Operators
and
Matrices
Volume 15, Number 4 (2021), 1227–1240 doi:10.7153/oam-2021-15-77

ON SOME GENERALIZED INVERSES OF $M_{\vee}$–MATRICES

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(*Communicated by L. Zhang*)

Abstract. In this paper we study a class of generalized M-matrices known as $M_{\vee}$-matrices. An $M_{\vee}$-matrix has the form $A = sI - B$, with $s \geqslant \rho(B)$ and B eventually nonnegative. An attempt is made to characterize $M_{\vee}$-matrices, by extending the results of Neumann and Plemmons for M-matrices. In particular, we characterize two different subclasses of $M_{\vee}$-matrices in terms of various types of generalized inverses.

1. Introduction

A real square M-matrix has the form $A = sI - B$ with entry-wise nonnegative matrix B and $s \geqslant \rho(B)$, the spectral radius of B. An extensive theory on the properties of nonnegative matrices and hence of M-matrices, has been developed due to their role in numerical analysis, modelling of the economy, optimization and Markov chains [1]. Ever since researchers are interested to generalize the class of M-matrices by generalizing the class of nonnegative matrices. For a general overview, we refer to [4, 5, 7, 11]. In this paper we consider a particular type of generalized M-matrices, known as $M_{\vee}$-matrices, which is obtained by generalizing nonnegative matrices to eventually nonnegative matrices. A matrix B is eventually nonnegative if there is a positive integer k_0 such that B^{k_0} is nonnegative and remains nonnegative afterwards. Eventually nonnegative matrices B with $\text{index}(B) \leqslant 1$, that is, the order of the largest Jordan block of B corresponding to the eigenvalue 0, is at most 1, play an important role in dynamical systems for qualitative information regarding state evaluation. In particular, this type of matrices arises in the linear differential systems $\dot{x}(t) = Ax(t)$, $A \in \mathbb{R}^{n,n}$, $x(0) = x_0 \in \mathbb{R}^n$, $t \geqslant 0$, whose solution become and remains nonnegative. For more detail, we refer [10].

A real square matrix A is called an $M_{\vee}$-matrix if it can be expressible as $A = sI - B$, where B is an eventually nonnegative matrix and $s \geqslant \rho(B)$. This class of matrices were first introduced in [11], and certainly generalizes the class of M-matrices. One of the well known properties of a nonsingular M-matrix is that it has a nonnegative inverse (see [6, 1]). In [4, 3], the authors generalized this inverse-nonnegativity property of M-matrices to characterize a subclass of nonsingular $M_{\vee}$-matrices, known as pseudo-M-matrices and another class of generalized M-matrices, known as GM-matrices. The purpose of this paper is to extend the inverse-nonnegativity property to

Mathematics subject classification (2020): 5A09, 15A18, 15B48.
Keywords and phrases: Eventually nonnegative, eventually positive, eventually monotone, Perron-Frobenius property, generalized inverse.

Paper OaM-15-77

a subclass of $M_\vee$-matrices and to characterize this subclass with the help of eventually positivity property of a generalized inverse. The generalized concepts of monotonicity and nonnegativity property of a matrix on a set are important in the characterization of singular M-matrices. These concepts are employed to characterize the $M_\vee$-matrices of the form $A = sI - B$ with $\text{index}(B) \leqslant 1$ after they are extended respectively, to eventually monotonic and eventually nonnegative property on a set S.

The paper is organized as follows: We begin with some basic notations and preliminary definitions in Section 2. In Section 3, we discuss various properties of $M_\vee$-matrices. In particular, we prove the existence of eventually positive generalized left-inverse for a special subclass of $M_\vee$-matrices. We further show that this property does not carry over to the entire class of $M_\vee$-matrices. Next we introduce the concepts of eventually monotonicity and eventually nonnegativity, which are used to characterize a subclass of $M_\vee$-matrices. One important subclass of M-matrices are M-matrices with 'property c', that is, M-matrices of the form $A = sI - B$ for which $\lim_{k\to\infty}(B/s)^k$ exists. In this case, the matrix $T = B/s$ is known as semiconvergent, and these matrices are considered as an important tool in investigating the convergent of iterative methods for singular systems. Finally, we consider analogous subclass of $M_\vee$-matrices to the subclass of M-matrices with 'property c' and characterize the subclass of $M_\vee$-matrices in terms of the nonnegativity property of various generalized inverses.

2. Preliminaries

This section contains basic notations and some preliminary definitions. We denote the set $\{1,2,\dots,n\}$ by $\langle n\rangle$. For a real $n\times m$ matrix $A = [a_{i,j}]$ we use the following terminologies and notations.

- $A \geqslant 0$ (A is nonnegative) if $a_{i,j} \geqslant 0$, for all $i \in \langle n\rangle$, $j \in \langle m\rangle$.
- $A > 0$ (A is strictly positive) if $a_{i,j} > 0$, for all $i \in \langle n\rangle$, $j \in \langle m\rangle$.
- $N(A)$, the nullspace of A, and by $n(A)$ the nullity of A.
- $\text{range}(A) = \{Ax \mid x \in \mathbb{R}^m\}$, the range of A.

If $n = m$, then we denote by

- $\sigma(A)$, the spectrum of A.
- $\rho(A) = \max_{\lambda\in\sigma(A)}\{|\lambda|\}$, the spectral radius of A.
- $\text{index}_\lambda(A)$, the order of the largest Jordan block associated with the eigenvalue λ, and we simply write $\text{index}_0(A)$ as $\text{index}(A)$.
- $V_A = \bigcap_{k=0}^{\infty} \text{range}(A^k)$.

We now provide the basic definitions related to $M_\vee$-matrices and various generalized inverses of a matrix.

DEFINITION 1. [6, 1] An $n \times n$ matrix A is called an *M-matrix* if it can be written as $A = sI - B$, where $B \geqslant 0$ and $s \geqslant \rho(B)$.

DEFINITION 2. [11, 4] A square matrix A is called an *eventually nonnegative (positive)* matrix if there is a positive integer n_0 such that $A^k \geqslant 0$ $(A^k > 0)$ for all $k \geqslant n_0$.

DEFINITION 3. [11] A square matrix A is called an *$M_\vee$-matrix* if it can be expressed as $A = sI - B$ with eventually nonnegative matrix B and $s \geqslant \rho(B)$.

Various types of generalized inverses have been defined and studied by several authors. The important classes of generalized inverses for our purpose are those that leave the subspace V_A invariant.

DEFINITION 4. [1] Let $A \in \mathbb{R}^{n,n}$ with $m = \text{index}(A)$. Then each $Y \in \mathbb{R}^{n,n}$ satisfying the condition,

$$YAx = x \text{ for all } x \in V_A \text{ with } V_A = \bigcap_{k=0}^{\infty} \text{range}(A^k) = \text{range}(A^m)$$

is called a *generalized left inverse of* A. Similarly, each $Z \in \mathbb{R}^{n,n}$ satisfying the condition,

$$x^T AZ = x^T \text{ for all } x \in V_A$$

is called a *generalized right inverse of* A.

Note that if Y is a generalized left inverse of A, then Y leaves V_A invariant, because any $v \in V_A$ can be written as $v = A^{m+1}u$ for some u and hence $Yv = YA(A^m u) = A^m u \in V_A$.

Some equivalent definitions of generalized left inverses are given in the following lemma.

LEMMA 1. [1, 8] *Let $A \in \mathbb{R}^{n,n}$. Then the following statements are equivalent for $Y \in \mathbb{R}^{n,n}$:*

(i) *Y is a generalized left inverse of A.*

(ii) $YA^{m+1} = A^m$, *where* $m = \text{index}(A)$.

(iii) $YA^{k+1} = A^k$, *where* $k \geqslant \text{index}(A)$.

(iv) $YA^{k+1} = A^k$, *for some* $k \geqslant 0$.

Similar characterizations can also be given for generalized right inverses.

DEFINITION 5. [1] Let $A, Y \in \mathbb{R}^{n,n}$. Consider the following conditions:

(1) $AYA = A$.

(2) $YAY = Y$.

(3) $AY = (AY)^T$.

(4) $YA = (YA)^T$.

(5) $AY = YA$.

(6) $YA^{m+1} = A^m, \qquad m = \text{index}(A)$.

Let λ be any subset of $\{1,2,3,4\}$ containing 1. Then a λ*-inverse* of A is a matrix Y which satisfies the condition (i) for each $i \in \lambda$. The *Drazin inverse* of A is a matrix Y which satisfies the conditions $(2),(5)$ and (6), hence it is a generalized left inverse.

3. Characterizations of some subclasses of $M_\vee$ matrices

In this section, we discuss various properties of $M_\vee$-matrices of the form $A = sI - B$ with $\text{index}(B) \leqslant 1$, with respect to generalized left inverse and in terms of eventually monotonicity and eventually nonnegativity property on the set V_A. Lastly, we generalize the concept of 'c-property' of M-matrices to $M_\vee$-matrices and provide characterizations for this subclass of $M_\vee$-matrices.

In [4], the authors characterized nonsingular pseudo-M-matrices in terms of inverse-eventually positivity. In the next theorem we extend the inverse-eventually positivity property to a subclass of $M_\vee$-matrices. More specifically, we provide the existence of an eventually positive generalized left inverse for a subclass of $M_\vee$-matrices.

THEOREM 1. [4] *Suppose that* $B \in \mathbb{R}^{n,n}$ *is an eventually nonnegative matrix with* $\text{index}(B) \leqslant 1$. *Then, there are positive right and left eigenvectors corresponding to* $\rho(B)$ *if and only if* B *is permutationally similar to a direct sum of irreducible matrices having the same spectral radius.*

THEOREM 2. *If* $A = sI - B$ *is an* $M_\vee$*-matrix where* B *is an irreducible eventually nonnegative matrix with* $\text{index}(B) \leqslant 1$, *then there always exists an eventually positive generalized left inverse of* A.

Proof. Let $A = XJX^{-1}$ be the Jordan canonical form of A. As $\text{index}(B) \leqslant 1$ and B is irreducible eventually nonnegative matrix, by Theorem 1 there exist positive vectors x and y such that $Ax = (s-\rho)x$ and $y^TA = (s-\rho)y^T$, where $\rho = \rho(B)$. Without loss of generality we may assume that $X = \left[x,\, X^{(1)}\right]$. Then

$$A = \left[x,\, X^{(1)}\right] \begin{bmatrix} s-\rho & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} y^T \\ Y^{(1)} \end{bmatrix},$$

where $D \in \mathbb{R}^{n-1,n-1}$ is the nonsingular part of the Jordan canonical form J of A.

Case-I. Let A be singular, that is, $s=\rho(B)$. Choose a large positive number α such that $\alpha > \frac{1}{|\lambda|}$ for all $\lambda(\neq 0)\in\sigma(A)$. Consider the matrix

$$Y_1=\begin{bmatrix}\alpha & 0\\ 0 & D^{-1}\end{bmatrix}.$$

Take $Y=XY_1X^{-1}$, so for any positive integer k, $Y^k=XY_1^kX^{-1}$. Then

$$\frac{1}{\alpha^k}Y^k=\left[x,\,X^{(1)}\right]\begin{bmatrix}1 & 0\\ 0 & \widetilde{D}(k)\end{bmatrix}\begin{bmatrix}y^T\\ Y^{(1)}\end{bmatrix},$$

where $\tilde{D}(k)=\frac{1}{\alpha^k}\left(D^{-1}\right)^k$ and any eigenvalue $\lambda(k)$ of $\widetilde{D}(k)$ is of absolute value less than 1. Hence it follows that $\lim\limits_{k\to\infty}\tilde{D}(k)=0$ and,

$$\lim_{k\to\infty}\frac{1}{\alpha^k}Y^k=xy^T>0.$$

This shows that there exists a positive integer k_0 such that $Y^k>0$ for all $k\geqslant k_0$, that is, Y is an eventually positive matrix. We now show that Y is a generalized left inverse of A. Let $m=\operatorname{index}(A)$. Then,

$$Y_1J^{m+1}=\begin{bmatrix}\alpha & 0\\ 0 & D^{-1}\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & D^{m+1}\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & D^{m}\end{bmatrix}=J^m.$$

Thus $XY_1X^{-1}XJ^{m+1}X^{-1}=XJ^mX^{-1}$, or, $YA^{m+1}=A^m$.

Case-II. Let $s>\rho(B)=\rho\,(\text{say})$. Set $\alpha=\frac{1}{s-\rho}$ and take Y as defined in Case-I. Note that ρ is simple, being B is irreducible, and if the eigenvalues μ_i of B are arranged as $\mu_1=\rho>|\mu_2|\geqslant\ldots|\mu_n|$, then eigenvalues of $\widetilde{D}(k)$ are

$$\lambda_i(k)=\left(\frac{s-\rho}{s-\mu_i}\right)^k \quad\text{for}\quad i=2,\ldots n$$

Furthermore, for $i=2,\ldots,n$, $\rho>|\mu_i|\geqslant\operatorname{Re}(\mu_i)$ and hence

$$\left|\frac{s-\rho}{s-\mu_i}\right|^2=\frac{(s-\rho)^2}{(s-\operatorname{Re}(\mu_i))^2+\operatorname{Im}(\mu_i)^2}<\frac{(s-\operatorname{Re}(\mu_i))^2}{(s-\operatorname{Re}(\mu_i))^2+\operatorname{Im}(\mu_i)^2}\leqslant 1$$

So $\lim\limits_{k\to\infty}\widetilde{D}(k)=0$ and hence as shown in the previous case, it can be verified that Y is a generalized left inverse of A, in fact Y is the inverse of A and Y is eventually positive. $\square$

Following theorem is a general case of Theorem 2, which covers the eventually nonnegative matrices in Theorem 1.

THEOREM 3. *Let $A = sI - B$ is a $M_\vee$-matrix with* $\operatorname{index}(B) \leqslant 1$ *and B is permutationally similar to a direct sum of irreducible matrices having the same spectral radius. Then A has an eventually positive generalized left inverse.*

Proof. As A is an $M_\vee$-matrix with $\operatorname{index}(B) \leqslant 1$, any permutational similar matrix of A must have the same properties. So we may assume that B is direct sum of irreducible matrices having the same spectral radius $\rho(B) = \rho$ (say). Write $B = B_1 \oplus B_2 \oplus \ldots \oplus B_m$ with the order of B_i is k_i, each B_i is irreducible and $\rho(B_i) = \rho$. Then for $i = 1, 2, \ldots, m$, the $k_i \times k_i$ matrix $A_i = sI - B_i$ is an $M_\vee$-matrix and each B_i is irreducible eventually nonnegative matrix with $\operatorname{index}(B_i) \leqslant 1$. Hence by Theorem 2, for each i, there exists eventually positive generalized left inverse Y_i of B_i. Choose k_0 such that $Y_i^k \geqslant 0$, for $k \geqslant k_0$ and for all $i = 1, 2, \ldots, m$. Set $Y = Y_1 \oplus Y_2 \oplus \ldots \oplus Y_m$. Then it can be easily checked from Lemma 1 that Y is a generalized left inverse of A and $Y^k \geqslant 0$ for all $k \geqslant k_0$. □

Note that from the above theorem we cannot conclude that every generalized left inverse of A is eventually positive. The following example illustrates the fact.

EXAMPLE 1. Consider the matrix

$$A = 8I - B = 8I - \begin{bmatrix} 3 & 2 & 3 \\ 3 & 6 & -1 \\ -1 & 2 & 7 \end{bmatrix}.$$

Then $B^k > 0$ for all $k \geqslant 3$ and so A is an $M_\vee$-matrix satisfying the conditions of the previous theorem. Let $A = XJX^{-1}$ be the Jordan canonical form of A where

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } X = \begin{bmatrix} 0.25 & 2 & 0.75 \\ 0.25 & -2 & -0.25 \\ 0.25 & 2 & -0.25 \end{bmatrix}. \tag{3.1}$$

Consider the generalized left (Drazin) inverse,

$$Y = X\widetilde{J}X^{-1} = X \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.25 & -0.0625 \\ 0 & 0 & 0.25 \end{bmatrix} X^{-1} = \begin{bmatrix} 0.0625 & -0.125 & 0.0625 \\ 0.0625 & 0.125 & -0.1875 \\ -0.1875 & -0.125 & 0.3125 \end{bmatrix}.$$

Then for any positive integer k, $Y^k = X\widetilde{J}^k X^{-1}$. By using induction on k, we can check that

$$\widetilde{J}^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4^k} & -\frac{k}{4^{k+1}} \\ 0 & 0 & \frac{1}{4^k} \end{bmatrix},$$

which implies

$$Y^k = X\widetilde{J}^k X^{-1} = \begin{bmatrix} \frac{1}{4} & 2 & \frac{3}{4} \\ \frac{1}{4} & -2 & -\frac{1}{4} \\ \frac{1}{4} & 2 & -\frac{1}{4} \end{bmatrix} \widetilde{J}^k X^{-1} = \begin{bmatrix} 0 & \frac{2}{4^k} & \frac{3-2k}{4^{k+1}} \\ 0 & -\frac{2}{4^k} & \frac{2k-1}{4^{k+1}} \\ 0 & \frac{2}{4^k} & -\frac{2k+1}{4^{k+1}} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 1 & 0 & -1 \end{bmatrix}.$$

This shows that for any positive integer k, the $(1,2)$-entry of Y^k is always negative. Hence Y is not an eventually positive matrix.

In Example 1, we observe that the Drazin inverse of A is not eventually positive. Next example illustrates Theorem 3, that is, the existence of eventually positive generalized left inverse of A in Example 1.

EXAMPLE 2. Consider the matrix A in Example 1.
Take $\widetilde{J} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0.25 & -0.0625 \\ 0 & 0 & 0.25 \end{bmatrix}$ and set $Y = X\widetilde{J}X^{-1}$, where X is defined by the equation (3.1), so that

$$Y = \begin{bmatrix} 2.5625 & 4.8750 & 2.5625 \\ 2.5625 & 5.1250 & 2.3125 \\ 2.3125 & 4.8750 & 2.8125 \end{bmatrix} > 0$$

Then $YA^2 = A$ and hence Y is the desired (eventually) positive generalized left inverse of A.

The following example shows that in Theorem 2 and 3, the condition $\operatorname{index}(B) \leqslant 1$ cannot be relaxed.

EXAMPLE 3. Consider the $M_{\vee}$-matrix $A = 2I - B$, where

$$B = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}.$$

Note that B is an irreducible eventually nonnegative matrix and $\operatorname{index}(B) = 2$. Let Y be any generalized left inverse of A. As $\operatorname{index}(A) = 1$, by Lemma 1, Y must satisfy the condition $YA^2 = A$, which implies that Y has the form

$$Y = \begin{bmatrix} \frac{1}{2} & 0 & a & a \\ 0 & \frac{1}{2} & b & b \\ \frac{1}{4} & \frac{1}{4} & c+\frac{1}{2} & c \\ -\frac{1}{4} & -\frac{1}{4} & d & d+\frac{1}{2} \end{bmatrix}$$

where a, b, c, d are some constants. Consider the following matrices

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \text{ and } E = Y - \frac{1}{4}F.$$

Note that $EF = \frac{1}{2}F$, $E^kF = \frac{1}{2^k}F$ (for any $k \in \mathbb{N}$), $FE = \frac{1}{2}F + (a+b)G$, $GE = (c + d + \frac{1}{2})G$ and $F^2 = 0 = GF$. We now show by induction that $Y^k = E^k + \alpha F + \beta G$, for some scalar α,β with $\alpha > 0$. For $k=1$, it is trivial. Assume that $Y^k = E^k + \alpha F + \beta G$. Now,

$$\begin{aligned} Y^{k+1} = Y^k.Y &= (E^k + \alpha F + \beta G)\left(E + \frac{1}{4}F\right) \\ &= E^{k+1} + \alpha FE + \beta GE + \frac{1}{4}E^kF + \frac{\alpha}{4}F^2 + \frac{\beta}{4}GF \\ &= E^{k+1} + \left(\frac{\alpha}{2} + \frac{1}{2^{k+1}}\right)F + \left(\alpha(a+b) + \beta\left(c+d+\frac{1}{2}\right)\right)G \\ &= E^{k+1} + \alpha'(k)F + \beta' G \end{aligned} \tag{3.2}$$

where $\alpha'(k) = \frac{\alpha}{2} + \frac{1}{2^{k+1}} > 0$ and $\beta' = \alpha(a+b) + \beta(c+d+\frac{1}{2})$.

As $Y^k = E^k + \alpha F + \beta G$ and $\alpha > 0$, so for any positive integer k, the $(4,1)$-entry and $(4,2)$-entry of Y^k are always negative. Hence Y is not an eventually nonnegative matrix.

The following theorem due to Neumann and Plemmons in [8] gives characterizations of M-matrices in terms of the monotonicity property of a matrix on a particular set and the nonnegativity property of its generalized left inverses.

THEOREM 4. [8] *Let $A = sI - B$ where $B \geqslant 0$ and $s > 0$. Then the following statements are equivalent:*

(i) *A is an M-matrix.*

(ii) *A has a nonnegative generalized left inverse Y.*

(iii) *A has a generalized left inverse Y, which is nonnegative on V_A, that is, $x \geqslant 0$ and $x \in V_A \Rightarrow Yx \geqslant 0$.*

(iv) *Every generalized left inverse is nonnegative on V_A.*

(v) *A is monotone on V_A, that is, $Ax \geqslant 0$ and $x \in V_A \Rightarrow x \geqslant 0$.*

Motivated by the above characterizations of M-matrices, we now introduce some new definitions which are generalizations of nonnegativity [Theorem 4(iii)] and monotonicity [Theorem 4(iv)] of a matrix on a subset of $\mathbb{R}^n$. These generalizations give some interesting characterizations of a subclass of $M_\vee$-matrices.

DEFINITION 6. Let $A \in \mathbb{R}^{n,n}$ and $S \subseteq \mathbb{R}^n$. We say that A is *eventually nonnegative on S*, if $x \in S$ and $x \geqslant 0$ imply that there exists a positive integer k_0, such that $A^k x \geqslant 0$, for all $k \geqslant k_0$.

REMARK 1. Note that if A is an eventually nonnegative matrix such that $A^k \geqslant 0$ for all $k \geqslant g$, then we can choose $k_0 = g$.

DEFINITION 7. Let $A \in \mathbb{R}^{n,n}$ and $S \subseteq \mathbb{R}^n$. Then we say that A is *eventually monotone on* S, if there exists a positive integer k_0, such that for any $x \in S$, $A^k x \geqslant 0$, for all $k \geqslant k_0$, implies $x \geqslant 0$.

The following is an example of a matrix which is eventually monotone on a subspace S of $\mathbb{R}^2$.

EXAMPLE 4. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}.$$

Take $S = \mathbb{R}^2$. Let $x \in S$ and there exists k_0 such that $A^k x \geqslant 0$ for all $k \geqslant k_0$ so that

$$\begin{bmatrix} 1 & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geqslant 0 \text{ for all } k \geqslant k_0,$$

which implies that $x_2 = 0$ and $x_1 \geqslant 0$, hence $x \geqslant 0$ on S. Thus the above matrix is eventually monotone on S.

Next theorem provides some characterizations of the subclass of $M_\vee$-matrices of the form $A = sI - B$ with $\text{index}(B) \leqslant 1$, in terms of eventually monotonicity and eventually nonnegativity property on V_A.

THEOREM 5. *Let $A = sI - B$ where B is an irreducible eventually nonnegative matrix with* $\text{index}(B) \leqslant 1$. *Then the following statements are equivalent:*

(i) *A is an $M_\vee$-matrix.*

(ii) *A has a generalized left inverse Y, which is eventually positive.*

(iii) *A has a generalized left inverse Y, which is eventually nonnegative on V_A.*

(iv) *Every generalized left inverse is eventually nonnegative on V_A.*

(v) *A is eventually monotone on V_A.*

Proof.

(i) $\Rightarrow$ (ii): Follows from Theorem 2.

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Leftrightarrow$ (iv): Assume that (iii) holds, that is, Y is a generalized inverse of A, which is eventually nonnegative on V_A. Let Z be any generalized inverse of A, and let $x \in V_A$ and $x \geqslant 0$. By our assumption there exists an integer k_0 such that $Y^k x \geqslant 0$ for all $k \geqslant k_0$. Choose an integer t such that $t \geqslant \max\{k_0, m\}$, where $m = \text{index}(A)$. Let $k \geqslant t$. Then x can be written as $x = A^{m+k} z$, for some z. Now

$$A^m z = Y^k A^{m+k} z = Y^k x \geqslant 0.$$

So $Z^k x = Z^k A^{m+k} z = A^m z \geqslant 0$. Thus $Z^k x \geqslant 0$ for all $k \geqslant t$. Hence Z is eventually nonnegative on V_A.

Converse part is obvious.

(iv) $\Leftrightarrow$ (v)**:** Let Y be a generalized left inverse of A such that if $x \in V_A$ and $x \geqslant 0$, then there exists a $\tilde{k}$ such that $Y^k x \geqslant 0$ for all $k \geqslant \tilde{k}$. To show that A is eventually monotone on V_A.

Let $x \in V_A$ and k_1 be a positive integer such that $A^k x \geqslant 0$ for all $k \geqslant k_1$ and let $m = \text{index}(A)$. Choose k_2 such that $k_2 \geqslant \max\{k_1, m\}$. Since $x \in V_A = R(A^m)$, there exists a z such that $x = A^m z$. Thus for any $k \geqslant k_2$, $A^k x = A^{m+k} z \in V_A$ and $A^k x \geqslant 0$. Again by assumption (iv), there exists a k_3 such that $Y^s A^k x \geqslant 0$ for all $s \geqslant k_3$ and for all $k \geqslant k_2$. Choose $k_0 \geqslant \max\{k_2, k_3\}$. Then for any $k \geqslant k_0$, $Y^k A^k x \geqslant 0$, i.e., $Y^k A^{m+k} z \geqslant 0$. Thus $YA^{k+1} = A^k$ for all $k \geqslant m$ implies that $YA^{m+1} z \geqslant 0$, or, $x = A^m z = YA^{m+1} z \geqslant 0$.

Conversely let A be eventually monotone on V_A, i.e., if $x \in V_A$ and there exists a k_0 such that $A^k x \geqslant 0$, for all $k \geqslant k_0$, then $x \geqslant 0$. Let Y be any generalized inverse of A. To show that Y is eventually nonnegative on V_A.

Let $y \in V_A$, $y \geqslant 0$ and $y = A^m x$ for some x. Write x as $x = u + v$, for some $u \in V_A$ and $A^m v = 0$. For any $k \geqslant m$ we have that $A^k v = 0$, and $u = A^k w$. Now

$$A^m x = A^m u = A^{m+k} w = A^k (A^m w) \geqslant 0.$$

Hence by our assumption $A^m w \geqslant 0$. Thus $Y^k y = Y^k A^m x = Y^k A^{m+k} w = A^m w \geqslant 0$. This shows that Y is eventually monotone on V_A.

(iii) $\Rightarrow$ (i)**:** Let Y be a generalized left inverse of A which is eventually nonnegative on V_A. To show that $s \geqslant \rho(B) = \rho$, and we let $s \neq \rho$.

Choose a nonnegative vector x such that $Bx = \rho x$ and thus for any positive integer k, $A^k x = (s-\rho)^k x$ and hence $x \in V_A$. So, by our assumption there exists a k_0 such that $Y^k x \geqslant 0$ for all $k \geqslant k_0$. Take $\tilde{k} = \max\{k_0, m\}$, where $m = \text{index}(A)$. Then for any $k \geqslant \tilde{k}$,

$$\begin{aligned}(s-\rho)^{m+k} x &= A^{m+k} x \\ &= YA^{m+k+1} x \\ &= Y^k A^{m+2k} x \\ &= (s-\rho)^{m+2k} Y^k x\end{aligned}.$$

Thus $(s-\rho)^k x = (s-\rho)^{2k} Y^k x$ for all $k \geqslant \tilde{k}$. As x and $Y^k x$ with $k \geqslant \tilde{k}$ are all nonnegative vectors, so we must have $s > \rho$. Thus $s \geqslant \rho$ and hence A is an $M_\vee$-matrix. □

REMARK 2. In Example 1, we have seen that the Drazin inverse Y is not eventually nonnegative. But we will verify that Y is eventually nonnegative on V_A. Note that

$m=1$ so that $V_A=\text{range}(A)$. It can be easily verified that $\{x : Ax \geqslant 0\} = \{x : x_1 = x_2 = x_3\} = \{x : Ax = 0\}$. Thus $V_A \cap \left(\mathbb{R}^3\right)^+ = \{0\}$. Thus Y is eventually nonnegative on V_A.

LEMMA 2. *Let A be any real square matrix of order n. The we have*

(a) *If Y is a $\{1\}$-inverse of A with* $\text{range}(YA)=\text{range}(A)$*, then*

  (i) $YAx = x$*, for all* $x \in \text{range}(A)$.

  (ii) $YA^{k+1} = A^k$*, for all* $k \geqslant 1$*. In particular,* $\text{index}(A) \leqslant 1$ *and Y is a generalized left inverse of A.*

(b) *If Z is a $\{1\}$-inverse of A with* $\text{range}(Z^TA^T)=\text{range}(A^T)$*, then*

  (i) $x^TAZ = x^T$*, for all* $x \in \text{range}(A)$.

  (ii) $A^{k+1}Z = A^k$*, for all* $k \geqslant 1$*. In particular,* $\text{index}(A) \leqslant 1$ *and Z is a generalized right inverse of A.*

Proof. We prove Part (a)

(i) Since range(YA) = range(A), so any $x \in \text{range}(A)$ can be written as $x = YAz$, for some z and hence $YAx = YAYAz = YAz = x$.

(ii) We prove it by induction on k. Let $k=1$ and $x \in \mathbb{R}^n$. Then $Ax \in \text{range}(A)$ and hence by the given hypothesis there exists a z such that $Ax = YAz$, which implies that $YA^2x = YAYAz = YAz = Ax$. Thus $YA^2 = A$. Now suppose that $k>1$ and $YA^{t+1} = A^t$, for all $t<k$. Then $YA^{k+1} = YA^k \cdot A = A^{k-1} \cdot A = A^k$.

   Suppose that, $m = \text{index}(A) > 1$. Then there exists an x, such that $x \in \text{range}(A)$ and $x \notin \text{range}(A^2)$. Hence $x = Ay$ for some $y \in \mathbb{R}^n$. Take $y = u+v$ with $u \in \text{range}(A^m)$ and $A^m v = 0$. So, $A^m y = A^m u$, or, $YA^m y = YA^m u$ and since $m>1$, so $A^{m-1}y = A^{m-1}u$. Repeating this process up to $(m-1)$ steps, we get $x = Ay = Au$, hence $x \in \text{range}(A^{m+1})$ and $m>1$ imply that $x \in \text{range}(A^2)$, a contradiction. Thus $\text{index}(A) \leqslant 1$.

(b) Proof is similar to that of Part (a) □

LEMMA 3. *Let A be any matrix with* $\text{index}(A) \leqslant 1$*. Then Y is a generalized left inverse of A if and only if Y is a $\{1\}$-inverse of A with* $\text{range}(YA) = \text{range}(Y)$.

Proof. If $\text{index}(A) < 1$, that is, A is nonsingular, then the result is obviously true, hence assume that $\text{index}(A) = 1$. The 'if' part follows from Lemma 2. Now for the 'only if part', let us assume that Y is a generalized left inverse of A. Any $x \in \mathbb{R}^n$ can be written as $x = u+v$ with $u \in \text{range}(A)$ and $Av = 0$. Then $Ax = Au$. Since Y is a left inverse and $\text{index}(A) = 1$, so we have $YAu = u$ and $AYAx = AYAu = Au = Ax$ and hence $AYA = A$.

Next if $x \in \text{range}(YA)$, then as in the earlier case, x can be written as $x = YAy$ for some $y \in \text{range}(A)$. Since Y is a left inverse, $x = y \in \text{range}(A)$. Conversely if $x \in \text{range}(A)$, then $x = YAx$ and so $x \in \text{range}(YA)$. Hence $\text{range}(YA) = \text{range}(A)$. □

An important subclass of M-matrices is the set of M-matrices with Property c, that is, matrices of the form $A = sI - B$, with $B \geqslant 0$, $s \geqslant \rho(B)$ such that $\lim_{k\to\infty}(B/s)^k$ exists. The following result from [8], gives characterizations of M-matrices with 'property c' in terms of some special types of generalized inverses.

THEOREM 6. [8] *Let $A = sI - B$ where $B \geqslant 0$ and $s > 0$. The following statements are equivalent:*

(i) *A is an M-matrix with 'property c'.*

(ii) *A has a $\{1\}$-inverse Y which is a nonnegative matrix and* $\text{range}(YA) = S$.

(iii) *A has a $\{1\}$-inverse Y with* $\text{range}(YA) = S$, *such that Y is nonnegative on S.*

(iv) *A has a $\{1,2\}$-inverse Z with* $\text{range}(Z) = S$, *such that Z is nonnegative on S.*

(v) *A is monotone on S.*

In Chapter 6 of [1], the authors proved that an M-matrix A has 'property c' if and only if $\text{index}(A) \leqslant 1$. In the next theorem we consider a similar subclass of $M_\vee$-matrices, that is, $M_\vee$-matrices A with $\text{index}(A) \leqslant 1$ and give analogous characterizations as described in Theorem 6 for the mentioned subclass of $M_\vee$-matrices.

THEOREM 7. *Let $A = sI - B$ where B is an eventually positive matrix with* $\text{index}(B) \leqslant 1$. *Then for $S = \text{range}(A)$, the following statements are equivalent:*

(i) *A is an $M_\vee$-matrix with* $\text{index}(A) \leqslant 1$.

(ii) *A has a $\{1\}$-inverse Y which is an eventually nonnegative matrix and* $\text{range}(YA) = S$.

(iii) *A has a $\{1\}$-inverse Y with* $\text{range}(YA) = S$, *such that Y is eventually nonnegative on S.*

(iv) *Every $\{1\}$-inverse Y of A with* $\text{range}(YA) = S$, *is eventually nonnegative on S*

(v) *A has a $\{1,2\}$-inverse Z with* $\text{range}(Z) = S$, *such that Z is eventually nonnegative on S.*

(vi) *A is eventually monotone on S.*

Proof. From Theorem 5 and Lemma 3, it follows that if $\text{index}(A) \leqslant 1$, then conditions (ii), (iii), (iv), (vi) are equivalent to the statement that "A is an $M_\vee$-matrix". Thus we have (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (vi). To complete the proof it is enough to show (iii) $\Rightarrow$ (v) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (v): Let Y be a $\{1\}$-inverse of A such that $\text{range}(YA) = \text{range}(A)$ and Y is eventually nonnegative on $\text{range}(A)$. Take $Z = YAY$. Then it can be easily checked that Z is a $\{1,2\}$-inverse of A. Since $Z = YAY$ and $\text{range}(YA) = \text{range}(A)$, so $\text{range}(Z) \subseteq \text{range}(A)$. Again if $x \in \text{range}(A)$, then by Lemma 2 (i), $x = YAx = ZAx$ and hence $\text{range}(Z) = \text{range}(A)$. In order to show that Z is eventually nonnegative on $\text{range}(A)$, it suffices to show that $Z^k x = Y^k x$ for all $x \in \text{range}(A)$, and for all positive integer k.

Let $x = Au$ for some $u \in \mathbb{R}^n$, then $Zx = YAYx = YAYAu = YAu = Yx$. Now assume that $k > 1$, and $Z^t x = Y^t x$, for all $x \in \text{range}(A)$ and for all $t < k$. Then $Z^k x = Z^{k-1}(Zx) = Z^{k-1}(Yx) = Y^{k-1}(Yx) = Y^k x$ and by induction on k, $Z^k x = Y^k x$ for all positive integer k.

(v) $\Rightarrow$ (i): Suppose that Z is a $\{1,2\}$-inverse of A such that $\text{range}(Z) = \text{range}(A)$ and Z is eventually nonnegative on $\text{range}(A)$. Then Z is a $\{1\}$-inverse implies that $\text{range}(ZA) = \text{range}(A)$ and hence by Lemma 2, $\text{index}(A) \leqslant 1$ and Z is a generalized left inverse of A. Hence the generalized left inverse Z is eventually nonnegative on $V_A = \text{range}(A)$ and (i) follows from Theorem 5.

Thus conditions (i) – (vi) are equivalent. □

REMARK 3. Similar results can be obtained for generalized right inverses Z, with $S = \text{range}(A^T)$ and $\text{range}(YA)$ replaced by $\text{range}(Z^T A^T)$ in the above statements.

4. Conclusion

The paper characterizes two different subclasses of $M_{\vee}$-matrices by extending results obtained by Neumann and Plemmons in [8] for M-matrices, to some subclasses of $M_{\vee}$-matrices. We characterized a subclass of $M_{\vee}$-matrices in terms of eventually positivity of generalized inverses. We also generalized the concepts of monotonicity and nonnegativity property on a set S and termed them as eventually monotonicity and eventually nonnegativity on S, respectively. We used these concepts to characterize another subclass of $M_{\vee}$-matrices.

Acknowledgement. The author would like to thank Dr. Sriparna Bandhopadhay, IIT Guwahati, for providing insight and expertise that greatly assisted the research. The author is grateful to the anonymous reviewers for their positive criticism and valuable suggestions that improved the quality of the paper.

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(Received July 18, 2020)

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