in \{\dot{T_h}\}$.

In the following Definition, we introduce the notion of uniform ergodicity for a family of operators of $C_b\left((0,1], \mathscr{B}(\mathscr{X})\right)$.

DEFINITION 1. We say that a family of operators $\{T_h\}_{h\in(0,1]} \in C_b\left((0,1],\ \mathscr{B}(\mathscr{X})\right)$ is uniformly ergodic if there exists $\{P_h\}_{h\in(0,1]} \in C_b\left((0,1],\ \mathscr{B}(\mathscr{X})\right)$ such that

$$\lim_{n\to\infty} \limsup_{h\to 0} \|M_n(T_h) - P_h\| = 0.$$

EXAMPLE 1. (i) If T_h is uniformly ergodic for any $h \in (0,1]$, then the family $\{T_h\}_{h\in(0,1]}$ does.

(ii) If $T_h = T$ for each $h \in (0,1]$ then, T is uniformly ergodic if and only if $\{T_h\}_{h\in(0,1]}$ does.

(iii) Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$. We consider the following Banach space

$$H^\infty(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < \infty \right\}.$$

For $\varphi_h(z) = \left(\frac{1}{2} - h\right) z,\ h \in (0,1]$, we have $\|\varphi_h^n\|_\infty$ converges to 0 as $n \to \infty$. Then, by [1, Theorem 3.3.], the composition operator $C_{\varphi_h} : H^\infty(\mathbb{D}) \longrightarrow H^\infty(\mathbb{D})$ defined by $C_{\varphi_h}(f) = f \circ \varphi_h$ is uniformly ergodic for all $h \in (0,1]$. By (i), then the family $\{C_{\varphi_h}\}_{h\in(0,1]}$ is uniformly ergodic.

In [11], Macovei showed that the set

$$\mathscr{X}_b\left((0,1], \mathscr{X}\right) = \Big\{ \{x_h\}_{h\in(0,1]} \subset \mathscr{X} : \{x_h\}_{h\in(0,1]} \text{ is a bounded sequence, i.e. } \sup_{h\in(0,1]} \|x_h\| < \infty \Big\},$$

is a Banach algebra with norm

$$\|\{x_h\}\| = \sup_{h\in(0,1]} \|x_h\|.$$

And

$$\mathscr{X}_0\left((0,1], \mathscr{X}\right) = \left\{ \{x_h\}_{h\in(0,1]} \subset \mathscr{X}_b\left((0,1], \mathscr{X}\right) : \lim_{h\to 0} \|x_h\| = 0 \right\}$$

is a closed bilateral ideal of $\mathscr{X}_b((0,1],\mathscr{X})$. The quotient space

$$\mathscr{X}_b((0,1],\mathscr{X})\,/\,\mathscr{X}_0((0,1],\mathscr{X}),$$

which will be denoted $\mathscr{X}_\infty$, is a Banach algebra with quotient norm

$$\begin{aligned}\left\|\dot{\{x_h\}}\right\| &= \inf_{\{u_h\}_{h\in(0,1]}\in\mathscr{X}_0((0,1],\mathscr{X})}\|\{x_h\}+\{u_h\}\| = \inf_{\{y_h\}_{h\in(0,1]}\in\dot{\{x_h\}}}\|\{y_h\}\| \\ &= \inf_{\{y_h\}_{h\in(0,1]}\in\dot{\{x_h\}}}\ \sup_{h\in(0,1]}\|y_h\|.\end{aligned}$$

In [11], it was shown that $\mathscr{B}_\infty\subset\mathscr{B}(\mathscr{X}_\infty)$, where $\mathscr{B}(\mathscr{X}_\infty)$ is the algebra of bounded linear operators on $\mathscr{X}_\infty$.

In the following Definition, we introduce the notion of uniform ergodicity for a family of operators of $\mathscr{B}_\infty$.

DEFINITION 2. We say that $\{\dot{T_h}\}\in\mathscr{B}_\infty$ is uniformly ergodic if there exists $\{\dot{P_h}\}\in\mathscr{B}_\infty$ such that

$$\lim_{n\to\infty}\left\|M_n(\{\dot{T_h}\})-\{\dot{P_h}\}\right\|=0,$$

where

$$M_n(\{\dot{T_h}\})-\{\dot{P_h}\}=\{M_n\dot{(T_h)}\}-\{\dot{P_h}\}=\{M_n\,(T_h)-P_h\}^{\cdot}.$$

3. Main results

In this section, we will extend the known uniform ergodic theorem of M. Lin [8] from the case of a bounded linear operator to the case of a family of bounded linear operators on a Banach space.

We start this section by the following Proposition.

PROPOSITION 1. *Let $\{\dot{T_h}\}\in\mathscr{B}_\infty$ be uniformly ergodic. Then any $\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}$ is also uniformly ergodic.*

Proof. Suppose that $\{\dot{T_h}\}$ is uniformly ergodic, then there exists $\{\dot{P_h}\}\in\mathscr{B}_\infty$ such that

$$\lim_{n\to\infty}\left\|M_n(\{\dot{T_h}\})-\{\dot{P_h}\}\right\|=0.$$

Let $\{S_h\}_{h\in(0,1]}\in\{\dot{T_h}\}$ be arbitrary. Then for $\{P_h\}_{h\in(0,1]}\in\{\dot{P_h}\}$, we have

$$\lim_{n\to\infty}\limsup_{h\to0}\|M_n\,(S_h)-P_h\|\leqslant\lim_{n\to\infty}\left\|M_n(\{T_h\})\dot{-}\{P_h\}\right\|=\lim_{n\to\infty}\left\|M_n(\{\dot{T_h}\})-\{\dot{P_h}\}\right\|=0.$$

Therefore, $\{S_h\}_{h\in(0,1]}$ is uniform ergodic. $\square$

In particular, we obtain the following result.

COROLLARY 1. *Let* $\{\dot{T_h}\} \in \mathscr{B}_\infty$ *be uniformly ergodic. Then* $\{T_h\}_{h\in(0,1]}$ *is also uniformly ergodic.*

The proof of $(d) \Rightarrow (a)$, in the principal theorem (Theorem 2), requires the following Lemma.

LEMMA 1. *Let* $\Phi : \mathscr{X}_\infty \to \mathscr{Y}_\infty$ *a linear map. Then the following assertions are equivalent:*

(1) Φ *is open from* $\mathscr{X}_\infty$ *onto* $\mathscr{Y}_\infty$*;*

(2) There exists $k > 0$ *such that for any* $\{\dot{y_h}\} \in \mathscr{Y}_\infty$*, there exists* $\{\dot{x_h}\} \in \mathscr{X}_\infty$ *with* $\Phi\left(\{\dot{x_h}\}\right) = \{\dot{y_h}\}$ *and* $\left\|\{\dot{x_h}\}\right\| \leqslant k\left\|\{\dot{y_h}\}\right\|$.

Proof. $(1) \Rightarrow (2)$ Since $\Phi(B_{\mathscr{X}_\infty})$ is open and $\{\dot{0}\} \in \Phi(B_{\mathscr{X}_\infty})$, (where $B_{\mathscr{X}_\infty}$ is the unit ball of $\mathscr{X}_\infty$), there exists $\delta > 0$ such that $\left\{\{\dot{y_h}\} \in \Phi(\mathscr{X}_\infty) : \left\|\{\dot{y_h}\}\right\| < \delta\right\} \subset \Phi(B_{\mathscr{X}_\infty})$. Then, for $\{\dot{y_h}\} \in \mathscr{X}_\infty$ such that $\{\dot{y_h}\} \neq \{\dot{0}\}$, it follows that $\frac{\delta\{\dot{y_h}\}}{2\left\|\{\dot{y_h}\}\right\|} \in \left\{\{\dot{y_h}\} \in \Phi(\mathscr{X}_\infty) : \left\|\{\dot{y_h}\}\right\| < \delta\right\}$. Thus, $\frac{\delta\{\dot{y_h}\}}{2\left\|\{\dot{y_h}\}\right\|} \in \Phi(B_{\mathscr{X}_\infty})$. Hence, there exists $\{\dot{z_h}\} \in B_{\mathscr{X}_\infty}$ such that $\Phi(\{\dot{z_h}\}) = \frac{\delta\{\dot{y_h}\}}{2\left\|\{\dot{y_h}\}\right\|}$. If we set $\{\dot{x_h}\} = \frac{2\left\|\{\dot{y_h}\}\right\|}{\delta}\{\dot{z_h}\}$, we have

$$\Phi\left(\{\dot{x_h}\}\right) = \Phi\left(\frac{2\left\|\{\dot{y_h}\}\right\|}{\delta}\{\dot{z_h}\}\right) = \frac{2\left\|\{\dot{y_h}\}\right\|}{\delta}\Phi\left(\{\dot{z_h}\}\right) = \frac{2\left\|\{\dot{y_h}\}\right\|}{\delta}.\frac{\delta\{\dot{y_h}\}}{2\left\|\{\dot{y_h}\}\right\|} = \{\dot{y_h}\},$$

Since $\left\|\{\dot{z_h}\}\right\| < 1$, then

$$\left\|\{\dot{x_h}\}\right\| \leqslant \frac{2}{\delta}\left\|\{\dot{y_h}\}\right\|.$$

Consequently by taking $k = \frac{2}{\delta} > 0$, (2) holds.

Conversely, fix an open set $U \in \mathscr{X}_\infty$ and $\{\dot{x_h}\} \in U$. There exists $r > 0$ such that $\{\dot{x_h}\} + B\left(\{\dot{0}\}, r\right) = B\left(\{\dot{x_h}\}, r\right) \subset U$. To show that $\Phi(U)$ is open, it suffices to prove that

$$\Phi\left(\{\dot{x_h}\} + B\left(\{\dot{0}\}, \frac{r}{k}\right)\right) = B\left(\Phi\left(\{\dot{x_h}\}\right), \frac{r}{k}\right) \subset \Phi(U).$$

Take $\{\dot{y_h}\} \in \mathscr{Y}_\infty$ with $\left\|\{\dot{y_h}\}\right\| < \frac{r}{k}$, then by (2) there exists $\{\dot{z_h}\} \in \mathscr{X}_\infty$ such that $\Phi\left(\{\dot{z_h}\}\right) = \{\dot{y_h}\}$ and $\left\|\{\dot{z_h}\}\right\| < k.\frac{r}{k} = r$. Thus, $\{\dot{x_h}\} + \{\dot{z_h}\} \in U$ and $\Phi\left(\{\dot{x_h}\}\right) + \{\dot{y_h}\} = \Phi\left(\{\dot{x_h}\}\right) + \Phi\left(\{\dot{z_h}\}\right) = \Phi\left(\{\dot{x_h}\} + \{\dot{z_h}\}\right) \in \Phi(U)$. Then $\Phi(U)$ is open. $\square$

THEOREM 2. *Let* $\{\dot{T_h}\} \in \mathscr{B}_\infty$ *satisfy* $\lim_{n\to\infty} \frac{1}{n}\left\|\left(\{\dot{T_h}\}\right)^n\right\| = 0$. *Then, the following assertions are equivalent:*

(a) $\{\dot{T_h}\}$ *is uniformly ergodic;*

(b) $\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty$ *is closed and*

$$\mathscr{X}_\infty = \left\{\{\dot{x_h}\} \in \mathscr{X}_\infty : \{\dot{T_h}\}(\{\dot{x_h}\}) = \{\dot{x_h}\}\right\} \oplus \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty;$$

(c) $\left(\{\dot{I}\} - \{\dot{T_h}\}\right)^2\mathscr{X}_\infty$ *is closed;*

(d) $\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty$ *is closed.*

Proof. Let $\mathfrak{Y} = \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty}$.

$(a) \Rightarrow (b)$. Suppose that there exists $\{\dot{P_h}\} \in \mathscr{B}_\infty$ such that

$$\left\|M_n\left(\{\dot{T_h}\}\right) - \{\dot{P_h}\}\right\| \longrightarrow 0 \text{ as } n \to \infty.$$

We start first by showing that $\{\dot{T_h}\}.\{\dot{P_h}\} = \{\dot{P_h}\}.\{\dot{T_h}\} = \{\dot{P_h}\}$, where $\{\dot{T_h}\}.\{\dot{P_h}\} = \{\dot{T_hP_h}\}$ (recall that $\mathscr{B}_\infty$ is a Banach algebra). We have

$$\begin{aligned}\left(\{\dot{I}\} - \{\dot{T_h}\}\right).M_n\left(\{\dot{T_h}\}\right) &= \frac{1}{n}\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\cdot\left(\{\dot{I_h}\} + \{\dot{T_h}\} + \ldots + \left(\{\dot{T_h}\}\right)^{n-1}\right)\\ &= \frac{1}{n}\left(\{\dot{I_h}\} - \{\dot{T_h}\}^n\right).\end{aligned}$$

Since $\frac{1}{n}\left\|\left(\{\dot{T_h}\}\right)^n\right\| \longrightarrow 0$ as $n \to \infty$. Hence by passing to limit as $n \to \infty$ we get $\left(\{\dot{I}\} - \{\dot{T_h}\}\right).\{\dot{P_h}\} = 0$. Thus $\{\dot{T_h}\}.\{\dot{P_h}\} = \{\dot{P_h}\}$. Analogously $\{\dot{P_h}\}.\{\dot{T_h}\} = \{\dot{P_h}\}$, which means that

$$\{\dot{P_h}\}.\left(\{\dot{I}\} - \{\dot{T_h}\}\right) = \left(\{\dot{I}\} - \{\dot{T_h}\}\right).\{\dot{P_h}\} = \{\dot{0}\}.$$

We will prove that $\{\dot{P_h}\}\mathscr{X}_\infty = \left\{\{\dot{x_h}\} \in \mathscr{X}_\infty : \{\dot{T_h}\}(\{\dot{x_h}\}) = \{\dot{x_h}\}\right\}$.

Let $\{\dot{z_h}\} \in \mathscr{X}_\infty$ such that $\{\dot{T_h}\}(\{\dot{z_h}\}) = \{\dot{z_h}\}$. Then $M_n\left(\{\dot{T_h}\}\right)\left(\{\dot{z_h}\}\right) = \{\dot{z_h}\}$, thus, by passing to limit when $n \to \infty$, we get $\{\dot{P_h}\}\left(\{\dot{z_h}\}\right) = \{\dot{z_h}\}$. Hence $\{\dot{z_h}\} \in \{\dot{P_h}\}\mathscr{X}_\infty$.

Conversely, let $\{\dot{y_h}\} \in \{\dot{P_h}\}\mathscr{X}_\infty$. Then there exists $\{\dot{x_h}\} \in \mathscr{X}_\infty$ such that $\{\dot{P_h}\}\left(\{\dot{x_h}\}\right) = \{\dot{y_h}\}$. The fact that $\{\dot{T_h}\}.\{\dot{P_h}\}\left(\{\dot{x_h}\}\right) = \{\dot{P_h}\}\left(\{\dot{x_h}\}\right)$ allows us to show easily that

$$\{\dot{P_h}\}\left(\{\dot{x_h}\}\right) = \{\dot{y_h}\} \in \left\{\{\dot{x_h}\} \in \mathscr{X}_\infty : \{\dot{T_h}\}\left(\{\dot{x_h}\}\right)) = \{\dot{x_h}\}\right\}.$$

Now, let us show that $\left(\{\dot{P}_h\}\right)^2 = \{\dot{P}_h\}$. Since $\{\dot{T}_h\}.\{\dot{P}_h\} = \{\dot{P}_h\}$, it follows that

$$M_n\left(\{\dot{T}_h\}\right).\{\dot{P}_h\} = \frac{1}{n}\left(\{\dot{P}_h\} + \{\dot{T}_h\}.\{\dot{P}_h\} + \ldots + \left(\{\dot{T}_h\}\right)^n.\{\dot{P}_h\}\right) = \{\dot{P}_h\}.$$

Hence $\left(\{\dot{P}_h\}\right)^2 = \{\dot{P}_h\}$.

Next, we will prove that $\mathscr{X}_\infty = \left(\{\dot{P}_h\}\right)\mathscr{X}_\infty \oplus \mathfrak{Y}$. Put $\{\dot{Q}_h\} = \{\dot{I}_h\} - \{\dot{P}_h\}$. First, we prove that $\left(\{\dot{Q}_h\}\right)\mathscr{X}_\infty \subset \mathfrak{Y}$.

Suppose that there exists $\{\dot{u_h}\} \notin \mathfrak{Y}$ but $\{\dot{u_h}\} \in \left(\{\dot{Q}_h\}\right)\mathscr{X}_\infty$. Then there exists a linear and continuous mapping $f : \mathscr{X}_\infty \longrightarrow \mathbb{C}$ such that

$$f\left(\{\dot{u_h}\}\right) = 1 \text{ and } f\left(\{\dot{y_h}\}\right) = 0 \text{ for all } \{\dot{y_h}\} \in \mathfrak{Y}.$$

Since $f\left(\left(\{\dot{I}_h\} - \{\dot{T}_h\}\right)\{\dot{z_h}\}\right) = 0$, for all $\{\dot{z_h}\} \in \mathscr{X}_\infty$, it results $f\left(\{\dot{I}_h\}\left(\{\dot{z_h}\}\right)\right) = f\left(\{\dot{T}_h\}\right)\left(\{\dot{z_h}\}\right) = f\left(M_n\left(\{\dot{T}_h\}\right)\{\dot{z_h}\}\right)$, for all $\{\dot{z_h}\} \in \mathscr{X}_\infty$. Passing to limit as $n \to \infty$ we obtain

$$f\left(M_n\left(\{\dot{T}_h\}\right)\{\dot{z_h}\}\right) \longrightarrow f\left(\{\dot{P}_h\}\{\dot{z_h}\}\right).$$

Thus $f\left(\{\dot{Q}_h\}\{\dot{z_h}\}\right) = 0$, for all $\{\dot{z_h}\} \in \mathscr{X}_\infty$. Let $\{\dot{v_h}\} \in \mathscr{X}_\infty$ such that $\{\dot{u_h}\} = \{\dot{Q}_h\}(\{\dot{v_h}\})$, thus $f(\{\dot{u_h}\}) = 0$, contradiction. Therefore $\{\dot{Q}_h\}\mathscr{X}_\infty \subset \mathfrak{Y}$.

Since $\{\dot{P}_h\}$ is a projection, we have the equality $\mathscr{X}_\infty = \{\dot{P}_h\}\mathscr{X}_\infty \oplus \{\dot{Q}_h\}\mathscr{X}_\infty$. Then $\mathscr{X}_\infty \subset \{\dot{P}_h\}\mathscr{X}_\infty \oplus \mathfrak{Y}$. Therefore, $\mathscr{X}_\infty = \{\dot{P}_h\}\mathscr{X}_\infty \oplus \mathfrak{Y}$.

Since

$$\{\dot{T}_h\}\mathfrak{Y} = \{\dot{T}_h\}\mathscr{X}_\infty - \{\dot{T}_h\}\{\dot{P}_h\}\mathscr{X}_\infty = \{\dot{T}_h\}\mathscr{X}_\infty - \{\dot{P}_h\}\mathscr{X}_\infty \subset \mathscr{X}_\infty - \{\dot{P}_h\}\mathscr{X}_\infty \subset \mathfrak{Y},$$

thus $\mathfrak{Y}$ is $\{\dot{T}_h\}$-invariant.

If we put $\{\dot{S}_h\} = \{\dot{T}_h\}_{|\mathfrak{Y}}$, then, by (1) we obtain

$$\left\|\frac{1}{n}\sum_{k=0}^{n-1}\{\dot{S}_h\}^k\right\| \longrightarrow 0 \text{ as } n \to \infty.$$

Fix $n_0 \in \mathbb{N}$ such that $\left\|\frac{1}{n_0}\sum_{k=0}^{n_0-1}\{\dot{S}_h\}^k\right\| < 1$. Then $\{\dot{I}\} - \frac{1}{n_0}\sum_{k=0}^{n_0-1}\{\dot{S}_h\}^k$ is invertible.

Using

$$\begin{aligned}\{\dot{I}\} - \frac{1}{n_0}\sum_{k=0}^{n_0-1}\{\dot{S}_h\}^k &= \{\dot{I}\} - \frac{1}{n_0}\{\dot{I}\} - \frac{1}{n_0}\{\dot{S}_h\} - \ldots - \frac{1}{n_0}\{\dot{S}_h\}^{n_0-1} \\ &= \frac{1}{n_0}\left(\{\dot{I}\} - \{\dot{S}_h\}\right)\left(\{\dot{I}\} + \left(\{\dot{I}\} + \{\dot{S}_h\}\right)\right. \\ &\quad \left. + \ldots + \left(\{\dot{I}\} + \{\dot{S}_h\} + \ldots + \{\dot{S}_h\}^{n_0-2}\right)\right),\end{aligned}$$

we deduce that $\{\dot{I}\} - \{\dot{S_h}\}$ is invertible. Thus,

$$\mathfrak{Y} = \left(\{\dot{I}\} - \{\dot{S_h}\}\right)\mathfrak{Y} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathfrak{Y} \subset \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty,$$

hence

$$\mathfrak{Y} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty.$$

Then $\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty$ is closed and (b) is verified.

$(b) \Rightarrow (c)$. Let $\mathfrak{Y} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty$, we want to prove the equality

$$\left(\{\dot{I}\} - \{\dot{T_h}\}\right)^2\mathscr{X}_\infty = \mathfrak{Y}.$$

Evidently we have

$$\left(\{\dot{I}\} - \{\dot{T_h}\}\right)^2\mathscr{X}_\infty = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty \subset \mathfrak{Y}.$$

Let $\{\dot{y_h}\} \in \mathfrak{Y}$, then there exists $\{\dot{x_h}\} \in \mathscr{X}_\infty$ such that $\{\dot{y_h}\} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\{\dot{x_h}\}$. By (b), we write $\{\dot{x_h}\}$ as follows

$$\{\dot{x_h}\} = \{\dot{u_h}\} + \{\dot{v_h}\} \text{ with } \{\dot{T_h}\}\{\dot{u_h}\} = \{\dot{u_h}\} \text{ and } \{\dot{v_h}\} \in \mathfrak{Y}.$$

Thus,

$$\begin{aligned}\{\dot{y_h}\} &= \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\{\dot{x_h}\} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\{\dot{v_h}\} \in \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathfrak{Y} \\ &= \left(\{\dot{I}\} - \{\dot{T_h}\}\right)^2\mathscr{X}_\infty,\end{aligned}$$

hence $\mathfrak{Y} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)^2\mathscr{X}_\infty$, therefore $\left(\{\dot{I}\} - \{\dot{T_h}\}\right)^2\mathscr{X}_\infty$ is closed.

$(c) \Rightarrow (d)$. Let $\mathfrak{Y} = \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty}$. We will prove that $\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty = \mathfrak{Y}$. It is easy to show that

$$\begin{aligned}\left(\{\dot{I}\} - \{\dot{T_h}\}\right)^2\mathscr{X}_\infty &= \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty \\ &\subset \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty} = \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathfrak{Y}.\end{aligned}$$

On the other hand, by (c), we have

$$\begin{aligned}\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathfrak{Y} &= \left(\{\dot{I}\} - \{\dot{T_h}\}\right)\overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)\mathscr{X}_\infty} \subset \overline{\left(\{\dot{I}\} - \{\dot{T_h}\}\right)^2\mathscr{X}_\infty} \\ &= \left(\{\dot{I}\} - \{\dot{T_h}\}\right)^2\mathscr{X}_\infty.\end{aligned}$$

Since $\left(\{\dot{I}\}-\{\dot{T_h}\}\right)^2 \mathscr{X}_\infty$ is closed thus $\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathfrak{Y}=\overline{\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathfrak{Y}}$. As in the proof of $(b)\Rightarrow(c)$, the restriction $\{\dot{S_h}\}=\{\dot{T_h}\}_{|\mathfrak{Y}}$ satisfies $\left\|\frac{1}{n}\sum_{k=0}^{n-1}\{\dot{S_h}\}^k\{\dot{y_h}\}\right\|\longrightarrow 0$ as $n\to\infty$, for any $\{\dot{y_h}\}\in\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathscr{X}_\infty$. Then

$$\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathscr{X}_\infty\subset\overline{\left(\{\dot{I}\}-\{\dot{S_h}\}\right)\mathfrak{Y}}=\left(\{\dot{I}\}-\{\dot{S_h}\}\right)\mathfrak{Y}=\left(\{\dot{I}\}-\{\dot{T_h}\}\right)^2\mathscr{X}_\infty.$$

Hence $\mathfrak{Y}\subset\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathscr{X}_\infty$. Therefore $\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathscr{X}_\infty$ is closed.

$(d)\Rightarrow(a)$. Let $\mathfrak{Y}=\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathscr{X}_\infty$, then $\mathfrak{Y}$ is a Banach space. The operator $\{\dot{I}\}-\{\dot{T_h}\}:\mathscr{X}_\infty\longrightarrow\mathfrak{Y}$ is surjective and continuous, then by the open mapping theorem, $\{\dot{I}\}-\{\dot{T_h}\}$ is open. Thus by the Lemma 1 there exists $k>0$ satisfying that for each $\{\dot{y_h}\}\in\mathfrak{Y}$, there exists $\{\dot{z_h}\}\in\mathscr{X}_\infty$ such that

$$\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\{\dot{z_h}\}=\{\dot{y_h}\},\text{ and }\left\|\{\dot{z_h}\}\right\|\leqslant k\left\|\{\dot{y_h}\}\right\|.$$

Hence, for $\{\dot{y_h}\}\in\mathfrak{Y}$, we have

$$\begin{aligned}\left\|\frac{1}{n}\sum_{k=0}^{n-1}\{\dot{T_h}\}^k\{\dot{y_h}\}\right\|&=\left\|\frac{1}{n}\sum_{k=0}^{n-1}\{\dot{T_h}\}^k\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\{\dot{x_h}\}\right\|=\left\|\frac{1}{n}\left(\{\dot{I}\}-\{\dot{T_h}\}^n\right)\{\dot{x_h}\}\right\|\\&\leqslant k\left\|\frac{1}{n}\left(\{\dot{I}\}-\{\dot{T_h}\}^n\right)\right\|\left\|\{\dot{y_h}\}\right\|.\end{aligned}$$

Let $\{\dot{S_h}\}=\{\dot{T_h}\}_{|\mathfrak{Y}}$, the restriction of $\{\dot{T_h}\}$ to $\mathfrak{Y}$. Then $\left\|\frac{1}{n}\sum_{k=0}^{n-1}\{\dot{S_h}\}^k\right\|\longrightarrow 0$ as $n\to\infty$. By the proof of $(a)\Rightarrow(b)$, we obtain that $\{\dot{I}\}-\{\dot{S_h}\}$ is invertible on $\mathfrak{Y}$ and

$$\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathscr{X}_\infty=\mathfrak{Y}=\left(\{\dot{I}\}-\{\dot{S_h}\}\right)\mathfrak{Y}=\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathfrak{Y}=\left(\{\dot{I}\}-\{\dot{S_h}\}\right)\mathscr{X}_\infty.$$

Then for $\{\dot{x_h}\}\in\mathscr{X}_\infty$ there exists $\{\dot{y_h}\}\in\mathfrak{Y}$ such that

$$\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\{\dot{x_h}\}=\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\{\dot{y_h}\},$$

consequently $\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\left(\{\dot{x_h}\}-\{\dot{y_h}\}\right)=\{\dot{0}\}$. Thus $M_n\left(\{\dot{T_h}\}\right)\left(\{\dot{x_h}\}-\{\dot{y_h}\}\right)=\left(\{\dot{x_h}\}-\{\dot{y_h}\}\right)$ for all $n\in\mathbb{N}$. Hence by the equality $\{\dot{x_h}\}=\left(\{\dot{x_h}\}-\{\dot{y_h}\}\right)+\{\dot{y_h}\}$, one can show that

$$\mathscr{X}_\infty=\left\{\{\dot{x_h}\}\in\mathscr{X}_\infty:\{\dot{T_h}\}(\{\dot{x_h}\})=\{\dot{x_h}\}\right\}\oplus\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\mathscr{X}_\infty.$$

Therefore, if we define the map $\{\dot{P_h}\}:\mathscr{X}_\infty\longrightarrow\mathscr{X}_\infty$ by $\{\dot{P_h}\}\{\dot{x_h}\}=\{\dot{x_h}\}-\{\dot{y_h}\}$ such that $\{\dot{y_h}\}$ is the element defined as $\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\{\dot{x_h}\}=\left(\{\dot{I}\}-\{\dot{T_h}\}\right)\{\dot{y_h}\}$, one can show that

$$\lim_{n\to\infty}\left\|M_n\left(\{\dot{T_h}\}\right)-\{\dot{P_h}\}\right\|=0.$$

and we obtain (a), so the proof is complete. □

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