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STRICT SINGULARITY OF WEIGHTED COMPOSITION OPERATORS ON DERIVATIVE HARDY SPACES

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Abstract. We prove that the weighted composition operator $W_{\phi,\varphi}$ fixes an isomorphic copy of ℓ^p if the operator $W_{\phi,\varphi}$ is not compact on the derivative Hardy space S^p. In particular, this implies that the strict singularity of the operator $W_{\phi,\varphi}$ coincides with the compactness of it on S^p. Moreover, when $p \neq 2$, we characterize the conditions for those weighted composition operators $W_{\phi,\varphi}$ on S^p which fix an isomorphic copy of ℓ^2.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Hardy space H^p is the space of functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p} := \left(\sup_{0 \leqslant r < 1} \int_{\partial\mathbb{D}} |f(r\xi)|^p dm(\xi)\right)^{1/p} < \infty,$$

where m is the normalized Lebesgue measure on $\partial\mathbb{D}$. From [25, Theorem 9.4], this norm is equal to the following norm:

$$\|f\|_{H^p} = \left(\int_{\partial\mathbb{D}} |f(\xi)|^p dm(\xi)\right)^{1/p},$$

where for any $\xi \in \partial\mathbb{D}$, $f(\xi)$ is the radial limit which exists almost every.

For $p = \infty$, the space H^∞ is defined by

$$H^\infty = \{f \in H(\mathbb{D}) : \|f\|_\infty := \sup_{z\in\mathbb{D}}\{|f(z)|\} < \infty\}.$$

We define the weighted composition operator $W_{\phi,\varphi}$ for $f \in H(\mathbb{D})$ by

$$W_{\phi,\varphi}(f)(z) = \phi(z) f \circ \varphi(z), \qquad z \in \mathbb{D},$$

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where $\phi \in H(\mathbb{D})$ and φ is an analytic self-map of $\mathbb{D}$. If $\phi(z) \equiv 1$, $W_{\phi,\varphi}$ becomes the composition operator C_φ while if $\varphi(z) \equiv z$, $W_{\phi,\varphi}$ becomes the multiplication operator M_ϕ. For weighted composition operators $W_{\phi,\varphi}$ on Hardy spaces H^p, we refer the readers to the literatures [4, 6, 9, 10]. It should be noticed that the complete characterizations for the boundedness and compactness of the weighted composition operator $W_{\phi,\varphi}$ on Dirichlet spaces are still unknown (for this, see [2] and the references therein).

We define the derivative Hardy space S^p by

$$S^p = \{f \in H(\mathbb{D}) : \|f\|_{S^p} := |f(0)| + \|f'\|_{H^p} < \infty\}.$$

For $1 \leqslant p \leqslant \infty$, S^p is a Banach algebra and there is an inclusion relation: $S^p \subset H^\infty$ (for the detailed structure of S^p spaces, see [7, 8, 13, 16, 17] for references).

In paper [24], Roan started the investigation of composition operators on the space S^p. After his work, MacCluer [19] gave the characterizations of the boundedness and the compactness of the composition operators on the space S^p in terms of Carleson measures. A remarkable result on the boundedness and the compactness of the weighted composition operators on S^p was obtained in [3], in which they are both characterized through the corresponding weighted composition operators on H^p. Furthermore, the isometries between S^p was obtained by Novinger and Oberlin in [23], in which they showed that the isometries were closely related to the weighted composition operator.

A bounded operator $T: X \to Y$ between Banach spaces is strictly singular if its restriction to any infinite-dimensional closed subspace is not an isomorphism onto its image. This notion was introduced by Kato [14].

A bounded operator $T: X \to Y$ between Banach spaces is said to fix a copy of the given Banach space E if there is a closed subspace $M \subset X$, linearly isomorphic to E, such that the restriction $T_{|M}$ defines an isomorphism from M onto $T(M)$. The bounded operator $T: X \to Y$ is called ℓ^p-singular if it does not fix any copy of ℓ^p.

Laitila, et al [15] recently investigated the strict singularity for the composition operators on H^p spaces. Following their ideas, Miihkinen [20] studied the strict singularity of the Volterra type operator on Hardy space H^p and showed that the strict singularity of the Volterra type operator coincides with its compactness on H^p, $1 \leqslant p < \infty$. Miihkinen [20] also post an open question which was resolved in [22] by utilizing the generalized Volterra operators. It should be noticed that Hernández, et al [12] investigated the interpolation and extrapolation of strictly singular operators between L_p spaces. Recently, Miihkinen, Pau, Perälä and Wang [21] considered the singularity of the Volterra type operator on Hardy space of the unit ball.

In this paper, we prove that the weighted composition operator $W_{\phi,\varphi}$ fixes an isomorphic copy of ℓ^p if the operator $W_{\phi,\varphi}$ is not compact on the derivative Hardy space S^p. In particular, this implies that the strict singularity of the operator $W_{\phi,\varphi}$ coincides with the compactness of it on S^p. Moreover, when $p \neq 2$, we characterize the conditions for those weighted composition operators $W_{\phi,\varphi}$ on S^p which fix an isomorphic copy of ℓ^2.

Our main results read as follows:

THEOREM 1. *Let* $1 \leqslant p < \infty$, $\phi \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *If the weighted composition operator* $W_{\phi,\varphi}\colon S^p \to S^p$ *is bounded but not compact, then* $W_{\phi,\varphi}$ *fixes an isomorphic copy of* ℓ^p *in* S^p. *In particular, the operator* $W_{\phi,\varphi}$ *is not strictly singular, that is, strict singularity of bounded operator* $W_{\phi,\varphi}$ *coincides with its compactness.*

REMARK 1. In the final section, we prove that the claims of theorem 1 are still true for the case of $p = \infty$.

Denote $E_\varphi = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$, where φ is an analytic self-map of $\mathbb{D}$. Then we have

THEOREM 2. *Let* $1 \leqslant p < \infty$, $\phi \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *Suppose that* $W_{\phi,\varphi}\colon S^p \to S^p$ *is bounded and* $m(E_\varphi) = 0$. *If* $W_{\phi,\varphi}$ *is bounded below on an infinite-dimensional subspace* $M \subset S^p$, *then the restriction* $W_{\phi,\varphi}$ *on* M *fixes an isomorphic copy of* ℓ^p *in* M. *In particular, if* $p \neq 2$, *the operator* $W_{\phi,\varphi}$ *does not fix any isomorphic copy of* ℓ^2 *in* S^p.

When $m(E_\varphi) > 0$, it holds that

THEOREM 3. *Let* $1 \leqslant p < \infty$, $\phi \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$. *Suppose that* $W_{\phi,\varphi}\colon S^p \to S^p$ *is bounded. If* $m(E_\varphi) > 0$ *and* $\phi\varphi' \neq 0$, *then the operator* $W_{\phi,\varphi}$ *fixes an isomorphic copy of* ℓ^2 *in* S^p.

Notation: throughout this paper, C will represent a positive constant which may be given different values at different occurrences.

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. First, the following Lemma 1 can be deduced from [3, Theorem 2.1] and [10, Theorem 2.2 and Theorem 2.3].

LEMMA 1. *Let* $1 \leqslant p < \infty$, $\phi \in H(\mathbb{D})$ *and* φ *is an analytic self-map of* $\mathbb{D}$. *Then* $W_{\phi,\varphi}\colon S^p \to S^p$ *is compact if and only if* $\phi \in S^p$ *and*

$$\lim_{|a|\to 1^-} \int_{\partial\mathbb{D}} \frac{1-|a|^2}{|1-\bar{a}\varphi(\omega)|^2} |\phi(\omega)\varphi'(\omega)|^p dm(\omega) = 0.$$

The following lemma 2 is proven in [3, Proposition 3.3(ii)].

LEMMA 2. *Let* $1 \leqslant p \leqslant \infty$, $\phi \in H^p$ *and* φ *is an analytic self-map of* $\mathbb{D}$. *Then* $W_{\phi,\varphi}\colon S^p \to H^p$ *is compact.*

We employ the test functions

$$f_a(z) = \int_0^z \frac{(1-|a|^2)^{1/p}}{(1-\bar{a}\omega)^{2/p}} d\omega, \quad z \in \mathbb{D},$$

where $a\in\mathbb{D}$. They all satisfy $\|f_a\|_{S^p}=1$ and f_a converges to 0 uniformly on compact subsets of $\mathbb{D}$, as $|a|\to 1^-$.

For φ is an analytic self-map of $\mathbb{D}$, let $L=\{\xi\in\partial\mathbb{D}:\ \text{the radial limit } \varphi(\xi) \text{ exists}\}$ and

$$E_\varepsilon:=\{\xi\in L:\ |1-\varphi(\xi)|<\varepsilon\}$$

for any given $\varepsilon>0$, then $m(\partial\mathbb{D}\setminus L)=0$. The proof of Theorem 1 relies on the following auxiliary lemma.

LEMMA 3. *Let $(a_n)\subset\mathbb{D}$ be a sequence such that $0<|a_1|<|a_2|<\ldots<1$ and $a_n\to 1$. If the bounded operator $W_{\phi,\varphi}:\ S^p\to S^p$ is not compact, then we have*

$$(1)\qquad \lim_{\varepsilon\to 0}\int_{E_\varepsilon}|(W_{\phi,\varphi}f_{a_n})'|^p dm=0\quad \textit{for every } n\in\mathbb{N}.$$

$$(2)\qquad \lim_{n\to\infty}\int_{\partial\mathbb{D}\setminus E_\varepsilon}|(W_{\phi,\varphi}f_{a_n})'|^p dm=0\quad \textit{for every } \varepsilon>0.$$

Proof. **(1)** For each fixed n, this follows immediately from the absolute continuity of Lebesgue measure, the boundedness of the operator $W_{\phi,\varphi}$ and the fact that $W_{\phi,\varphi}$ is not compact (which implies that φ is not identically 1).

(2) For any given $\varepsilon>0$, let $\xi\in L\setminus E_\varepsilon$. Then there exists an $N>0$ such that whenever $n>N$, it holds that

$$\begin{aligned}|1-\bar a_n\varphi(\xi)|&=|1-\varphi(\xi)+\varphi(\xi)-\bar a_n\varphi(\xi)|\\&\geqslant|1-\varphi(\xi)|-|\varphi(\xi)-\bar a_n\varphi(\xi)|\\&\geqslant|1-\varphi(\xi)|-|1-\bar a_n|>\frac{\varepsilon}{2}.\end{aligned}$$

Now, by definition, we have

$$\int_{\partial\mathbb{D}\setminus E_\varepsilon}|(W_{\phi,\varphi}f_{a_n})'|^p dm\leqslant C\left(\int_{\partial\mathbb{D}\setminus E_\varepsilon}|\phi' f_{a_n}(\varphi)|^p dm+\int_{\partial\mathbb{D}\setminus E_\varepsilon}|\phi\varphi' f_{a_n}'(\varphi)|^p dm\right).$$

Since $W_{\phi,\varphi}$ is bounded, it follows that $\phi\in S^p$, that is, $\phi'\in H^p$. By Lemma 2, $W_{\phi',\varphi}:\ S^p\to H^p$ is compact, which implies that

$$\lim_{n\to\infty}\int_{\partial\mathbb{D}\setminus E_\varepsilon}|\phi' f_{a_n}(\varphi)|^p dm\leqslant\lim_{n\to\infty}\int_{\partial\mathbb{D}}|W_{\phi',\varphi}f_{a_n}|^p dm=0.$$

For the estimate of the second integral, we have

$$\begin{aligned}\int_{\partial\mathbb{D}\setminus E_\varepsilon}|\phi\varphi' f_{a_n}'(\varphi)|^p dm&=\int_{\partial\mathbb{D}\setminus E_\varepsilon}|\phi\varphi'|^p\frac{1-|a_n|^2}{|1-\bar a_n\varphi|^2}dm\\&=\int_{\partial\mathbb{D}\setminus E_\varepsilon}|\phi\varphi'|^p\frac{1-|a_n|^2}{|1-\bar a_n\varphi|^2}dm\\&\leqslant\frac{4(1-|a_n|^2)}{\varepsilon^2}\int_{\partial\mathbb{D}}|\phi\varphi'|^p dm,\end{aligned}$$

where $\int_{\partial\mathbb{D}} |\phi\varphi'|^p dm$ is finite due to the boundedness of $W_{\phi,\varphi} : S^p \to S^p$ and [3, Theorem 2.1] and [9, Theorem 4].

Therefore,

$$\lim_{n\to\infty} \int_{\partial\mathbb{D}\setminus E_\varepsilon} |\phi\varphi' f'_{a_n}(\varphi)|^p dm = 0,$$

The proof is complete. □

Now, we are ready to give a proof of Theorem 1.

Proof of Theorem 1. First, we prove that there exists a sequence $(a_n) \subset \mathbb{D}$ with $0 < |a_1| < |a_2| < \ldots < 1$ and $a_n \to \omega \in \partial\mathbb{D}$, such that there is a positive constant h such that

$$\|(W_{\phi,\varphi} f_{a_n})'\|_{H^p} \geqslant h > 0$$

holds for all $n \in \mathbb{N}$.

Since $W_{\phi,\varphi} : S^p \to S^p$ is not compact, by Lemma 1, there exists a sequence $(a_n) \subset \mathbb{D}$ with $0 < |a_1| < |a_2| < \ldots < 1$ and $a_n \to \omega \in \partial\mathbb{D}$, such that there is a positive constant h such that $\|\phi\varphi' f'_{a_n}(\varphi)\|_{H^p} \geqslant 2h > 0$ holds for all $n \in \mathbb{N}$. Note that

$$\|(W_{\phi,\varphi} f_{a_n})'\|_{H^p} \geqslant \|\phi\varphi' f'_{a_n}(\varphi)\|_{H^p} - \|\phi' f_{a_n}(\varphi)\|_{H^p}.$$

By Lemma 2, $W_{\phi',\varphi} :\ S^p \to H^p$ is compact, which implies that

$$\lim_{n\to\infty} \|\phi' f_{a_n}(\varphi)\|_{H^p} = 0.$$

Hence, there exists a subsequence of (a_n) (denoted still by (a_n)) such that the above claim holds. We assume without loss of generality that $a_n \to 1$ as $n \to \infty$ by utilizing a suitable rotation.

Then by Lemma 3 and the induction method, we are able to choose a decreasing positive sequence (ε_n) such that $E_{\varepsilon_1} = \partial\mathbb{D}$ and $\lim_{n\to\infty} \varepsilon_n = 0$, and a subsequence $(b_n) \subset (a_n)$ such that the following three conditions hold:

$$\left(\int_{E_{\varepsilon_n}} |(W_{\phi,\varphi} f_{b_k})'|^p dm\right)^{1/p} < 4^{-n}\delta h, \quad k = 1,\ldots,n-1; \tag{1}$$

$$\left(\int_{\partial\mathbb{D}\setminus E_{\varepsilon_n}} |(W_{\phi,\varphi} f_{b_n})'|^p dm\right)^{1/p} < 4^{-n}\delta h; \tag{2}$$

$$\left(\int_{E_{\varepsilon_n}} |(W_{\phi,\varphi} f_{b_n})'|^p dm\right)^{1/p} > \frac{h}{2} \tag{3}$$

for every $n \in \mathbb{N}$, where $\delta > 0$ is a small constant whose value will be determined later.

Now we are ready to prove that there is a $C > 0$ such that the inequality

$$\|\sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j})\|_{S^p} \geqslant C\|(c_j)\|_{\ell^p}$$

holds. By the triangle inequality in L^p, we have

$$\begin{aligned}\|\sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j})\|_{S^p}^p &\geqslant \|\sum_{j=1}^{\infty}\Big(c_j W_{\phi,\varphi}(f_{b_j})\Big)'\|_{H^p}^p\\ &= \sum_{n=1}^{\infty}\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}}\left|\sum_{j=1}^{\infty}\Big(c_j W_{\phi,\varphi}(f_{b_j})\Big)'\right|^p dm\\ &\geqslant \sum_{n=1}^{\infty}\left(|c_n|\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}}|\big(W_{\phi,\varphi}(f_{b_n})\big)'|^p dm\right)^{\frac{1}{p}}\right.\\ &\quad\left.-\sum_{j\neq n}|c_j|\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}}|\Big(W_{\phi,\varphi}(f_{b_j})\Big)'|^p dm\right)^{\frac{1}{p}}\right)^p.\end{aligned}$$

Observe that for every $n\in\mathbb{N}$, we have

$$\begin{aligned}&\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}}|\big(W_{\phi,\varphi}(f_{b_n})\big)'|^p dm\right)^{\frac{1}{p}}\\ &=\left(\int_{E_{\varepsilon_n}}|\big(W_{\phi,\varphi}(f_{b_n})\big)'|^p dm-\int_{E_{\varepsilon_{n+1}}}|\big(W_{\phi,\varphi}(f_{b_n})\big)'|^p dm\right)^{1/p}\\ &\geqslant\left(\left(\frac{h}{2}\right)^p-\big(4^{-n-1}\delta h\big)^p\right)^{1/p}\\ &\geqslant\frac{h}{2}-4^{-n-1}\delta h\end{aligned}$$

according to conditions (1) and (3) above, where the last estimate holds for $1\leqslant p<\infty$.

Moreover, by condition (1) and (2), it holds that

$$\left(\int_{E_{\varepsilon_n}\setminus E_{\varepsilon_{n+1}}}|\Big(W_{\phi,\varphi}(f_{b_j})\Big)'|^p dm\right)^{\frac{1}{p}}<2^{-n-j}\delta h\quad\text{for } j\neq n.$$

Consequently, we obtain that

$$\begin{aligned}\|\sum_{j=1}^{\infty}c_j W_{\phi,\varphi}(f_{b_j})\|_{S^p} &\geqslant\left(\sum_{n=1}^{\infty}\left(|c_n|\left(\frac{h}{2}-4^{-n-1}\delta h\right)-2^{-n}\delta h\|(c_j)\|_{\ell^p}\right)^p\right)^{1/p}\\ &\geqslant\left(\sum_{n=1}^{\infty}\left(|c_n|\left(\frac{h}{2}\right)-2^{-n+1}\delta h\|(c_j)\|_{\ell^p}\right)^p\right)^{1/p}\\ &\geqslant\frac{h}{2}\|(c_j)\|_{\ell^p}-\delta h\|(c_j)\|_{\ell^p}\left(\sum_{n=1}^{\infty}2^{-(n-1)p}\right)^{1/p}\\ &\geqslant h\left(\frac{1}{2}-\delta\left(1-2^{-p}\right)^{-1/p}\right)\|(c_j)\|_{\ell^p}\geqslant C\|(c_j)\|_{\ell^p},\end{aligned}$$

where the last inequality holds when we choose δ small enough.

On the other hand, we are to prove the converse inequality:

$$\|\sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j})\|_{S^p} \leqslant C\|(c_j)\|_{\ell^p}.$$

By definition,

$$\|\sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j})\|_{S^p} = |\sum_{j=1}^{\infty} c_j \phi(0) f_{b_j}(\varphi(0))| + \|\sum_{j=1}^{\infty}\Big(c_j W_{\phi,\varphi}(f_{b_j})\Big)'\|_{H^p}.$$

First, we note that a straightforward variant of the above procedure also gives

$$\|\sum_{j=1}^{\infty}\Big(c_j W_{\phi,\varphi}(f_{b_j})\Big)'\|_{H^p} \leqslant C\|(c_j)\|_{\ell^p}.$$

Next, when $p=1$, since $\lim_{j\to\infty} f_{b_j}(\varphi(0))=0$, it is trivial that

$$|\sum_{j=1}^{\infty} c_j \phi(0) f_{b_j}(\varphi(0))| \leqslant C\|(c_j)\|_{\ell^1}.$$

When $1<p<\infty$, we can choose a subsequence of (b_j) (still denoted by (b_j)) such that $\{(1-|b_j|^2)^{1/p}\}_{j=1}^{\infty} \in \ell^q$, where $1/p+1/q=1$. Then by Hölder's inequality,

$$|\sum_{j=1}^{\infty} c_j \phi(0) f_{b_j}(\varphi(0))| \leqslant C\|(c_j)\|_{\ell^p}.$$

Accordingly, the desired inequality follows.

By choosing $\phi=1$ and $\varphi=z$, we obtain that

$$C\|(c_j)\|_{\ell^p} \leqslant \|\sum_{j=1}^{\infty} c_j f_{b_j}\|_{S^p} \leqslant C\|(c_j)\|_{\ell^p}.$$

Thus, we have

$$\|\sum_{j=1}^{\infty} c_j W_{\phi,\varphi}(f_{b_j})\|_{S^p} \geqslant C\|\sum_{j=1}^{\infty} c_j f_{b_j}\|_{S^p}$$

The proof is complete. $\square$

3. Proof of Theorem 2

In this section, we give the proof of Theorem 2.

Proof of Theorem 2. Since M is the infinite-dimensional subspace of S^p and polynomials are dense in S^p (see [16]), there exists a sequence (f_n) of unit vectors in M

such that f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$. Since $W_{\phi,\varphi}$ is bounded below on $M \subset S^p$, there exists $h > 0$ such that

$$\|W_{\phi,\varphi} f_n\|_{S^p} > h,$$

for all $n \in \mathbb{N}$. For $k \geqslant 1$, denote $E_k := \{\xi \in \partial\mathbb{D} : \ |\varphi(\xi)| \geqslant 1 - 1/k\}$. Since by assumption, $\lim_{k\to\infty} m(E_k) = m(E_\varphi) = 0$, it holds that

$$\lim_{k\to\infty} \int_{E_k} |(W_{\phi,\varphi} f_n)'|^p dm = 0$$

for every $n \in \mathbb{N}$. Moreover, since f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$, it follows that

$$\lim_{n\to\infty} \int_{\partial\mathbb{D}\setminus E_k} |(W_{\phi,\varphi} f_n)'|^p dm = 0$$

for every $k \in \mathbb{N}$.

The remainder of the proof is an argument that goes exactly as the proof of Theorem 1, so we omit it. Thus, the proof is complete. □

4. Proof of Theorem 3

In this last section, we give a proof for Theorem 3.

Proof of Theorem 3. We first note that the integral operator $T_z : f \mapsto \int_0^z f(\zeta)d\zeta$ is bounded from H^p into S^p. Based on this relation, we consider the following operator:

$$T := \frac{d}{dz} \circ W_{\phi,\varphi} \circ T_z = W_{\phi',\varphi} \circ T_z + W_{\phi\varphi',\varphi} \text{ on } H^p.$$

By Lemma 2, [3, Theorem 2.1] and the expression of the operator T, we see that the boundedness of $W_{\phi,\varphi}$ on S^p is equivalent to the boundedness of the operator T on H^p.

Now, for the operator $W_{\phi\varphi',\varphi}$ on H^p, we can deduce from the proof of [18, Theorem 2] that there exists a sequence of integers (n_k) satisfying $\inf_k(n_{k+1}/n_k) > 1$ and a positive constant C such that

$$\|\sum_k c_k W_{\phi\varphi',\varphi}(e_{n_k})\|_{H^p} \geqslant C\|(c_k)\|_{\ell^2},$$

where $e_{n_k} := z^{n_k}$ is the unit vector in H^p. Since the operator $W_{\phi',\varphi} \circ T_z : H^p \to H^p$ is compact (it is equivalent to the compactness of $W_{\phi',\varphi} : S^p \to H^p$, which is claimed by Lemma 2), then for any $\varepsilon > 0$, there exists a subsequence of (n_k) (still denoted as (n_k)) such that

$$\|\sum_k c_k W_{\phi',\varphi} \circ T_z(e_{n_k})\|_{H^p} \leqslant \varepsilon\|(c_k)\|_{\ell^2}.$$

Thus,

$$\|\sum_k c_k T(e_{n_k})\|_{H^p} \geqslant C\|(c_k)\|_{\ell^2},$$

which implies that the weighted composition operator $W_{\phi,\varphi}$ on S^p is bounded below on the closed linear span of $\{g_{n_k} : k \geqslant 1\}$:

$$\|\sum_k c_k W_{\phi,\varphi}(g_{n_k})\|_{S^p} \geqslant C\|(c_k)\|_{\ell^2},$$

where $g_{n_k} := T_z(e_{n_k})$ is the unit vector in S^p.

Since Paley's theorem (see [11]) implies that the closed linear span $M := \{e_{n_k} : k \geqslant 1\}$ in H^p is isomorphic to ℓ^2, this implies that the closed linear span of $T_z(M) = \{g_{n_k} : k \geqslant 1\}$ in S^p is isomorphic to ℓ^2. Hence, $W_{\phi,\varphi}$ fixes an isomorphic copy of ℓ^2 in S^p.

Accordingly, it follows that $W_{\phi,\varphi}$ fixes an isomorphic copy of ℓ^2 in S^p, which is the desired result. □

5. The strict singularity of $W_{\phi,\varphi}$ on S^∞

Here we show that the claims of theorem 1 is still true for the case of $p = \infty$. By Lemma 2 and [3, Theorem 2.1], we have known that the boundedness of the weighted composition operator $W_{\phi,\varphi}$ on S^∞ is equivalent to the boundedness of the operator

$$T := W_{\phi',\varphi} \circ T_z + W_{\phi\varphi',\varphi} \text{ on } H^\infty.$$

It follows from [5] that any weakly compact weighted composition operator on H^∞ is compact. Since by Lemma 2 the operator $W_{\phi\varphi',\varphi}$ is compact on H^∞, it holds that T is weakly compact on H^∞ is and only if T is compact on H^∞. Moreover, Bourgain [1] established that a bounded linear operator on H^∞ is weakly compact if and only if it does not fix any copy of ℓ^∞. Thus, T is compact on H^∞ if and only if it does not fix any copy of ℓ^∞.

Therefore, the weighted composition operator $W_{\phi,\varphi}$ on S^∞ is compact if and only if it does not fix any copy of ℓ^∞. In particular, the noncompact operator $W_{\phi,\varphi}$ on S^∞ is not strictly singular, that is, strict singularity of bounded operator $W_{\phi,\varphi}$ on S^∞ coincides with its compactness.

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