Operators
and
Matrices
Volume 15, Number 4 (2021), 1257–1266
doi:10.7153/oam-2021-15-79

WEIGHTED OPERATOR–VALUED FUNCTION SPACES APPLIED TO THE STABILITY OF DELAY SYSTEMS

ASMAHAN E. ALAJYAN AND JONATHAN R. PARTINGTON*

(*Communicated by J. Behrndt*)

Abstract. This paper extends the theory of Zen spaces (weighted Hardy/Bergman spaces on the right-hand half-plane) to the Hilbert-space valued case, and describes the multipliers on them; it is shown that the methods of H^∞ control can therefore be extended to a family of weighted L^2 input and output spaces. Next, the particular case of retarded delay systems with operator-valued transfer functions is analysed, and the dependence of H^∞ structure on the delay is determined by developing an extension of the Walton–Marshall technique used in the scalar case. The method is illustrated with examples.

1. Introduction

There is an extensive literature on the use of the H^∞ norm of an analytic (operator-valued) function on the right-hand half-plane $\mathbb{C}_+$, which describes the gain of a linear time-invariant system from (vector-valued) $L^2(0,\infty)$ inputs to $L^2(0,\infty)$ outputs; we mention here some well-known books on the subject, namely, [6, 18, 16]. Some new contributions to the theory will be presented here. There are two themes: first we extend the recent theory of Zen spaces [7, 8, 11] to functions taking values in a Hilbert space H, where the Laplace transform $\mathscr{L}$ provides an isometric embedding from a weighted function space $L^2(0,\infty,w(t)dt;H)$ into a space $A^2_\nu(\mathbb{C}_+,H)$ of analytic operator-valued functions (all notation will be defined below). In this context we prove a theorem showing that the H^∞ norm can be used to measure the gain (operator norm) of a linear system defined in the context of a wide variety of weighted L^2 spaces; thus we show that various notions of stability are equivalent.

Second, we apply the analysis to one particular case, namely the H^∞ stability of retarded delay systems with transfer functions of the form

$$G(s) = (P(s)I + Q(s)e^{-sh}A)^{-1}, \tag{1}$$

where P and Q are real polynomials and A is a bounded operator on a Banach space X. These arise from delay-differential equations of the form

$$\sum_{j=0}^{n} a_j \frac{\partial^j x(t)}{\partial t^j} + A\sum_{k=0}^{n} b_k \frac{\partial^k x(t-h)}{\partial t^k} = u(t), \qquad x(0)=0,$$

Mathematics subject classification (2020): 30H10, 30H20, 44A10, 46E15, 47N70, 93B36.
Keywords and phrases: Hardy space, Bergman space, Zen space, Plancherel theorem, H^∞ control, retarded delay system, stability, subnormal operator.

* Corresponding author.

where the a_j and b_k are scalars, by taking Laplace transforms of each side, so that $\mathscr{L}x(s) = G(s)\mathscr{L}u(s)$ where G has the form given in (1). We refer to [2, 13] for further details and other ways of formulating such systems by differential equations.

The example that will illustrate most of our results arises from the delay-differential equation

$$\dot{x}(t) + Ax(t-h) = u(t), \quad x(0) = 0, \tag{2}$$

and similar equations, with $x(t) \in X, u(t) \in U$ (where X and U are Banach spaces) and h is the delay. In this case the operator-valued transfer function is $(sI + e^{-sh}A)^{-1}$ and L^2-to-L^2 stability holds if and only if the operator-valued function is bounded in the right-hand half-plane $\mathbb{C}_+$. The Walton–Marshall method [17, 13] gives such an analysis in the purely scalar case $A = a$, say: the method involves increasing h to see where the zeros of $P(s) + aQ(s)e^{-sh}$ cross the imaginary axis. Additionally, at such points the direction in which the zeros cross the axis can be identified. Here we develop the Walton–Marshall method further to apply it to bounded operators.

We recall that a classification of delay systems into retarded, neutral and advanced type can be found in [2]. We shall show that, even in the operatorial case, for systems of retarded type ($\deg P > \deg Q$) invertibility of $P(s)I + Q(s)e^{-sh}A$ is equivalent to the inverse function being in H^∞: this is true for retarded systems, but not for systems of neutral type ($\deg P = \deg Q$). Systems of advanced type ($\deg P < \deg Q$) are never stable.

We write H^∞ for the Hardy space of bounded analytic functions on the right-hand half-plane $\mathbb{C}_+$, $\mathrm{L}(U,X)$ for the bounded operators from U to X, and $H^\infty(\mathbb{C}_+, \mathrm{L}(U,X))$ or simply $H^\infty(\mathrm{L}(U,X))$ for the space of bounded $\mathrm{L}(U,X)$-valued functions, with norm

$$\|F\| = \sup_{s \in \mathbb{C}_+} \|F(s)\|.$$

We shall mostly be able to take $U = X$, and then we write $\mathrm{L}(X)$ for $\mathrm{L}(X,X)$.

2. Stability on weighted L^2 spaces

In this section we show that H^∞ methods can be applied to stability questions in a wide variety of weighted $L^2(0,\infty)$ spaces.

Let $w(t)$ be a positive measurable function. Then for a separable Hilbert space H we write $L^2(0,\infty,w(t)dt,H)$ for the space of measurable H-valued functions f such that the norm $\|f\|$, given by

$$\|f\|^2 = \int_0^\infty \|f(t)\|^2 w(t)\, dt,$$

is finite. We start by giving conditions on w for the Laplace transform to induce an isometry between $L^2(0,\infty,w(t)dt,H)$ and a space of H-valued analytic functions on $\mathbb{C}_+$.

Let ν be a positive regular Borel measure satisfying the doubling condition

$$R := \sup_{t>0} \frac{\nu[0,2t)}{\nu[0,t)} < \infty.$$

The Zen space $A^2_\nu(H)$ is defined to consist of all analytic H-valued functions F on $\mathbb{C}_+$ such that the norm, given by

$$\|F\|^2 = \sup_{\varepsilon>0} \int_{\mathbb{C}_+} \|F(s+\varepsilon)\|^2 \, d\nu(x)\, dy$$

is finite, where we write $s = x+iy$ for $x \geqslant 0$ and $y \in \mathbb{R}$.

The best-known examples here are:

1. For $\nu = \delta_0$, a Dirac mass at 0, we obtain the Hardy space $H^2(\mathbb{C}_+, H)$;
2. For ν equal to Lebesgue measure (dx), we obtain the Bergman space $A^2(\mathbb{C}_+, H)$.

Often we shall have $\nu\{0\} = 0$, in which case $\|F\|^2$ can be written simply as

$$\int_{\mathbb{C}_+} \|F(s)\|^2 \, d\nu(x)\, dy.$$

THEOREM 1. *Suppose that w is given as a weighted Laplace transform*

$$w(t) = 2\pi \int_0^\infty e^{-2rt} \, d\nu(r), \qquad (t>0). \tag{3}$$

Then the Laplace transform provides an isometric map

$$\mathscr{L} : L^2(0,\infty, w(t)dt, H) \to A^2_\nu(H). \tag{4}$$

Proof. This result was given in the scalar case $H = \mathbb{C}$ in [11] (see also [12], where applications to admissibility and controllability were given, and [7, 8] for earlier related work). The general case follows using the standard method for proving the Hilbert space-valued case of Plancherel's theorem [1, Thm. 1.8.2]: let $(e_n)_{n=1}^\infty$ be an orthonormal basis for H, and write

$$f(t) = \sum_{n=1}^\infty f_n(t) e_n,$$

where $f_n \in L^2(0,\infty, w(t)dt, \mathbb{C})$. Then $F := \mathscr{L} f = \sum_{n=1}^\infty F_n e_n$, where $F_n = \mathscr{L} f_n \in A^2_\nu(\mathbb{C})$ and $\|f_n\| = \|F_n\|$ from [11, Prop. 2.3].

Now $\|f\|^2 = \sum_{n=1}^\infty \|f_n\|^2$ and $\|F\|^2 = \sum_{n=1}^\infty \|F_n\|^2$, so the result follows. □

In the case that $\nu = \delta_0$, we have the vectorial version of the well-known Paley–Wiener result linking $L^2(0,\infty)$ and the Hardy space $H^2(\mathbb{C}_+)$; for ν equal to Lebesgue measure, we recover the fact that the weighted signal space $L^2(0,\infty, dt/t)$ is isometric (within a constant) to the Bergman space on $\mathbb{C}_+$.

We now have a result for input–output stability.

THEOREM 2. *Let $G \in H^\infty(\mathbb{C}_+, \mathrm{L}(H))$. Then the multiplication operator M_G defined by*

$$(M_G F)(s) = G(s)F(s) \qquad (s \in \mathbb{C}_+, \quad F \in A^2_\nu(H))$$

is bounded on $A^2_\nu(H)$ with $\|M_G\| \leqslant \|G\|_\infty$. In the case when the Laplace transform (4) is surjective onto $A^2_\nu(H)$ we have equality.

Proof. It is clear that

$$\sup_{\varepsilon>0}\int_{\mathbb{C}_+}\|G(s+\varepsilon)\|^2\|F(s+\varepsilon)\|^2\,d\nu(x)\,dy \leqslant \|G\|_\infty^2 \sup_{\varepsilon>0}\int_{\mathbb{C}_+}\|F(s+\varepsilon)\|^2\,d\nu(x)\,dy,$$

so that $\|M_G\| \leqslant \|G\|_\infty$.

For the converse inequality we begin by noting that by (3) we have the inequality $w(t) \geqslant 2\pi e^{-2\varepsilon t}\nu[0,\varepsilon)$ for every $\varepsilon>0$. Hence, if $z=x+iy\in\mathbb{C}_+$, we have for $0<\varepsilon<x$ the inequality

$$\int_0^\infty |e^{-\bar{z}t}/w(t)|^2\,w(t)dt \leqslant \int_0^\infty e^{-2xt}\frac{1}{2\pi\nu[0,\varepsilon)}e^{2\varepsilon t}\,dt<\infty.$$

Thus the function $k_z : t\mapsto e^{-\bar{z}t}/w(t)$ lies in $L^2(0,\infty,w(t)dt)$ for every $z\in\mathbb{C}_+$, and we have

$$\mathscr{L}f(z)=\langle f,k_z\rangle_{L^2(0,\infty,w(t)dt)}$$

for all $f\in L^2(0,\infty,w(t)dt)$. That is, $A^2_\nu=\mathscr{L}L^2(0,\infty,w(t)dt)$ is a reproducing kernel Hilbert space with kernel $K_z:=\mathscr{L}k_z$ (see, for example, [14] for more on such spaces). For $x\in H$ we write $K_z\otimes x$ for the function $s\mapsto K_z(s)x$ in $A^2_\nu(H)$ and note that for a function $F\in A^2_\nu(H)$ we have $\langle F,K_z\otimes x\rangle_{A^2_\nu(H)}=\langle F(z),x\rangle_H$. Moreover $\|K_z\otimes x\|_{A^2_\nu(H)}=\|K_z\|_{A^2_\nu}\|x\|_H$.

Now for $F\in A^2_\nu(H)$ and $G\in H^\infty(\mathrm{L}(H))$ we have, for every $x\in H$ and $z\in\mathbb{C}_+$, that

$$\begin{aligned}\langle F,M_G^*(K_z\otimes x)\rangle_{A^2_\nu(H)} &= \langle M_GF,K_z\otimes x\rangle_{A^2_\nu(H)}=\langle G(z)F(z),x\rangle_H\\ &=\langle F(z),G(z)^*x\rangle_H=\langle F,K_z\otimes G(z)^*x\rangle,\end{aligned}$$

and so $M_G^*(K_z\otimes x)=K_z\otimes G(z)^*x$, and $\|M_G\|=\|M_G^*\|\geqslant\|G^*\|_\infty=\|G\|_\infty$. □

This result will be fundamental to the analysis in the next section.

3. Stability of retarded delay systems

3.1. General results

In Section 2 we gave necessary and sufficient conditions for an operator-valued function to give a bounded operator on various function spaces. As indicated in the introduction, we shall apply these results in the context of delay equations. We begin by reviewing the classical (scalar) case, before providing a generalization to the operator-valued case.

The 1980s result of Walton and Marshall on the location of the zeros of a scalar function $G(s)=P(s)+Q(s)e^{-sh}$ is the following.

PROPOSITION 1. [17],[13, p.132] Let $P(s)$ and $Q(s)$ be real polynomials. If

$$P(s)+Q(s)\mathrm{e}^{-sh}, \tag{5}$$

where $h > 0$, has a zero at point $s \in i\mathbb{R}$, then such an s satisfies the equation

$$P(s)P(-s) = Q(s)Q(-s). \tag{6}$$

Moreover, if $P(s), Q(s)$ are not zero at s, then the direction in which the zeros cross the axis with increasing h is given by

$$\operatorname{sgn}\operatorname{Re}\frac{ds}{dh} = \operatorname{sgn}\operatorname{Re}\frac{1}{s}\left[\frac{Q'(s)}{Q(s)} - \frac{P'(s)}{P(s)}\right].$$

Our first result analyses the H^∞ stability of an operator-valued transfer function $(P(s) + Q(s)e^{-sh}A)^{-1}$, linking it to to properties of the function $P(s) + \lambda Q(s)e^{-sh}$, where λ is in the spectrum $\sigma(A)$ of A.

THEOREM 3. *If A is a bounded operator on a Banach space X, and $h \geqslant 0$ and $P(s), Q(s)$ complex polynomials with $\deg P > \deg Q$, then the following three statements are equivalent:*

(i) $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1} \in H^\infty(\mathrm{L}(X))$.

(ii) $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)$ *is invertible* $\forall s \in \overline{\mathbb{C}}_+$.

(iii) $P(s)I + \lambda Q(s)\mathrm{e}^{-sh} \neq 0 \ \forall s \in \overline{\mathbb{C}}_+, \ \forall \lambda \in \sigma(A)$.

Proof. $(i) \implies (ii)$: This is clear.

$(ii) \implies (iii)$: the operator $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)$ is invertible if and only if $0 \notin \sigma\left[P(s)I + Q(s)A\mathrm{e}^{-sh}\right]$; but for fixed s, we get

$$\sigma\left[P(s)I + Q(s)A\mathrm{e}^{-sh}\right] = \left\{P(s)I + Q(s)\lambda\mathrm{e}^{-sh} : \lambda \in \sigma(A)\right\},$$

which means that $P(s)I + Q(s)\lambda\mathrm{e}^{-sh} \neq 0 \ \ \forall s \in \mathbb{C}_+, \ \forall\lambda \in \sigma(A)$.

$(iii) \implies (i)$: Suppose $P(s)I + Q(s)\lambda\mathrm{e}^{-sh} \neq 0 \ \ \forall s \in \overline{\mathbb{C}}_+, \ \forall\lambda \in \sigma(A)$ and so $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)$ is invertible $\forall s \in \overline{\mathbb{C}}_+$. We show that the inverse is bounded as a function of s.

First: there is an $R > 0$ such that for $s \in \overline{\mathbb{C}}_+$ with $|s| > R$, we have

$$|P(s)| > |Q(s)|\|A\||\mathrm{e}^{-sh}| + 1,$$

and so for $x \in X$ we get

$$\|P(s)x\| > \left(|Q(s)|\|A\||\mathrm{e}^{-sh}|\right)\|x\| + \|x\|,$$

and so

$$\|P(s)Ix + Q(s)A\mathrm{e}^{-sh}x\| \geqslant \|P(s)x\| - \left(|Q(s)|\|A\||\mathrm{e}^{-sh}|\right)\|x\| \geqslant \|x\|.$$

That means
$$\| \left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1} \| \leqslant 1,$$
and so $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1}$ is bounded for $|s| > R$, $s \in \overline{\mathbb{C}}_+$.

Second, to prove $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1}$ is uniformly bounded for $|s| \leqslant R$, $s \in \overline{\mathbb{C}}_+$, we suppose not, so $\exists (x_n) \subset X$, $\|x_n\| = 1$ and a sequence $(s_n) \subset S$ where $S = \{s \in \overline{\mathbb{C}}_+ : |s| \leqslant R\}$ such that
$$\left(P(s_n)I + Q(s_n)A\mathrm{e}^{-s_n h}\right) x_n \to 0,$$
and because S is a compact set then there is a subsequence $(s_{n_k})_{k\geqslant 0}$ and $s_0 \in S$ such that $(s_{n_k}) \to s_0$. Now
$$\|P(s_n)I + Q(s_n)A\mathrm{e}^{-s_n h} - P(s_0)I - Q(s_0)A\mathrm{e}^{-s_0 h}\| \to 0$$
and so $\left(P(s_0)I + Q(s_0)A\mathrm{e}^{-s_0 h}\right) x_n \to 0$, which means that
$$0 \in \sigma\left(P(s_0)I + Q(s_0)A\mathrm{e}^{-s_0 h}\right),$$
so there exists a $\lambda \in \sigma(A)$ such that $P(s_0)I + Q(s_0)\lambda\mathrm{e}^{-s_0 h} = 0$. $\square$

REMARK 1. (i) The result does not hold in general if A is unbounded (and in this case the linear system may even be destabilised by an arbitrarily small delay). For example, if A is a diagonal operator on a Hilbert space with orthonormal eigenvectors and eigenvalues $\lambda_n = ni + 1/n$ $(n \geqslant 1)$, then $s + \lambda_n$ has no zeros in $\mathbb{C}_+$ but $(sI + A)^{-1}$ is unbounded on $\mathbb{C}_+$.

(ii) The location of the poles of a neutral delay system ($\deg P = \deg Q$) does not determine its stability; as the following example indicates [3].

Consider $G(s) = \dfrac{1}{s + 1 + s\mathrm{e}^{-s}}$. If $\operatorname{Re} s > 0$ then we cannot have $\mathrm{e}^{-s} = -1 - \dfrac{1}{s}$, since the left-hand side has modulus < 1 and the right-hand side has modulus strictly > 1; thus this system has no poles in $\mathbb{C}_+$, nor indeed on $i\mathbb{R}$ (as is easily verified), although it does have a sequence of poles z_n with $\operatorname{Im} z_n \approx (2n+1)\pi$ and $\operatorname{Re} z_n \to 0$. The system is not stable, as an analysis of its values at $s = i\left[(2n+1)\pi + \dfrac{1}{(2n+1)\pi}\right]$, $n \in \mathbb{Z}$, shows that it is not in H^∞.

Since real matrices may have complex spectrum, we require a complex version of Proposition 1, as follows:

PROPOSITION 2. Let $P(s)$ and $Q(s)$ be real polynomials. If
$$P(s) + \lambda Q(s)\mathrm{e}^{-sh}$$
has a zero for some $h \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $s \in i\mathbb{R}$, then such an s satisfies the equation
$$P(s)P(-s) = |\lambda|^2 Q(s)Q(-s). \tag{7}$$

Moreover if $P(s), Q(s)$ are not zero at s and $\lambda \neq 0$, then we have

$$\operatorname{sgn}\operatorname{Re}\frac{ds}{dh} = \operatorname{sgn}\operatorname{Re}\frac{1}{s}\left[\frac{Q'(s)}{Q(s)} - \frac{P'(s)}{P(s)}\right]. \tag{8}$$

Proof. From $P(s) + \lambda Q(s)e^{-sh} = 0$, we obtain by taking complex conjugates and noting that $\overline{s} = -s$, that $P(-s) + \overline{\lambda} Q(-s)e^{sh} = 0$. This establishes (7) on eliminating the exponential term from the equations.

Next, elementary calculus (performing differentiation with respect to h) gives us

$$(P'(s) + \lambda Q'(s)e^{-sh} - h\lambda Q(s)e^{-sh})\frac{ds}{dh} - \lambda s Q(s)e^{-sh} = 0,$$

and using the fact that $\lambda e^{-sh} = -P(s)/Q(s)$ gives us

$$\left[P'(s) - \frac{P(s)Q'(s)}{Q(s)} + P(s)h\right]\frac{ds}{dh} = -sP(s)$$

or

$$\frac{ds}{dh} = -s\left[\frac{P'(s)}{P(s)} - \frac{Q'(s)}{Q(s)} + h\right]^{-1}$$

Now $\operatorname{sgn}\operatorname{Re} u = \operatorname{sgn}\operatorname{Re} u^{-1}$ for every $u \neq 0$, and h/s is purely imaginary, so the result follows. $\square$

REMARK 2. Although it is physically less relevant, we may also consider the case of complex polynomials P and Q. In this case (7) is replaced by $|P(s)|^2 = |\lambda|^2 |Q(s)|^2$.

3.2. Matrices, normal and subnormal operators

In many of the systems occurring in applications we can determine the spectrum of A, and this makes the analysis above easier to perform explicitly. For systems with inputs and outputs in $\mathbb{C}^n$ or $\mathbb{R}^n$, rather than general Hilbert spaces, we have that A is a matrix, while, as we explain below, there are classes of operators on infinite-dimensional spaces for which the analysis can also be performed directly.

A classical theorem of Schur (see, for example, [9, p. 79]) asserts that any finite square matrix can be transformed into an upper triangular matrix by conjugation with a unitary matrix. This means that when working with norm estimates for $G(s)$ as in (1) we may perform calculations using upper triangular matrices (and the spectrum is the set of diagonal elements when the matrix is in triangular form).

Another important class of infinite-dimensional operators A is the class of normal operators, such that $A^*A = AA^*$. These are unitarily equivalent to multiplication operators $M_g : f \mapsto fg$ on an $L^2(\Omega, \mu)$ space, and the spectrum is simply the closure of the range of M_g. We shall illustrate this by an example in Section 3.3.

Going beyond that we may consider the class of subnormal operators A, for which good references are [4, 5]. These may be regarded up to unitary equivalence as restrictions of normal operators N to invariant subspaces $\mathscr{M}$ (that is, $N(\mathscr{M}) \subseteq \mathscr{M}$); as for example the unilateral shift operator, or multiplication by the independent variable on the Hardy space H^2 of the disc. Such operators have been considered in a systems-theory context in [10].

A subnormal operator A has a minimal normal extension N (that is, no proper restriction of N is a normal extension of A). We mention this because we then have $\sigma(N) \subseteq \sigma(A) \subseteq \sigma(N) \cup H(N)$, where $H(N)$ is the union of the bounded components of $\mathbb{C} \setminus \sigma(N)$ (that is, the "holes" in $\sigma(N)$).

We therefore have the following immediate Corollary of Theorem 3.

COROLLARY 1. *Under the hypotheses of Theorem 3 if N is the minimal normal extension of A and $P(s)+Q(s)\lambda \mathrm{e}^{-sh} \neq 0$ for all $\lambda \in \sigma(N) \cup H(N)$ then $(P(s)+Q(s)A\mathrm{e}^{-sh})^{-1} \in H^\infty(\mathrm{L}(X))$. Conversely, if $(P(s)+Q(s)A\mathrm{e}^{-sh})^{-1} \in H^\infty(\mathrm{L}(X))$ then $P(s)+Q(s)\lambda \mathrm{e}^{-sh} \neq 0$ for all $\lambda \in \sigma(N)$.*

3.3. Examples

We conclude by showing some examples in which the results of Sections 2 and 3 can be applied directly to stability analysis.

EXAMPLE 1. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. To apply Theorem 3, with $P(s) = s, Q(s) = 1$ and the eigenvalues $\lambda_k \in \{1,2\}$ we check the zero sets of $s + e^{-sh}$ and $s + 2e^{-sh}$. The equations (7) giving the points where zeros cross the axis with increasing h are $-s^2 = 1$ and $-s^2 = 4$, respectively, and from $s + \lambda e^{-sh} = 0$ we arrive at stability ranges $[0, \pi/2)$ and $[0, \pi/4)$, respectively. Thus the system (2) is stable for $0 \leqslant h < \pi/4$ (it is easily verified using (8) that poles move from left to right as h increases).

EXAMPLE 2. For the normal matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ we have the transformation

$$T = R^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} R = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix},$$

where $R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$, which is unitary.

Now we have to consider the equation (7), obtaining $-s^2 = 2$ and for each eigenvalue λ we perform the analysis for $s + \lambda e^{-sh} = 0$. Because the λ are not real, we obtain different values of h for each eigenvalue, which we summarise now:

- For $\lambda = 1+i$, $s = i\sqrt{2}$, we have $h = \frac{3\pi}{4\sqrt{2}}$.
- For $\lambda = 1+i$, $s = -i\sqrt{2}$, we have $h = \frac{\pi}{4\sqrt{2}}$.
- For $\lambda = 1-i$, $s = i\sqrt{2}$, we have $h = \frac{\pi}{4\sqrt{2}}$.

- For $\lambda = 1-i$, $s = -i\sqrt{2}$, we have $h = \frac{3\pi}{4\sqrt{2}}$.

Again it may be checked that the zeros cross from left to right with increasing h. Thus, we can deduce that system (2) is stable when $0 \leqslant h < \frac{\pi}{4\sqrt{2}}$.

EXAMPLE 3. Examples involving subnormal operators are harder to analyse, since the spectrum of a non-normal subnormal operator cannot be contained in any simple closed curve [15]. As an example, consider again $P(s)I + Q(s)Ae^{-sh}$, and let $P(s) = s+1$, $Q(s) = 1$, and $A = 1+S$, where S is the unilateral shift operator; thus A is unitarily equivalent to the operator of multiplication by $1+z$ on the Hardy space H^2 and has spectrum $\sigma(A) = \{z \in \mathbb{C} : |z-1| \leqslant 1\}$.

From Proposition 2 we have at a zero-crossing for λ, that $1-s^2 = |\lambda|^2$, and thus we need only consider the points $\lambda \in \sigma(A)$ with $1 \leqslant |\lambda| \leqslant 2$ (a crescent-shaped set). We see from Proposition 2 that the system is stable for $h = 0$ and that the poles cross from left to right as we increase h.

Setting $s = iy$, with $-\sqrt{3} \leqslant y \leqslant \sqrt{3}$, we then need to work with the equation $e^{-sh} = -(s+1)/\lambda$, or equivalent $e^{sh} = (s-1)/\overline{\lambda}$.

For each values of s is easily verified that the λ leading to the smallest stability margin lies on the circle $\{z \in \mathbb{C} : |z-1| \leqslant 1\}$, and then it is an exercise to verify that the extreme value $\lambda = 2$ corresponding to $s = i\sqrt{3}$ gives the minimal margin of $0 \leqslant h < 2\pi/(3\sqrt{3})$.

REFERENCES

[1] W. ARENDT, C. J. K. BATTY, M. HIEBER AND F. NEUBRANDER, *Vector-valued Laplace transforms and Cauchy problems*, Second edition, Monographs in Mathematics, 96, Birkhäuser/Springer Basel AG, Basel, 2011.

[2] R. BELLMAN AND K. L. COOKE, *Differential-difference equations*, Academic Press, New York-London, 1963.

[3] C. BONNET AND J. R. PARTINGTON, *Analysis of fractional delay systems of retarded and neutral type*, Automatica J. IFAC 38 (2002), no. 8, 1133–1138.

[4] J. BRAM, *Subnormal operators*, Duke Math. J. 22 (1955), 75–94.

[5] J. B. CONWAY, *Subnormal operators*, Research Notes in Mathematics, 51. Pitman (Advanced Publishing Program), Boston, Mass. London, 1981.

[6] B. A. FRANCIS, *A course in H_∞ control theory*, Lecture Notes in Control and Information Sciences, Springer-Verlag, Berlin, 1987.

[7] Z. HARPER, *Boundedness of convolution operators and input-output maps between weighted spaces*, Complex Anal. Oper. Theory 3 (2009), no. 1, 113–146.

[8] Z. HARPER, *Laplace transform representations and Paley–Wiener theorems for functions on vertical strips*, Doc. Math. 15 (2010), 235–254.

[9] R. A. HORN AND C. R. JOHNSON, *Matrix analysis*, Corrected reprint of the 1985 original, Cambridge University Press, Cambridge, 1990.

[10] B. JACOB, A. MIRONCHENKO, J. R. PARTINGTON AND F. WIRTH, *Noncoercive Lyapunov functions for input-to-state stability of infinite-dimensional systems*, SIAM J. Control. Optim. 58 (2020), no. 5, 2952–2978.

[11] B. JACOB, J. R. PARTINGTON AND S. POTT, *On Laplace–Carleson embedding theorems*, J. Funct. Anal. 264 (2013), no. 3, 783–814.

[12] B. JACOB, J. R. PARTINGTON AND S. POTT, *Applications of Laplace–Carleson embeddings to admissibility and controllability*, SIAM J. Control Optim. 52 (2014), no. 2, 1299–1313.

[13] J. R. PARTINGTON, *Linear operators and linear systems. An analytical approach to control theory*, London Mathematical Society Student Texts, 60, Cambridge University Press, Cambridge, 2004
[14] V. I. PAULSEN AND M. RAGHUPATHI, *An introduction to the theory of reproducing kernel Hilbert spaces*, Cambridge Studies in Advanced Mathematics, 152.
[15] C. R. PUTNAM, *The spectra of subnormal operators*, Proc. Amer. Math. Soc. 28 (1971), 473–477.
[16] M. VIDYASAGAR, *Control System Synthesis: A Factorization Approach*, MIT Press Series in Signal Processing, Optimization and Control, 7, 1985.
[17] K. WALTON AND J. E. MARSHALL, *Direct method for TDS stability analysis*, IEE proceedings D, control theory and applications 134 (1987), 101–107.
[18] K. ZHOU, J. C. DOYLE AND K. GLOVER, *Robust and Optimal Control*, Prentice-Hall, Inc., Upper Saddle River, 1996.

(Received July 23, 2020)

Asmahan E. Alajyan
Department of Mathematics and Statistics
College of Science, King Faisal University
Al-Ahsa, Saudi Arabia and School of Mathematics
University of Leeds, Leeds LS2 9JT, U.K.
e-mail: ml15a3a@leeds.ac.uk

Jonathan R. Partington
School of Mathematics
University of Leeds
Leeds LS2 9JT, U.K.
e-mail: j.r.partington@leeds.ac.uk