Operators and Matrices
Volume 15, Number 4 (2021), 1319–1336 doi:10.7153/oam-2021-15-83

THE A–MODEL WITH MUTUALLY EQUAL MODEL PARAMETERS CAN LEAD TO A HILBERT SPACE MODEL

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(*Communicated by J. Behrndt*)

Abstract. It is known that the A-model for higher order singular perturbations can be considered as a Hilbert space model if the model parameters are mutually distinct, and that it is necessarily a Pontryagin space model if otherwise. In this note we demonstrate that the A-model with mutually equal model parameters can nonetheless lead to a Hilbert space model if the extensions in the model space are instead described by suitable linear relations.

1. Introduction

As it is known from [17], the A-model for rank one perturbations of class $\mathfrak{H}_{-m-2}\setminus\mathfrak{H}_{-m-1}$, $m\in\mathbb{N}$, of a lower semibounded self-adjoint operator L in $\mathfrak{H}_0$ is considered in general from the perspective of an indefinite inner product space (Pontryagin space), which we denote by $\mathcal{H}_{\mathrm{A}}$. Here $(\mathfrak{H}_n,\langle\cdot,\cdot\rangle_n)_{n\in\mathbb{Z}}$ is the scale of Hilbert spaces associated with L, and the $\mathfrak{H}_n$-scalar product is defined via an operator $b_n(L):=\prod_{j=1}^{n}(L-z_j)$ with some fixed model parameters $z_j\in\operatorname{res}L\cap\mathbb{R}$: $\langle\cdot,\cdot\rangle_n:=\langle\cdot,b_n(L)\cdot\rangle_0$. The rank of indefiniteness of $\mathcal{H}_{\mathrm{A}}$ depends on the Gram matrix $\mathcal{G}_{\mathrm{A}}$ that determines an indefinite inner product $[\cdot,\cdot]_{\mathrm{A}}$ in $\mathcal{H}_{\mathrm{A}}$. By definition it is assumed that $\mathcal{G}_{\mathrm{A}}$ is invertible and Hermitian, but for perturbations of class $\mathfrak{H}_{-4}$ or higher (*i.e.* $m\geqslant 2$), this is not sufficient in order to apply the extension theory of operators in $\mathcal{H}_{\mathrm{A}}$. It appears that for such perturbations additional restrictions imposed on $\mathcal{G}_{\mathrm{A}}$ are needed; for example, for mutually equal model parameters z_j, the Gram matrix $\mathcal{G}_{\mathrm{A}}=([\mathcal{G}_{\mathrm{A}}]_{jj'})$ must be of an anti-triangular form:

$$\begin{aligned}
[\mathcal{G}_{\mathrm{A}}]_{jj'} &=[\mathcal{G}_{\mathrm{A}}]_{j'j}\in\mathbb{R}, \quad j,j'\in\{1,\dots,m\},\\
[\mathcal{G}_{\mathrm{A}}]_{jj'} &=0, \quad j\in\{1,\dots,m-1\}, \quad j'\in\{1,\dots,m-j\},\\
[\mathcal{G}_{\mathrm{A}}]_{jm} &=[\mathcal{G}_{\mathrm{A}}]_{j+1,m-1}, \quad j\in\{1,\dots,m-1\}.
\end{aligned} \tag{1.1}$$

More generally ([17, Theorem 3.2]), if at least two of the z_j's are equal, then $\mathcal{H}_{\mathrm{A}}$ must have a nontrivial rank of indefiniteness; see also [11, Remark 4.10] with $z_j=0$. In contrast, if the points z_j are all mutually distinct, then $\mathcal{H}_{\mathrm{A}}$ can be considered as a

Mathematics subject classification (2020): 47A56, 47B25, 47B50, 35P05.

Keywords and phrases: Finite rank higher order singular perturbation, cascade (A) model, Hilbert space, scale of Hilbert spaces, Pontryagin space, ordinary boundary triple, Krein Q-function, Weyl function, gamma field, symmetric linear relation, proper extension, resolvent.

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Hilbert space, *i.e.* there exists a positive matrix $\mathcal{G}_{\mathrm{A}}$ satisfying all necessary conditions required for the application of the theory of extensions to $\mathcal{H}_{\mathrm{A}}$ of L.

The main goal of this note is to demonstrate that, for equal z_j's, we still can extract a Hilbert space model from the A-model provided that

$$[\mathcal{G}_{\mathrm{A}}]_{mm} > 0\,, \quad [\mathcal{G}_{\mathrm{A}}]_{m-1,m} = [\mathcal{G}_{\mathrm{A}}]_{m,m-1} \in \mathbb{R} \tag{1.2}$$

for $m \geqslant 2$. In fact, we consider rank-d perturbations, with an arbitrary $d \in \mathbb{N}$, so that actually we have that $\mathcal{G}_{\mathrm{A}} = ([\mathcal{G}_{\mathrm{A}}]_{\sigma j,\sigma' j'})$ is a $dm \times dm$ Gram matrix; the indices σ, σ' range over an index set $\mathcal{S}$ of cardinality $d \in \mathbb{N}$. The conditions in (1.1), (1.2) are then modified appropriately (see (2.5) and (3.1)).

In the A-model, singular perturbations of L in $\mathcal{H}_{\mathrm{A}}$ are specified by the extensions of a densely defined, closed, symmetric operator $A_{\min}$ in $\mathcal{H}_{\mathrm{A}}$, provided an invertible Hermitian $\mathcal{G}_{\mathrm{A}}$ satisfies appropriate conditions (for equal z_j's these are as in (1.1)). We recall that $A_{\min}$ is the adjoint in $\mathcal{H}_{\mathrm{A}}$ of the restriction $A_{\max} \supseteq A_{\min}$ to $\mathcal{H}_{\mathrm{A}}$ of the triplet adjoint $L_{\max}$ of $L_{\min}$. The triplet adjoint is taken with respect to the Hilbert triple $\mathfrak{H}_m \subseteq \mathfrak{H}_0 \subseteq \mathfrak{H}_{-m}$. The operator $L_{\min}$ is densely defined, closed, symmetric in $\mathfrak{H}_m$, has defect numbers (d,d), and is essentially self-adjoint in $\mathfrak{H}_0$, whose closure is L. As is usual in extension theory, an extension $A_\Theta \in \mathrm{Ext}(A_{\min})$ is parametrized by a linear relation Θ in $\mathbb{C}^d$ according to $\mathrm{dom} A_\Theta = \{f \in \mathrm{dom} A_{\max} \,|\, \Gamma f \in \Theta\}$, where $\Gamma := (\Gamma_0,\Gamma_1)\colon\ \mathrm{dom} A_{\max} \to \mathbb{C}^d \times \mathbb{C}^d$ defines the boundary triple $(\mathbb{C}^d,\Gamma_0,\Gamma_1)$ for $A_{\max} = A^*_{\min}$.

To explain our main idea, let us now consider the A-model with equal model parameters, $z_j = z_1$. For simplicity we let $d=1$. Let $\mathcal{H}^{\min}_{\mathrm{A}} := \mathcal{H}_{\mathrm{A}} \cap \mathfrak{H}_{m-2}$. The subscript "min", indicating the minimality of the space, is due to the following fact. Because $\mathcal{H}_{\mathrm{A}}$ is the direct sum of $\mathfrak{H}_m$ and an m-dimensional space $\mathfrak{K}_{\mathrm{A}}$ spanned by the singular elements $h_j \in \mathfrak{H}_{-m-2+2j} \setminus \mathfrak{H}_{-m-1+2j}$, we have that $\mathfrak{K}^{\min}_{\mathrm{A}} \subseteq \mathfrak{K}_{\mathrm{A}} \subseteq \mathfrak{H}_{-m}$, where $\mathfrak{K}^{\min}_{\mathrm{A}} := \mathfrak{K}_{\mathrm{A}} \cap \mathfrak{H}_{m-2}$ is a minimal subset contained in $\mathfrak{K}_{\mathrm{A}}$ in the sense that $\mathfrak{K}_{\mathrm{A}} \cap \mathfrak{K}_{m-1} = \{0\}$.

Consider the domain restriction $A_{\max}|_{\mathcal{H}^{\min}_{\mathrm{A}}}$ to $\mathcal{H}^{\min}_{\mathrm{A}} = \mathfrak{H}_m \dotplus \mathfrak{K}^{\min}_{\mathrm{A}}$ of $A_{\max}$. Let $B_{\max}$ denote a linear relation in $\mathcal{H}_{\mathrm{A}}$ defined by the componentwise sum of (the graph of) $A_{\max}|_{\mathcal{H}^{\min}_{\mathrm{A}}}$ and $\{0\} \times \mathcal{H}^{\perp}_{\mathrm{A}}$. Here $\mathcal{H}^{\perp}_{\mathrm{A}}$ denotes the orthogonal complement in $\mathcal{H}_{\mathrm{A}}$ of $\mathcal{H}^{\min}_{\mathrm{A}}$, which is a subset of $\mathfrak{K}_{\mathrm{A}}$. By the construction, the adjoint $B_{\min} := B^*_{\max}$ in $\mathcal{H}_{\mathrm{A}}$ is a linear relation given by the componentwise sum of (the graph of) $A_{\min}|_{\mathcal{H}^{\min}_{\mathrm{A}}}$ and $\{0\} \times \mathcal{H}^{\perp}_{\mathrm{A}}$. Assuming only the invertibility and the Hermiticity of $\mathcal{G}_{\mathrm{A}}$, the operator $A_{\min}$ differs from $A'_{\min} := A_{\max}|_{\ker\Gamma}$ (although $\mathrm{dom} A_{\min} = \mathrm{dom} A'_{\min}$), *i.e.* $A_{\min}$ is not symmetric; the symmetry of $A_{\min} = A'_{\min}$ is ensured by (1.1). Now the key point is that, without assumption (1.1), but instead assuming $[\mathcal{G}_{\mathrm{A}}]_{m-1,m} = [\mathcal{G}_{\mathrm{A}}]_{m,m-1}$ (the second condition in (1.2)), it holds

$$(A_{\min} - A'_{\min})(\mathrm{dom} A_{\min} \cap \mathcal{H}^{\min}_{\mathrm{A}}) \subseteq \mathcal{H}^{\perp}_{\mathrm{A}}$$

i.e. $B_{\min}$ is a symmetric linear relation in $\mathcal{H}_{\mathrm{A}}$. By the same reasoning one shows that $B_{\min}$ is also closed. Sequentially, one can apply the extension theory for $B_{\min}$, as is done for $A_{\min}$.

For $\mathcal{G}_{\mathrm{A}}$ as in (1.1), the Weyl function corresponding to a boundary triple for $A_{\max}$ determined by Γ is the sum of the Krein Q-function q of $L_{\min}$ and a generalized Nevanlinna function r (see *e.g.* [2, Section 4] for the terminology) defined by

$$r(z) := -\sum_{j=1}^{m} \frac{[\mathcal{G}_{\mathrm{A}}]_{mj}}{(z-z_1)^{m-j+1}}, \quad z \in \mathbb{C} \setminus \{z_1\}.$$

Likewise, for $\mathcal{G}_{\mathrm{A}}$ as in (1.2), the Weyl function corresponding to the boundary triple for $B_{\max}$, which is determined by restriction to $\operatorname{dom} B_{\max}$ of Γ, is the sum of the same Krein Q-function q and now a Nevanlinna function $\hat{r}$ defined by

$$\hat{r}(z) := \frac{[\mathcal{G}_{\mathrm{A}}]_{mm}}{\hat{\Delta}-z}, \quad z \in \mathbb{C} \setminus \{\hat{\Delta}\}$$

with some real number $\hat{\Delta}$. The strict inequality $[\mathcal{G}_{\mathrm{A}}]_{mm} > 0$ in (1.2) is closely related to the fact that the subspace $\mathcal{H}_{\mathrm{A}}^{\min} = (\mathfrak{H}_m \dotplus \mathfrak{K}_{\mathrm{A}}^{\min}, [\cdot,\cdot]_{\mathrm{A}})$ of $\mathcal{H}_{\mathrm{A}}$ is a Hilbert space iff $[\mathcal{G}_{\mathrm{A}}]_{mm} > 0$. Thus, for example, one may take $\mathcal{G}_{\mathrm{A}}$ as the Gram matrix of vectors h_j generating $\mathfrak{K}_{\mathrm{A}}$, in which case $[\mathcal{G}_{\mathrm{A}}]_{jj'} = \langle h_j, h_{j'} \rangle_{-m}$, and the conditions in (1.2) are all satisfied. In contrast, the so defined $\mathcal{G}_{\mathrm{A}}$ does not satisfy (1.1). We remark that, for $m=1$, we have $\hat{\Delta} = z_1$, and hence $\hat{r} = r$, as it should follow from $\mathcal{H}_{\mathrm{A}}^{\min} = \mathcal{H}_{\mathrm{A}}$. We also remark that an analogous development of extension theory for $B_{\min}$ takes place in the peak model for singular perturbations, *cf.* [22].

Because the Weyl function $q+\hat{r}$ of $B_{\min}$ is a (uniformly strict) Nevanlinna function, it follows from [24, Theorem 2.2] that $q+\hat{r}$ is the Weyl function of some closed simple symmetric operator, corresponding to a certain boundary triple. Following the terminology in [18], it is precisely in this sense what we mean by saying that the A-model with mutually equal model parameters leads to a Hilbert space model (of the function $q+\hat{r}$). For example, a simple symmetric operator may be considered as the operator of multiplication by an independent variable in a reproducing kernel Hilbert space induced by the Nevanlinna pair $(1, q+\hat{r})$; see *e.g.* [3, Theorem 6.1], [2, Theorem 4.10], [9, Remark 2.6].

Having determined the extensions to $\mathcal{H}_{\mathrm{A}}$ of $L_{\min}$ one then interprets singular perturbations of L by means of the compressions to $\mathfrak{H}_m$ of their resolvents. Thus, for $d=1$, $B_\Theta \in \operatorname{Ext}(B_{\min})$, $\Theta \in \mathbb{C} \cup \{\infty\}$, the compressed resolvent of B_Θ is represented in the generalized sense according to

$$P_{\mathfrak{H}_m}(B_\Theta - z)^{-1}|_{\mathfrak{H}_m} = (L-z)^{-1} + \frac{\langle g(\bar{z}), \cdot \rangle (L-z)^{-1} h_m}{\Theta - q(z) - \hat{r}(z)}$$

for a suitable $z \in \operatorname{res} L$. Here $P_{\mathfrak{H}_m}$ is a projection in $\mathcal{H}_{\mathrm{A}}$ onto $\mathfrak{H}_m$, $g(\bar{z}) \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1}$ is the eigenvector of $L_{\max}$ corresponding to the eigenvalue $\bar{z}$ (in particular $h_1 = g(z_1)$), and $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathfrak{H}_{-m}$ and $\mathfrak{H}_m$. By the above resolvent formula one concludes that the spectral properties of (super) singular perturbations in the A-model with equal model parameters can be described by Nevanlinna functions.

The reasoning behind the above mentioned interpretation of singular perturbations is that there exists a bijective correspondence between Nevanlinna families and generalized resolvents of $L_{\min}$, and the correspondence is established via a generalized

Krein–Naimark resolvent formula. Thus, to a rational Nevanlinna function $\hat{r}-\Theta$, with a real Θ, there corresponds a self-adjoint extension $\widetilde{B}$ of $L_{\min}$ in some larger Hilbert space $\widetilde{\mathfrak{H}} \supseteq \mathfrak{H}_m$, and such that $\widetilde{B} \cap L = L_{\min}$. For more details the reader may refer to [4, 18, 13, 8, 7, 14].

2. A brief overview of the A-model with equal model parameters

Here we restate the main results from [23, 17]; see also [11]. The main tools and terminology used in the theory of boundary relations of symmetric operators (or linear relations) are as in [4, 15, 10, 20, 16, 2, 19, 21, 8] and in references therein.

We consider a lower semibounded self-adjoint operator L in a Hilbert space $\mathfrak{H}_0$, and we let $(\mathfrak{H}_n)_{n\in\mathbb{Z}}$ be the scale of Hilbert spaces associated with L. The scalar product in $\mathfrak{H}_n$ is conjugate linear in the first factor and is defined via the scalar product $\langle\cdot,\cdot\rangle_0$ in $\mathfrak{H}_0$ according to

$$\langle\cdot,\cdot\rangle_n := \langle b_n(L)^{1/2}\cdot, b_n(L)^{1/2}\cdot\rangle_0\,, \quad b_n(L) := (L-z_1)^n$$

for some fixed model parameter $z_1 \in \operatorname{res} L \cap \mathbb{R}$ ($\operatorname{res} L$ denotes the resolvent set of L, and similarly for other operators). To $L = L_0$ one associates a self-adjoint operator $L_n := L|_{\mathfrak{H}_{n+2}}$ in $\mathfrak{H}_n$, and satisfying $L_{n+1} \subset L_n$ and $\operatorname{res} L_n = \operatorname{res} L$. For the reasons just described we sometimes omit the subscript n in L_n.

Let us fix m, $d \in \mathbb{N}$. Let $\{\varphi_\sigma \in \mathfrak{H}_{-m-2} \setminus \mathfrak{H}_{-m-1}\}$ be the family of linearly independent functionals; σ ranges over an index set $\mathcal{S}$ of cardinality d. The symmetric restriction $L_{\min}$ of L to the domain of $f \in \mathfrak{H}_{m+2}$ such that $\langle\varphi_\sigma, f\rangle = 0$, for all σ, is a densely defined, closed, symmetric operator in $\mathfrak{H}_m$, and has defect numbers (d,d). It is also essentially self-adjoint operator in $\mathfrak{H}_0$. The duality pairing $\langle\cdot,\cdot\rangle$ is defined via the $\mathfrak{H}_0$-scalar product in a usual way. We also define a vector valued functional φ via $\langle\varphi,\cdot\rangle = (\langle\varphi_\sigma,\cdot\rangle)\colon \mathfrak{H}_{m+2} \to \mathbb{C}^d$; hence $L_{\min} = L_m|_{\{f\in\mathfrak{H}_{m+2}\,|\,\langle\varphi,f\rangle=0\}}$.

The triplet adjoint $L_{\max}$ of $L_{\min}$ corresponding to the Hilbert triple $\mathfrak{H}_m \subset \mathfrak{H}_0 \subset \mathfrak{H}_{-m}$ is the operator extending L_{-m} to the domain $\mathfrak{H}_{-m+2} \dotplus \mathfrak{N}_z(L_{\max})$ (direct sum) for $z \in \operatorname{res} L$. The eigenspace $\mathfrak{N}_z(L_{\max})$ ($:= \ker(L_{\max}-z)$) is the linear span of the elements $g_\sigma(z)$ defined in the generalized sense according to

$$g_\sigma(z) := (L-z)^{-1}\varphi_\sigma \in \mathfrak{H}_{-m} \setminus \mathfrak{H}_{-m+1}\,.$$

Define an md-dimensional linear space

$$\mathfrak{K}_{\mathrm{A}} := \operatorname{span}\{h_\alpha \,|\, \alpha = (\sigma,j) \in \mathcal{S}\times J\}\,, \quad J := \{1,2,\dots,m\}$$

spanned by the elements

$$h_{\sigma j} := b_j(L)^{-1}\varphi_\sigma \in \mathfrak{H}_{-m-2+2j} \setminus \mathfrak{H}_{-m-1+2j}\,.$$

From here it follows that $\mathfrak{K}_{\mathrm{A}}^{\min} \subseteq \mathfrak{K}_{\mathrm{A}} \subseteq \mathfrak{H}_{-m}$ with

$$\mathfrak{K}_{\mathrm{A}}^{\min} := \mathfrak{K}_{\mathrm{A}} \cap \mathfrak{H}_{m-2} = h_m(\mathbb{C}^d)\,, \quad h_m(c) := \sum_\sigma c_\sigma h_{\sigma m}\,, \quad c = (c_\sigma) \in \mathbb{C}^d$$

and that in particular $\mathfrak{K}_{\mathrm{A}}^{\min} = \mathfrak{K}_{\mathrm{A}}$ for $m = 1$. Note that $\mathfrak{K}_{\mathrm{A}} \cap \mathfrak{H}_{m-1} = \{0\}$.

Because the system $\{h_\alpha\}$ is linearly independent, the matrix

$$\widetilde{\mathcal{G}}_{\mathrm{A}} = ([\widetilde{\mathcal{G}}_{\mathrm{A}}]_{\alpha\alpha'}) \in [\mathbb{C}^{md}], \quad [\widetilde{\mathcal{G}}_{\mathrm{A}}]_{\alpha\alpha'} := \langle h_\alpha, h_{\alpha'} \rangle_{-m}$$

is the Gram matrix of vectors generating $\mathfrak{K}_{\mathrm{A}}$; hence it is positive definite, Hermitian. One establishes a bijective correspondence

$$\mathfrak{K}_{\mathrm{A}} \ni k \leftrightarrow d(k) = (d_\alpha(k)) \in \mathbb{C}^{md}$$

via

$$k = \sum_\alpha d_\alpha(k) h_\alpha, \quad d(k) = \widetilde{\mathcal{G}}_{\mathrm{A}}^{-1} \langle h, k \rangle_{-m}, \quad \langle h, \cdot \rangle_{-m} = (\langle h_\alpha, \cdot \rangle_{-m}).$$

Here and in what follows $d(\cdot)$ is interpreted as a (bounded) vector valued functional from $\mathfrak{K}_{\mathrm{A}}$ to $\mathbb{C}^{md}$.

Let us define the matrix

$$\widetilde{\mathcal{G}}_{\mathrm{A}}^{\min} = ([\widetilde{\mathcal{G}}_{\mathrm{A}}^{\min}]_{\sigma\sigma'}) \in [\mathbb{C}^{d}], \quad [\widetilde{\mathcal{G}}_{\mathrm{A}}^{\min}]_{\sigma\sigma'} := \langle h_{\sigma m}, h_{\sigma' m} \rangle_{-m}$$

which is the Gram matrix of vectors generating $\mathfrak{K}_{\mathrm{A}}^{\min}$. Thus $\widetilde{\mathcal{G}}_{\mathrm{A}}^{\min}$ is also positive definite, Hermitian, and one therefore establishes a bijective correspondence

$$\mathfrak{K}_{\mathrm{A}}^{\min} \ni h_m(c) \leftrightarrow c \in \mathbb{C}^d$$

via

$$c = (\widetilde{\mathcal{G}}_{\mathrm{A}}^{\min})^{-1} \langle h_m, h_m(c) \rangle_{-m}, \quad \langle h_m, \cdot \rangle_{-m} = (\langle h_{\sigma m}, \cdot \rangle_{-m}).$$

On the other hand, because $\mathfrak{K}_{\mathrm{A}}^{\min} \subseteq \mathfrak{K}$, to each $k = h_m(c) \in \mathfrak{K}_{\mathrm{A}}^{\min}$ there corresponds $d(k) = \eta(c) \in \mathbb{C}^{md}$, where

$$\eta(c) := (\delta_{jm} c_\sigma).$$

Consider an indefinite inner product space

$$\mathcal{H}_{\mathrm{A}} := (\mathfrak{H}_m \dot{+} \mathfrak{K}_{\mathrm{A}}, [\cdot,\cdot]_{\mathrm{A}})$$

equipped with an indefinite metric

$$[f + k, f' + k']_{\mathrm{A}} := \langle f, f' \rangle_m + \langle d(k), \mathcal{G}_{\mathrm{A}} d(k') \rangle_{\mathbb{C}^{md}}$$

for f, $f' \in \mathfrak{H}_m$ and k, $k' \in \mathfrak{K}_{\mathrm{A}}$. The matrix $\mathcal{G}_{\mathrm{A}} = ([\mathcal{G}_{\mathrm{A}}]_{\alpha\alpha'})$ is called the Gram matrix of the A-model; it is initially assumed to be invertible and Hermitian, but otherwise arbitrary. Thus in particular $\mathcal{G}_{\mathrm{A}} \neq 0$. Clearly if $\mathcal{G}_{\mathrm{A}}$ is positive, then $\mathcal{H}_{\mathrm{A}}$ becomes a Hilbert space. Otherwise $\mathcal{H}_{\mathrm{A}}$ is a Pontryagin space.

For an appropriate $\mathcal{G}_{\mathrm{A}}$, the extensions to $\mathcal{H}_{\mathrm{A}}$ of $L_{\min}$ are the restrictions to $\mathcal{H}_{\mathrm{A}}$ of the triplet adjoint $L_{\max}$. Let

$$A_{\max} := L_{\max} \cap \mathcal{H}_{\mathrm{A}}^2.$$

Here and in what follows operators are frequently identified with their graphs. The operator $A_{\max}$ admits the following representation:

$$A_{\max} =\{(f^{\#}+h_{m+1}(c)+k, L_m f^{\#}+z_1 h_{m+1}(c)+\widetilde{k}) \,|\, f^{\#}\in\mathfrak{H}_{m+2}\,; \\ c\in\mathbb{C}^d\,; k,\widetilde{k}\in\mathfrak{K}_{\mathrm{A}}\,; d(\widetilde{k})=\mathfrak{M}_d d(k)+\eta(c)\}\,.$$

An element $h_{m+1}(c)\in\mathfrak{H}_m\setminus\mathfrak{H}_{m+1}$ is defined by

$$h_{m+1}(c):=\sum_{\sigma}c_\sigma h_{\sigma,m+1}\,,\quad h_{\sigma,m+1}:=b_{m+1}(L)^{-1}\varphi_\sigma\,.$$

The matrix $\mathfrak{M}_d:=\mathfrak{M}\oplus\cdots\oplus\mathfrak{M}$ (d times) is the matrix direct sum of d matrices $\mathfrak{M}=(\mathfrak{M}_{jj'})\in[\mathbb{C}^m]$ defined as follows: For $m\geqslant 2$

$$\mathfrak{M}_{jj'}:=1_{J\setminus\{m\}}(j)(\delta_{jj'}z_1+1_{J\setminus\{1\}}(j')\delta_{j+1,j'})+\delta_{jm}\delta_{j'm}z_1$$

for $j,\ j'\in J$; here 1_X is the characteristic function of a set X. For $m=1$, $\mathfrak{M}:=z_1$.

By direct computation, the boundary form of $A_{\max}$ is represented in the form

$$[f,A_{\max}g]_{\mathrm{A}}-[A_{\max}f,g]_{\mathrm{A}}=\langle d(k),(\mathcal{G}_{\mathfrak{M}}-\mathcal{G}_{\mathfrak{M}}^{*})d(k')\rangle_{\mathbb{C}^{md}} \\ +\langle\Gamma_0 f,\Gamma_1 g\rangle_{\mathbb{C}^d}-\langle\Gamma_1 f,\Gamma_0 g\rangle_{\mathbb{C}^d}\,,$$

$$\mathcal{G}_{\mathfrak{M}}:=\mathcal{G}_{\mathrm{A}}\mathfrak{M}_d$$

with $f=f^{\#}+h_{m+1}(c)+k\in\operatorname{dom}A_{\max}$; $g=g^{\#}+h_{m+1}(c')+k'\in\operatorname{dom}A_{\max}$; $f^{\#},g^{\#}\in\mathfrak{H}_{m+2}$; $c,c'\in\mathbb{C}^d$; $k,k'\in\mathfrak{K}_{\mathrm{A}}$. The operator $\Gamma:=(\Gamma_0,\Gamma_1)$ from $\operatorname{dom}A_{\max}$ to $\mathbb{C}^d\times\mathbb{C}^d$ is defined by

$$\Gamma_0(f^{\#}+h_{m+1}(c)+k):=c\,,\\ \Gamma_1(f^{\#}+h_{m+1}(c)+k):=\langle\varphi,f^{\#}\rangle-[\mathcal{G}_{\mathrm{A}}d(k)]_m$$

with

$$[\mathcal{G}_{\mathrm{A}}d(k)]_m:=([\mathcal{G}_{\mathrm{A}}d(k)]_{\sigma m})\in\mathbb{C}^d\,.$$

In the next lemma we give a description of the adjoint of $A_{\max}$ and, moreover, we show that Γ is surjective. By considering Γ as a single-valued linear relation from $\mathcal{H}_{\mathrm{A}}^2$ to $\mathbb{C}^{2d}$ with $\operatorname{dom}\Gamma=A_{\max}$, *i.e.*

$$\Gamma=\{\big((f,A_{\max}f),(\Gamma_0 f,\Gamma_1 f)\big)\,|\,f\in\operatorname{dom}A_{\max}\}$$

we recall (*e.g.* [2, Section 3.1]) that its Krein space adjoint $\Gamma^{[*]}$ is a linear relation from $\mathbb{C}^{2d}$ to $\mathcal{H}_{\mathrm{A}}^2$, and it consists of $\big((\chi,\chi'),(g,g')\big)$ such that $(\forall f\in\operatorname{dom}A_{\max})$

$$[f,g']_{\mathrm{A}}-[A_{\max}f,g]_{\mathrm{A}}=\langle\Gamma_0 f,\chi'\rangle_{\mathbb{C}^d}-\langle\Gamma_1 f,\chi\rangle_{\mathbb{C}^d}\,. \tag{2.1}$$

LEMMA 2.1. *Similar to* $A_{\max}$, *define the operator* $A'_{\max}$ *in* $\mathcal{H}_{\mathrm{A}}$ *by*

$$A'_{\max}:=\{(f^{\#}+h_{m+1}(c)+k, L_m f^{\#}+z_1 h_{m+1}(c)+\widetilde{k}') \,|\, f^{\#}\in\mathfrak{H}_{m+2}\,; \\ c\in\mathbb{C}^d\,; k,\widetilde{k}'\in\mathfrak{K}_{\mathrm{A}}\,; d(\widetilde{k}')=\mathcal{G}_{\mathrm{A}}^{-1}\mathcal{G}_{\mathfrak{M}}^{*}d(k)+\eta(c)\}\,.$$

The following statements hold:

(i) *Consider* $\Gamma = (\Gamma_0, \Gamma_1)$ *as a single-valued linear relation with* $\operatorname{dom}\Gamma = A_{\max}$. *Let* $\Gamma^{[*]}$ *be its Krein space adjoint. Then the inverse* $(\Gamma^{[*]})^{-1} = (\Gamma_0, \Gamma_1)$ *is a single-valued linear relation with* $\operatorname{ran}\Gamma^{[*]} = A'_{\max}$. *Moreover,* Γ *is closed and surjective.*

(ii) *The adjoint in* $\mathcal{H}_{\mathrm{A}}$ *of a closed operator* $A_{\max}$ *is the operator*

$$A_{\min} := A^*_{\max} = A'_{\max}|_{\ker\Gamma}\,.$$

(iii) *Define the operator*

$$A'_{\min} := A_{\max}|_{\ker\Gamma}$$

in $\mathcal{H}_{\mathrm{A}}$. *Then* $A'_{\min}$ *is closed, and its adjoint in* $\mathcal{H}_{\mathrm{A}}$ *is* $A'^*_{\min} = A'_{\max}$.

Proof. First we remark that

$$\operatorname{ran}\mathcal{G}^*_{\mathfrak{M}} \subseteq \operatorname{ran}\mathcal{G}_{\mathrm{A}} \tag{2.2}$$

so that $A'_{\max}$ is defined correctly. The inclusion in (2.2) is equivalent to the statement that

$$(\forall\xi \in \mathbb{C}^{md})\,(\exists\xi' \in \mathbb{C}^{md})\;\mathcal{G}^*_{\mathfrak{M}}\xi = \mathcal{G}_{\mathrm{A}}\xi'\,. \tag{2.3}$$

For $m=1$, $\mathcal{G}_{\mathfrak{M}} = z_1\mathcal{G}_{\mathrm{A}}$, so $\xi' = z_1\xi$ solves (2.3) for an arbitrary Hermitian $\mathcal{G}_{\mathrm{A}}$. For $m \geqslant 2$ we have

$$[\mathcal{G}_{\mathfrak{M}}]_{\sigma j,\sigma' j'} = z_1[\mathcal{G}_{\mathrm{A}}]_{\sigma j,\sigma' j'} + 1_{J\smallsetminus\{1\}}(j')[\mathcal{G}_{\mathrm{A}}]_{\sigma j;\sigma',j'-1}$$

and hence

$$[\mathcal{G}^*_{\mathfrak{M}}]_{\sigma j,\sigma' j'} = z_1[\mathcal{G}_{\mathrm{A}}]_{\sigma j,\sigma' j'} + 1_{J\smallsetminus\{1\}}(j)[\mathcal{G}_{\mathrm{A}}]_{\sigma,j-1;\sigma' j'}\,.$$

Then $\mathcal{G}^*_{\mathfrak{M}}\xi = \mathcal{G}_{\mathrm{A}}\xi'$ reads

$$[\mathcal{G}_{\mathrm{A}}(\xi' - z_1\xi)]_{\sigma j} = 1_{J\smallsetminus\{1\}}(j)[\mathcal{G}_{\mathrm{A}}\xi]_{\sigma,j-1}\,.$$

Put

$$\mathring{\mathcal{G}}_{\mathrm{A}} = ([\mathring{\mathcal{G}}_{\mathrm{A}}]_{\alpha\alpha'}) \in [\mathbb{C}^{md}], \quad [\mathring{\mathcal{G}}_{\mathrm{A}}]_{\sigma j,\sigma' j'} := 1_{J\smallsetminus\{1\}}(j)[\mathcal{G}_{\mathrm{A}}]_{\sigma,j-1;\sigma' j'}\,.$$

Then

$$\operatorname{ran}\mathring{\mathcal{G}}_{\mathrm{A}} \subseteq \operatorname{ran}\mathcal{G}_{\mathrm{A}} \quad \text{and} \quad \mathcal{G}_{\mathrm{A}}(\xi' - z_1\xi) = \mathring{\mathcal{G}}_{\mathrm{A}}\xi$$

and therefore

$$\xi' = (z_1 + \mathcal{G}_{\mathrm{A}}^{-1}\mathring{\mathcal{G}}_{\mathrm{A}})\xi$$

solves (2.3) for an arbitrary invertible Hermitian $\mathcal{G}_{\mathrm{A}}$.

(i) Letting $g = g^\natural + k_g$ and $g' = g'^\natural + k_{g'}$ in (2.1) for some $g^\natural$, $g'^\natural \in \mathfrak{H}_m$ and k_g, $k_{g'} \in \mathfrak{K}_{\mathrm{A}}$, and using that

$$\langle\langle\varphi, f^\#\rangle, \chi\rangle_{\mathbb{C}^d} = \langle(L - z_1)f^\#, h_{m+1}(\chi)\rangle_m$$

and

$$\langle [\mathcal{G}_{\mathrm{A}} d(k)]_m, \chi \rangle_{\mathbb{C}^d} = \langle d(k), \mathcal{X}\chi \rangle_{\mathbb{C}^{md}}$$

with

$$\mathcal{X} = ([\mathcal{X}]_{\alpha\sigma}) \in [\mathbb{C}^d, \mathbb{C}^{md}], \quad [\mathcal{X}]_{\alpha\sigma} := [\mathcal{G}_{\mathrm{A}}]_{\alpha,\sigma' m}$$

we find that $(\forall f^{\#} \in \mathfrak{H}_{m+2})$ $(\forall c \in \mathbb{C}^d)$ $(\forall k \in \mathfrak{K}_{\mathrm{A}})$

$$\begin{aligned} 0 = & \langle f^{\#}, g'^{\natural} - z_1 h_{m+1}(\chi) \rangle_m - \langle L f^{\#}, g^{\natural} - h_{m+1}(\chi) \rangle_m \\ & + \langle c, \langle h_{m+1}, g'^{\natural} - z_1 g^{\natural} \rangle_m - [\mathcal{G}_{\mathrm{A}} d(k_g)]_m - \chi' \rangle_{\mathbb{C}^d} \\ & + \langle d(k), \mathcal{G}_{\mathrm{A}} d(k_{g'}) - \mathcal{G}_{\mathfrak{M}}^{*} d(k_g) - \mathcal{X}\chi \rangle_{\mathbb{C}^{md}} \end{aligned} \tag{2.4}$$

with

$$\langle h_{m+1}, \cdot \rangle_m = (\langle h_{\sigma, m+1}, \cdot \rangle_m).$$

Thus it follows that

$$g^{\natural} = g^{\#} + h_{m+1}(\chi), \quad g^{\#} \in \mathfrak{H}_{m+2}, \quad g'^{\natural} = L_m g^{\#} + z_1 h_{m+1}(\chi)$$

and

$$\chi' = \langle h_{m+1}, (L - z_1) g^{\#} \rangle_m - [\mathcal{G}_{\mathrm{A}} d(k_g)]_m = \langle \varphi, g^{\#} \rangle - [\mathcal{G}_{\mathrm{A}} d(k_g)]_m$$

and

$$d(k_{g'}) = \mathcal{G}_{\mathrm{A}}^{-1} \mathcal{G}_{\mathfrak{M}}^{*} d(k_g) + \mathcal{G}_{\mathrm{A}}^{-1} \mathcal{X}\chi, \quad \mathcal{G}_{\mathrm{A}}^{-1} \mathcal{X}\chi = \eta(\chi).$$

This shows that

$$(\Gamma^{[*]})^{-1} = \{ ((f, A'_{\max} f), (\Gamma_0 f, \Gamma_1 f)) \mid f \in \operatorname{dom} A_{\max} \}.$$

Because $\ker \Gamma^{[*]} = \operatorname{mul}(\Gamma^{[*]})^{-1} = \{0\}$, it follows that $\overline{\operatorname{ran}} \Gamma = \operatorname{ran} \overline{\Gamma} = \mathbb{C}^{2d}$, and it therefore remains to verify that Γ is closed.

The closure $\overline{\Gamma}$ is the Krein space adjoint of $\Gamma^{[*]}$. Thus it consists $((g, g'), (\chi, \chi')) \in \mathcal{H}_{\mathrm{A}}^2 \times \mathbb{C}^{2d}$ such that $(\forall f \in \operatorname{dom} A_{\max})$ equation (2.4) holds, but with $\mathcal{G}_{\mathfrak{M}}^{*}$ replaced by $\mathcal{G}_{\mathfrak{M}}$. By repeating the subsequent steps as above, one finds that $\overline{\Gamma} = \Gamma$.

(ii) The adjoint linear relation $A_{\min}$ consists of $(g, g') \in \mathcal{H}_{\mathrm{A}}^2$ such that (2.4) holds, but with $\chi = 0 = \chi'$; therefore it is the operator as stated in the lemma.

(iii) By the arguments as in the proof of (i), $A'^{*}_{\max} = A'_{\min}$; thus $A'_{\min}$ is a closed operator whose adjoint in $\mathcal{H}_{\mathrm{A}}$ is as stated in the lemma. □

For $m = 1$, the matrix $\mathcal{G}_{\mathfrak{M}} = z_1 \mathcal{G}_{\mathrm{A}}$ is automatically Hermitian, while for $m \geqslant 2$, we have $\mathcal{G}_{\mathfrak{M}}^{*} = \mathcal{G}_{\mathfrak{M}}$ iff

$$\begin{aligned} [\mathcal{G}_{\mathrm{A}}]_{\sigma j, \sigma' j'} &= [\mathcal{G}_{\mathrm{A}}]_{\sigma j', \sigma' j}, \quad j, j' \in J, \\ [\mathcal{G}_{\mathrm{A}}]_{\sigma j, \sigma' j'} &= 0, \quad j \in J \setminus \{m\}, \quad j' \in \{1, \dots, m - j\}, \\ [\mathcal{G}_{\mathrm{A}}]_{\sigma j, \sigma' m} &= [\mathcal{G}_{\mathrm{A}}]_{\sigma, j+1; \sigma', m-1}, \quad j \in J \setminus \{m\}. \end{aligned} \tag{2.5}$$

Note that the entries of $\mathcal{G}_{\mathrm{A}}$ in (2.5), which are diagonal in $\sigma \in \mathcal{S}$, are real numbers. Note also that $\widetilde{\mathcal{G}}_{\mathrm{A}}$ does not satisfy (2.5), because $[\widetilde{\mathcal{G}}_{\mathrm{A}}]_{\sigma 1, \sigma 1} > 0$.

For an Hermitian $\mathcal{G}_{\mathfrak{M}}$ we have $A'_{\max} = A_{\max}$, $A'_{\min} = A_{\min}$, and Γ is a unitary operator, $\Gamma^{-1} = \Gamma^{[*]}$. Subsequently [12, Corollary 2.4], the triple $(\mathbb{C}^d, \Gamma_0, \Gamma_1)$ is a boundary triple for the adjoint $A_{\max} = A^*_{\min}$ of a densely defined, closed, and symmetric operator $A_{\min}$ in a Pontryagin space $\mathcal{H}_{\mathrm{A}}$; the reader may refer to [8, Definition 2.1], [7, Definition 3] for the formal definition of a boundary triple. An extension $A_\Theta \in \operatorname{Ext}(A_{\min})$ of $A_{\min}$, *i.e.* an operator satisfying $A_{\min} \subseteq A_\Theta \subseteq A_{\max}$, is parametrized by a linear relation Θ in $\mathbb{C}^d$ according to

$$\operatorname{dom} A_\Theta = \{f \in \operatorname{dom} A_{\max} \,|\, \Gamma f \in \Theta\}.$$

In particular, A_Θ is self-adjoint in $\mathcal{H}_{\mathrm{A}}$ iff Θ is self-adjoint in $\mathbb{C}^d$, because the adjoint A^*_Θ in $\mathcal{H}_{\mathrm{A}}$ of A_Θ is given by A_{Θ^*}, where Θ^* is the adjoint in $\mathbb{C}^d$ of Θ. The Krein–Naimark resolvent formula for A_Θ reads ([7, Theorem 2(iii)])

$$(A_\Theta - z)^{-1} = (A_0 - z)^{-1} + \gamma_\Gamma(z)(\Theta - M_\Gamma(z))^{-1}\gamma_\Gamma(\overline{z})^*$$

for $z \in \operatorname{res} A_0 \cap \operatorname{res} A_\Theta$. The self-adjoint operator A_0 corresponds to the self-adjoint linear relation $\{0\} \times \mathbb{C}^d$ in $\mathbb{C}^d$, and its resolvent is given by

$$(A_0 - z)^{-1}(f + k) = (L_m - z)^{-1} f + \sum_\alpha [(\mathfrak{M}_d - z)^{-1} d(k)]_\alpha h_\alpha$$

for $f \in \mathfrak{H}_m$, $k \in \mathfrak{K}_{\mathrm{A}}$, and $z \in \operatorname{res} A_0 = \operatorname{res} L \setminus \{z_1\}$. The γ-field γ_Γ and the Weyl function M_Γ corresponding to $(\mathbb{C}^d, \Gamma_0, \Gamma_1)$ are described by

$$\gamma_\Gamma(z)\mathbb{C}^d = \mathfrak{N}_z(A_{\max}) = \{\sum_\sigma c_\sigma F_\sigma(z) \,|\, c_\sigma \in \mathbb{C}\}, \quad F_\sigma(z) := \frac{g_\sigma(z)}{(z - z_1)^m}$$

and

$$M_\Gamma(z) = q(z) + r(z) \quad \text{on} \quad \mathbb{C}^d$$

for $z \in \operatorname{res} A_0$. The Krein Q-function q of $L_{\min}$ is defined by

$$q(z) = ([q(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [q(z)]_{\sigma\sigma'} := (z - z_1)\langle \varphi_\sigma, (L - z)^{-1} h_{\sigma', m+1}\rangle$$

for $z \in \operatorname{res} L$, and the generalized Nevanlinna function r is defined by

$$r(z) = ([r(z)]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [r(z)]_{\sigma\sigma'} := -\sum_j \frac{[\mathcal{G}_{\mathrm{A}}]_{\sigma m, \sigma' j}}{(z - z_1)^{m-j+1}}$$

for $z \in \mathbb{C} \setminus \{z_1\}$.

The compressed resolvent of A_Θ is represented in the generalized sense according to

$$\begin{aligned} P_{\mathfrak{H}_m}(A_\Theta - z)^{-1}|_{\mathfrak{H}_m} =& (L - z)^{-1} \\ &+ \sum_\sigma [(\Theta - M_\Gamma(z))^{-1}\langle \varphi, (L - z)^{-1} \cdot \rangle]_\sigma (L - z)^{-1} h_{\sigma m} \end{aligned} \tag{2.6}$$

for $z \in \operatorname{res} A_0 \cap \operatorname{res} A_\Theta$. As expected, in the A-model with equal model parameters the spectral properties of singular rank-d perturbations of class $\mathfrak{H}_{-4}$ or higher are described by a generalized Nevanlinna function M_Γ.

3. Extensions which are linear relations

Let $j_\star \in J$; then
$$\mathfrak{H}_m \cap \mathfrak{H}_{-m-2+2j_\star} = \mathfrak{H}_m$$
while
$$\mathfrak{K}_{\mathrm{A}} \cap \mathfrak{H}_{-m-2+2j_\star} = \operatorname{span}\{h_{\sigma j} \,|\, (\sigma, j) \in \mathcal{S} \times \{j_\star, \dots, m\}\}$$
is a $d(m-j_\star+1)$-dimensional linear space. Choosing $j_\star = m$ we therefore construct a d-dimensional subspace $\mathfrak{K}_{\mathrm{A}}^{\min}$ of $\mathfrak{K}_{\mathrm{A}}$, which is minimal in the sense that $\mathfrak{K}_{\mathrm{A}} \cap \mathfrak{H}_{m-1} = \{0\}$. Let
$$\mathcal{H}_{\mathrm{A}}^{\min} := (\mathfrak{H}_m \dotplus \mathfrak{K}_{\mathrm{A}}^{\min}, [\cdot,\cdot]_{\mathrm{A}}).$$
That is, $\mathcal{H}_{\mathrm{A}}^{\min}$ is a subspace of $\mathcal{H}_{\mathrm{A}}$ equipped with an indefinite metric
$$[f + h_m(c), f' + h_m(c')]_{\mathrm{A}} = \langle f, f' \rangle_m + \langle c, \mathcal{G}_{\mathrm{A}}^{\min} c' \rangle_{\mathbb{C}^d}$$
for f, $f' \in \mathfrak{H}_m$ and c, $c' \in \mathbb{C}^d$. The matrix
$$\mathcal{G}_{\mathrm{A}}^{\min} = ([\mathcal{G}_{\mathrm{A}}^{\min}]_{\sigma\sigma'}) \in [\mathbb{C}^d], \quad [\mathcal{G}_{\mathrm{A}}^{\min}]_{\sigma\sigma'} := [\mathcal{G}_{\mathrm{A}}]_{\sigma m, \sigma' m}$$
where, as previously, $\mathcal{G}_{\mathrm{A}}$ is the Gram matrix of the A-model; *i.e.* it is invertible and Hermitian. The matrix $\mathcal{G}_{\mathrm{A}}^{\min}$ is Hermitian, and the space $\mathcal{H}_{\mathrm{A}}^{\min}$ is a Hilbert space iff an Hermitian $\mathcal{G}_{\mathrm{A}}^{\min}$ is positive definite. In this case $\mathcal{H}_{\mathrm{A}}^{\min}$ becomes a subspace of the positive subspace of the Pontryagin space $\mathcal{H}_{\mathrm{A}}$.

LEMMA 3.1. *Let $\mathcal{H}_{\mathrm{A}}^{\perp}$ denote the orthogonal complement in $\mathcal{H}_{\mathrm{A}}$ of $\mathcal{H}_{\mathrm{A}}^{\min}$. Then:*

(i) *$\mathcal{H}_{\mathrm{A}}^{\perp}$ is a subset of $\mathfrak{K}_{\mathrm{A}}$ given by*
$$\mathcal{H}_{\mathrm{A}}^{\perp} = \{k \in \mathfrak{K}_{\mathrm{A}} \,|\, [\mathcal{G}_{\mathrm{A}} d(k)]_m = 0\}.$$

(ii) *Assume that*
$$[\mathcal{G}_{\mathrm{A}}]_{\sigma,m-1;\sigma' m} = [\mathcal{G}_{\mathrm{A}}]_{\sigma m;\sigma',m-1}, \quad \sigma, \sigma' \in \mathcal{S} \tag{3.1}$$
if $m \geqslant 2$. Then
$$(A'_{\max} - A_{\max})\mathfrak{K}_{\mathrm{A}}^{\min} \subseteq \mathcal{H}_{\mathrm{A}}^{\perp}.$$

Recall that $A'_{\max} = A_{\max}$ if $m = 1$.

Proof. (i) $\mathcal{H}_{\mathrm{A}}^{\perp}$ is the set of $g + k \in \mathfrak{H}_m \dotplus \mathfrak{K}_{\mathrm{A}}$ such that $(\forall f \in \mathfrak{H}_m)$ $(\forall c \in \mathbb{C}^d)$
$$0 = \langle f, g \rangle_m + \langle \eta(c), \mathcal{G}_{\mathrm{A}} d(k) \rangle_{\mathbb{C}^{md}} = \langle f, g \rangle_m + \langle c, [\mathcal{G}_{\mathrm{A}} d(k)]_m \rangle_{\mathbb{C}^d};$$
hence such that $g = 0$ and $[\mathcal{G}_{\mathrm{A}} d(k)]_m = 0$.

(ii) We have $(\forall c \in \mathbb{C}^d)$
$$(A'_{\max} - A_{\max})h_m(c) = \widetilde{k}'' \in \mathfrak{K}_{\mathrm{A}}, \quad d(\widetilde{k}'') = (\mathcal{G}_{\mathrm{A}}^{-1}\mathcal{G}_{\mathfrak{M}}^{*} - \mathfrak{M}_d)\eta(c).$$

Then $(\forall\sigma\in\mathcal{S})$

$$\begin{aligned}[\mathcal{G}_{\mathrm{A}}d(\widetilde{k}'')]_{\sigma m} =&[(\mathcal{G}_{\mathfrak{M}}^{*}-\mathcal{G}_{\mathfrak{M}})\eta(c)]_{\sigma m}\\ =&\sum_{\sigma'}([\mathcal{G}_{\mathrm{A}}]_{\sigma,m-1;\sigma'm}-[\mathcal{G}_{\mathrm{A}}]_{\sigma m;\sigma',m-1})c_{\sigma'}\,.\end{aligned}$$

By hypothesis one therefore sees that $\widetilde{k}''\in\mathcal{H}_{\mathrm{A}}^{\perp}$. □

Define a linear relation $B_{\max}$ in a (generally) Pontryagin space $\mathcal{H}_{\mathrm{A}}$ by

$$B_{\max}:=A_{\max}\,|_{\,\mathrm{dom}A_{\max}\cap\mathcal{H}_{\mathrm{A}}^{\min}}\,\widehat{+}\,(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})$$

(the componentwise sum), where

$$A_{\max}\,|_{\,\mathrm{dom}A_{\max}\cap\mathcal{H}_{\mathrm{A}}^{\min}}=A_{\max}\cap(\mathcal{H}_{\mathrm{A}}^{\min}\times\mathcal{H}_{\mathrm{A}})$$

is the domain restriction to $\mathrm{dom}A_{\max}\cap\mathcal{H}_{\mathrm{A}}^{\min}$ of $A_{\max}$. Let also

$$B_{\min}:=B_{\max}^{*}$$

be the adjoint in $\mathcal{H}_{\mathrm{A}}$ of $B_{\max}$.

For $m=1$ we have $\mathcal{H}_{\mathrm{A}}^{\min}=\mathcal{H}_{\mathrm{A}}$ and $\mathcal{H}_{\mathrm{A}}^{\perp}=\{0\}$, corresponding to $\mathfrak{K}_{\mathrm{A}}^{\min}=\mathfrak{K}_{\mathrm{A}}$. In this case $B_{\max}=A_{\max}$ and $B_{\min}=A_{\min}$ are operators. But for $m\geqslant 2$, $B_{\max}$ has a nontrivial multivalued part $\mathrm{mul}\,B_{\max}=\mathcal{H}_{\mathrm{A}}^{\perp}$. The multivalued part of $B_{\min}$ is also $\mathcal{H}_{\mathrm{A}}^{\perp}$, which is seen from $\mathrm{mul}\,B_{\min}=(\mathrm{dom}\,B_{\max})^{\perp}$ and using that $\mathfrak{H}_{m+2}$ is dense in $\mathfrak{H}_{m}$. We have, moreover, the next lemma.

LEMMA 3.2. *Assume* (3.1) *if* $m\geqslant 2$. *Then*

$$B_{\max}=A_{\max}'\,|_{\,\mathrm{dom}A_{\max}\cap\mathcal{H}_{\mathrm{A}}^{\min}}\,\widehat{+}\,(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})$$

and

$$\begin{aligned}B_{\min}=&A_{\min}\,|_{\,\mathrm{dom}A_{\min}\cap\mathcal{H}_{\mathrm{A}}^{\min}}\,\widehat{+}\,(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})\\ =&A_{\min}'\,|_{\,\mathrm{dom}A_{\min}\cap\mathcal{H}_{\mathrm{A}}^{\min}}\,\widehat{+}\,(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})\,.\end{aligned}$$

Moreover, $B_{\min}$ *is a closed symmetric linear relation in* $\mathcal{H}_{\mathrm{A}}$, *whose adjoint in* $\mathcal{H}_{\mathrm{A}}$ *is the linear relation* $B_{\min}^{*}=B_{\max}$.

Proof. For $m=1$ the statements of the lemma follow from Lemma 2.1, so in what follows we let $m\geqslant 2$.

The representation of $B_{\max}$, as stated, is due to Lemma 3.1. The adjoint of $B_{\max}$ is given by (recall *e.g.* [19, Lemma 2.6])

$$B_{\min}=(A_{\max}\,|_{\,\mathrm{dom}A_{\max}\cap\mathcal{H}_{\mathrm{A}}^{\min}})^{*}\cap(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})^{*}$$

with

$$(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})^{*}=\mathcal{H}_{\mathrm{A}}^{\min}\times\mathcal{H}_{\mathrm{A}}\,.$$

Because $A_{\max}$ and $\mathcal{H}_{\mathrm{A}}^{\min}$ are closed, by the same argument we also get that

$$\begin{aligned}(A_{\max}\,|_{\operatorname{dom}A_{\max}\cap\mathcal{H}_{\mathrm{A}}^{\min}})^{*}=&[A_{\max}\cap(\mathcal{H}_{\mathrm{A}}^{\min}\times\mathcal{H}_{\mathrm{A}})]^{*}\\=&\overline{A_{\min}\,\widehat{+}\,(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})}\,.\end{aligned}$$

Because

$$A_{\min}^{*}\,\widehat{+}\,(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})^{*}=A_{\max}\,\widehat{+}\,(\mathcal{H}_{\mathrm{A}}^{\min}\times\mathcal{H}_{\mathrm{A}})=\mathcal{H}_{\mathrm{A}}^{2}$$

is a closed linear relation, we have by [19, Lemma 2.10] that

$$\overline{A_{\min}\,\widehat{+}\,(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})}=A_{\min}\,\widehat{+}\,(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})$$

is also closed. Combining all together we deduce the first representation of $B_{\min}$ as stated in the lemma. By using this representation and noting that $A_{\min}\subseteq A'_{\max}$ and $A'_{\min}\subseteq A_{\max}$ (Lemma 2.1), we deduce also the second formula for $B_{\min}$ by applying Lemma 3.1. The computation of the adjoint $B_{\min}^{*}$ uses the same arguments as that of $B_{\max}^{*}$, and one concludes that $B_{\min}$ is a closed symmetric linear relation. □

The boundary value space of $B_{\min}$ is characterized by the next theorem.

THEOREM 3.3. *Assume* (3.1) *if* $m\geqslant 2$, *and let* $\mathcal{G}_{\mathrm{A}}^{\min}$ *be positive definite. Define the operator* $\Gamma':=(\Gamma'_0,\Gamma'_1)\colon B_{\max}\to\mathbb{C}^{2d}$ *by*

$$\Gamma'_0\widehat{f}:=c\,,\quad \Gamma'_1\widehat{f}:=\langle\varphi,f^{\#}\rangle-\mathcal{G}_{\mathrm{A}}^{\min}\chi$$

for $\widehat{f}=(f,f')\in B_{\max}$*; that is*

$$\begin{aligned}f=&f^{\#}+h_{m+1}(c)+h_m(\chi)\,,\quad f^{\#}\in\mathfrak{H}_{m+2}\,,\quad c,\chi\in\mathbb{C}^d\,,\\f'=&L_mf^{\#}+z_1h_{m+1}(c)+\widetilde{k}+k_{\perp}\,,\quad \widetilde{k}\in\mathfrak{K}_{\mathrm{A}}\,,\quad k_{\perp}\in\mathcal{H}_{\mathrm{A}}^{\perp}\,,\end{aligned}$$

$$d(\widetilde{k})=\mathfrak{M}_d\eta(\chi)+\eta(c)\,.$$

Then $(\mathbb{C}^d,\Gamma'_0,\Gamma'_1)$ *is a boundary triple for* $B_{\max}$*. The corresponding* γ*-field* $\gamma_{\Gamma'}$ *and the Weyl function* $M_{\Gamma'}$ *are bounded analytic operator functions given by*

$$\gamma_{\Gamma'}(z)\mathbb{C}^d=\mathfrak{N}_z(B_{\max})=\{(L-z)^{-1}h_m(c)+h_m(\chi)\,|\,\chi=(z-\hat{\Delta})^{-1}c\,;\,c\in\mathbb{C}^d\}\,,$$

$$\hat{\Delta}:=(\mathcal{G}_{\mathrm{A}}^{\min})^{-1}\Delta\in[\mathbb{C}^d]\,,\quad \Delta=(\Delta_{\sigma\sigma'})=\Delta^{*}\in[\mathbb{C}^d]\,,$$

$$\Delta_{\sigma\sigma'}:=[\mathcal{G}_{\mathfrak{M}}]_{\sigma m,\sigma' m}=z_1[\mathcal{G}_{\mathrm{A}}^{\min}]_{\sigma\sigma'}+1_{\mathbb{N}_{\geqslant 2}}(m)[\mathcal{G}_{\mathrm{A}}]_{\sigma,m-1;\sigma' m}$$

and

$$M_{\Gamma'}(z)=q(z)+\hat{r}(z)\,,\quad \hat{r}(z):=\mathcal{G}_{\mathrm{A}}^{\min}(\hat{\Delta}-z)^{-1}$$

for $z\in\operatorname{res}L\cap\operatorname{res}\hat{\Delta}$*. Moreover,* $M_{\Gamma'}$ *is a uniformly strict Nevanlinna function.*

Proof. Step 1. In this step we argue as in the proof of Lemma 3.2. Consider Γ' as a single-valued linear relation with $\operatorname{dom}\Gamma' = B_{\max}$:

$$\Gamma' = \{(\widehat{f},(\Gamma'_0\widehat{f},\Gamma'_1\widehat{f}))\,|\,\widehat{f}\in B_{\max}\}\,.$$

Likewise, consider Γ as a single-valued linear relation with $\operatorname{dom}\Gamma = A_{\max}$:

$$\Gamma = \{(\widehat{f},(\Gamma_0\widehat{f},\Gamma_1\widehat{f}))\,|\,\widehat{f}\in A_{\max}\}\,.$$

Then by definition

$$\Gamma' = (\Gamma\cap\mathfrak{M})\,\widehat{+}\,\mathfrak{N}\,,\quad \mathfrak{M} := (\mathcal{H}_{\mathrm{A}}^{\min}\times\mathcal{H}_{\mathrm{A}})\times\mathbb{C}^{2d}\,,\quad \mathfrak{N} := (\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp})\times\{0\}\,.$$

Then the Krein space adjoint of Γ' is given by

$$(\Gamma')^{[*]} = (\Gamma\cap\mathfrak{M})^{[*]}\cap\mathfrak{N}^{[*]}\,,\quad \mathfrak{N}^{[*]} = \mathbb{C}^{2d}\times(\mathcal{H}_{\mathrm{A}}^{\min}\times\mathcal{H}_{\mathrm{A}}) = \mathfrak{M}^{-1}\,.$$

Because $\mathfrak{M}$ is a closed linear relation, and so is Γ by Lemma 2.1(i), the Krein space adjoint of $\Gamma\cap\mathfrak{M}$ is given by

$$(\Gamma\cap\mathfrak{M})^{[*]} = \overline{\Gamma^{[*]}\,\widehat{+}\,\mathfrak{M}^{[*]}}\,,\quad \mathfrak{M}^{[*]} = \{0\}\times(\{0\}\times\mathcal{H}_{\mathrm{A}}^{\perp}) = \mathfrak{N}^{-1}\,.$$

Because $\Gamma\,\widehat{+}\,\mathfrak{M} = \mathcal{H}_{\mathrm{A}}^2\times\mathbb{C}^{2d}$, it follows that

$$(\Gamma\cap\mathfrak{M})^{[*]} = \Gamma^{[*]}\,\widehat{+}\,\mathfrak{N}^{-1} = [(\Gamma^{[*]})^{-1}\,\widehat{+}\,\mathfrak{N}]^{-1}$$

and therefore

$$(\Gamma')^{[*]} = [(\Gamma^{[*]})^{-1}\,\widehat{+}\,\mathfrak{N}]^{-1}\cap\mathfrak{M}^{-1} = \{[(\Gamma^{[*]})^{-1}\cap\mathfrak{M}]\,\widehat{+}\,\mathfrak{N}\}^{-1}\,.$$

By applying Lemma 2.1(i) and Lemma 3.1(ii), this leads to $(\Gamma')^{[*]} = (\Gamma')^{-1}$.

Since Γ' is single-valued, unitary, and with closed domain, we conclude that Γ' is surjective, and then the triple $(\mathbb{C}^d,\Gamma'_0,\Gamma'_1)$ is a boundary triple for $B_{\max}$.

Step 2. We compute the eigenspace of $B_{\max}$. For $f\in\mathfrak{N}_z(B_{\max})$, $z\in\mathbb{C}$, we have

$$0 = (L-z)f^{\#} + (z_1-z)h_{m+1}(c)\,,\quad 0 = \widetilde{k} + k_{\perp} - zh_m(\chi)\,.$$

Then, for $z\in\operatorname{res}L$, the first equation leads to

$$f^{\#} = (z-z_1)(L-z)^{-1}h_{m+1}(c) = -h_{m+1}(c) + (L-z)^{-1}h_m(c)$$

The second equation implies that

$$0 = d(\widetilde{k}) + d(k_{\perp}) - z\eta(\chi) \quad\text{or else}\quad d(k_{\perp}) = (z-\mathfrak{M}_d)\eta(\chi) - \eta(c)\,.$$

Because $k_{\perp}\in\mathcal{H}_{\mathrm{A}}^{\perp}$, we have that $[\mathcal{G}_{\mathrm{A}}d(k_{\perp})]_m = 0$; hence

$$0 = \mathcal{G}_{\mathrm{A}}^{\min}(z\chi - c) - [\mathcal{G}_{\mathfrak{M}}\eta(\chi)]_m\,,\quad [\mathcal{G}_{\mathfrak{M}}\eta(\chi)]_m = \Delta\chi\,.$$

Because by hypothesis an Hermitian $\mathcal{G}_{\mathrm{A}}^{\min}$ is positive definite, the latter shows that

$$0=(z-\hat{\Delta})\chi-c \quad \Rightarrow \quad \chi=(z-\hat{\Delta})^{-1}c\,, \quad z\in\operatorname{res}\hat{\Delta}\,.$$

Step 3. By definition $\gamma_{\Gamma'}(z)c=f\in\mathfrak{N}_z(B_{\max})$; thus by Step 2, we get $\gamma_{\Gamma'}(z)$ as claimed. Again by definition $M_{\Gamma'}(z)c=\Gamma_1'(f,zf)$, $f\in\mathfrak{N}_z(B_{\max})$; thus by Step 2, we get $M_{\Gamma'}(z)$ as stated in the lemma.

Step 4. Because q is the Weyl function corresponding to the boundary triple $(\mathbb{C}^d,\mathring{\Gamma}_0,\mathring{\Gamma}_1)$ for the adjoint in $\mathfrak{H}_m$ of $L_{\min}$, where ([23, Corollary 7.4])

$$\mathring{\Gamma}_0(f^{\#}+h_{m+1}(c)):=c\,, \quad \mathring{\Gamma}_1(f^{\#}+h_{m+1}(c)):=\langle\varphi,f^{\#}\rangle\,,$$

we have by *e.g.* [15, Theorem 1.4] that q is a uniformly strict Nevanlinna function.

By hypothesis imposed on $\mathcal{G}_{\mathrm{A}}$, the matrix Δ is Hermitian, so the matrix function $\hat{r}$ is symmetric with respect to the real axis, $\hat{r}(z)^*=\hat{r}(\overline{z})$, $z\in\operatorname{res}\hat{\Delta}$. We prove that $\operatorname{res}\hat{\Delta}\supseteq\mathbb{C}\setminus\mathbb{R}$. Because $\hat{r}$ is analytic on $\operatorname{res}\hat{\Delta}$, and moreover the matrix

$$\frac{\Im\hat{r}(z)}{\Im z}=AB(z)\,, \quad \Im z\neq 0\,,$$

$$A:=(\mathcal{G}_{\mathrm{A}}^{\min})^{-2}>0\,, \quad B(z):=\hat{r}(z)^*(\mathcal{G}_{\mathrm{A}}^{\min})^{-1}\hat{r}(z)>0$$

is similar to the positive definite matrix $B(z)^{1/2}AB(z)^{1/2}$, this would imply that $\hat{r}$ is a uniformly strict Nevanlinna function.

The spectrum of $\hat{\Delta}$ consists of $z\in\mathbb{C}$ such that the determinant $\det(\hat{\Delta}-z)=0$. Because $\hat{\Delta}$ is the product of two Hermitian matrices, using their spectral decompositions we get that z solves $\det(Y-z)=0$, where the matrix $Y:=\Lambda^{-1}X$, Λ is the positive definite diagonal matrix with the eigenvalues of $\mathcal{G}_{\mathrm{A}}^{\min}$ on its diagonal, and X is an Hermitian matrix. Because $Y=\Lambda^{-1/2}Y'\Lambda^{1/2}$ is similar to an Hermitian matrix $Y':=\Lambda^{-1/2}X\Lambda^{-1/2}$, we get that z is an eigenvalue of Y', and hence belongs to $\mathbb{R}$. Consequently, $\operatorname{res}\hat{\Delta}\supseteq\mathbb{C}\setminus\mathbb{R}$ as claimed.

The sum $M_{\Gamma'}$ of two uniformly strict Nevanlinna functions q and $\hat{r}$ is itself of the same class, as can be deduced from [5, Lemma 2.6] [6, Proposition 3.2], and this accomplishes the proof of the theorem. □

Under assumptions of Theorem 3.3, consider Γ' as a (unitary) single-valued linear relation with $\operatorname{dom}\Gamma'=B_{\max}$. According to [2, Theorem 4.8], if Γ' is minimal, *i.e.* if the closed linear span

$$\mathcal{H}_s:=\overline{\operatorname{span}}\{\mathfrak{N}_z(B_{\max})\,|\,z\in\operatorname{reg}B_{\min}\}$$

($\operatorname{reg}B_{\min}$ is the regularity domain of $B_{\min}$; see *e.g.* [1, Eq. (6.14)]) coincides with $\mathcal{H}_{\mathrm{A}}$, then $M_{\Gamma'}$ must be a generalized Nevanlinna function with a generally nontrivial number κ of negative squares (where κ is equal to the rank of indefiniteness of the Pontryagin space $\mathcal{H}_{\mathrm{A}}$). Recall that $\mathcal{H}_s=\mathcal{H}_{\mathrm{A}}$ means also that a closed symmetric linear relation $B_{\min}$ is simple. If, however, Γ' is not minimal, then $M_{\Gamma'}$ is a generalized Nevanlinna function with $\kappa'\leqslant\kappa$ negative squares. By Theorem 3.3 we have $\kappa'=0$, and by the next proposition this corresponds to the fact that Γ' is not a minimal boundary relation for $B_{\max}$ for at least $m\geqslant 2$, unless $\mathcal{H}_{\mathrm{A}}^{\perp}=\{0\}$; if the latter holds then by our hypothesis on $\mathcal{G}_{\mathrm{A}}$ the space $\mathcal{H}_{\mathrm{A}}=\mathcal{H}_{\mathrm{A}}^{\min}$ is a Hilbert space (for all $m\geqslant 1$), and hence $\kappa=0$.

THEOREM 3.4. *Under assumptions of Theorem 3.3,* $\emptyset \neq \mathcal{H}_s \subseteq \mathcal{H}_{\mathrm{A}}^{\min}$*. Moreover, if the only solutions* $f \in \mathfrak{H}_m$ *and* $\chi \in \mathbb{C}^d$ *to*

$$(\forall z \in \mathbb{C} \setminus \mathbb{R})\ \langle \varphi, (L-z)^{-1} f \rangle = \hat{r}(z)\chi \tag{3.2}$$

are $f = 0$ *and* $\chi = 0$*, then* $\mathcal{H}_s = \mathcal{H}_{\mathrm{A}}^{\min}$.

Proof. First we prove the next lemma.

LEMMA 3.5. $(\forall k \in \mathfrak{K}_{\mathrm{A}})\ (\exists \chi \in \mathbb{C}^d)\ (\exists k_\perp \in \mathcal{H}_{\mathrm{A}}^\perp)\ d(k) = \eta(\chi) + d(k_\perp)$.

Proof. Because every $f \in \mathcal{H}_{\mathrm{A}}$ is of the form $f = f' + k_\perp$, for some $f' \in \mathcal{H}_{\mathrm{A}}^{\min}$ and $k_\perp \in \mathcal{H}_{\mathrm{A}}^\perp$, we have that $f' = f'' + h_m(\chi)$, for some $f'' \in \mathfrak{H}_m$ and $\chi \in \mathbb{C}^d$. Choosing $f'' = 0$ the claim follows. □

That $\mathcal{H}_s$ is nonempty follows from the following lemma (recall that $\operatorname{res} L \cap \operatorname{res} \hat{\Delta} \supseteq \mathbb{C} \setminus \mathbb{R}$).

LEMMA 3.6. $\operatorname{reg} B_{\min} \supseteq \operatorname{res} L \cap \operatorname{res} \hat{\Delta}$.

Proof. We show that, for $z \in \operatorname{res} L \cap \operatorname{res} \hat{\Delta}$, the eigenspace $\mathfrak{N}_z(B_{\min}) = \{0\}$ and the range $\operatorname{ran}(B_{\min} - z)$ is closed, from which the statement of the lemma follows.

The linear relation $B_{\min} = \ker \Gamma'$ explicitly reads

$$\begin{aligned} B_{\min} =& \{(f^\# + h_m(\chi), L_m f^\# + \widetilde{k} + k_\perp) \mid f^\# \in \mathfrak{H}_{m+2}; \chi \in \mathbb{C}^d; \\ & k_\perp \in \mathcal{H}_{\mathrm{A}}^\perp; \widetilde{k} \in \mathfrak{K}_{\mathrm{A}}; d(\widetilde{k}) = \mathfrak{M}_d \eta(\chi); \langle \varphi, f^\# \rangle = \mathcal{G}_{\mathrm{A}}^{\min} \chi\}. \end{aligned}$$

Therefore $f \in \mathfrak{N}_z(B_{\min})$ solves

$$0 = (L_m - z) f^\#, \quad 0 = (\hat{\Delta} - z)\chi, \quad \langle \varphi, f^\# \rangle = \mathcal{G}_{\mathrm{A}}^{\min} \chi.$$

Since $z \in \operatorname{res} L_m = \operatorname{res} L$, this leads to $f = 0$.

By applying Lemma 3.5 $\widetilde{k} = h_m(\hat{\Delta}\chi) + k'_\perp$, $k'_\perp \in \mathcal{H}_{\mathrm{A}}^\perp$. Therefore the range

$$\begin{aligned} \operatorname{ran}(B_{\min} - z) =& \{(L_m - z) f^\# + h_m((\hat{\Delta} - z)\chi) + k_\perp \mid f^\# \in \mathfrak{H}_{m+2}; \chi \in \mathbb{C}^d; \\ & k_\perp \in \mathcal{H}_{\mathrm{A}}^\perp; \langle \varphi, f^\# \rangle = \mathcal{G}_{\mathrm{A}}^{\min} \chi\} \quad (z \in \mathbb{C}) \\ =& \{(L_m - z) f^\# + h_m(\chi) + k_\perp \mid f^\# \in \mathfrak{H}_{m+2}; \chi \in \mathbb{C}^d; \\ & k_\perp \in \mathcal{H}_{\mathrm{A}}^\perp; \langle \varphi, f^\# \rangle = \hat{r}(z)\chi\} \quad (z \in \operatorname{res} \hat{\Delta}). \end{aligned}$$

On the other hand, the closure $\overline{\operatorname{ran}}(B_{\min} - z)$, $\overline{z} \in \operatorname{res} L \cap \operatorname{res} \hat{\Delta}$, is the orthogonal complement in $\mathcal{H}_{\mathrm{A}}$ of $\mathfrak{N}_{\overline{z}}(B_{\max})$; hence

$$\begin{aligned} \overline{\operatorname{ran}}(B_{\min} - z) =& \{f + k \in \mathfrak{H}_m \dotplus \mathfrak{K}_{\mathrm{A}} \mid (\forall c \in \mathbb{C}^d) \\ & 0 = \langle f, (L - \overline{z})^{-1} h_m(c) \rangle_m + \langle d(k), \mathcal{G}_{\mathrm{A}} \eta(\chi) \rangle_{\mathbb{C}^{md}}; \\ & \chi = (\overline{z} - \hat{\Delta})^{-1} c\}. \end{aligned}$$

Note that $\operatorname{res} L \cap \operatorname{res} \hat{\Delta} \supseteq \mathbb{C} \setminus \mathbb{R}$ implies that also $z \in \operatorname{res} L \cap \operatorname{res} \hat{\Delta}$.

We have

$$\langle f, (L-\overline{z})^{-1} h_m(c) \rangle_m = \langle \langle \varphi, (L-z)^{-1} f \rangle, c \rangle_{\mathbb{C}^d} ,$$

$$\begin{aligned} \langle d(k), \mathcal{G}_{\mathrm{A}} \eta(\chi) \rangle_{\mathbb{C}^{md}} &= \langle (z-\hat{\Delta}^*)^{-1} [\mathcal{G}_{\mathrm{A}} d(k)]_m, c \rangle_{\mathbb{C}^d} \\ &= - \langle \hat{r}(z) (\mathcal{G}_{\mathrm{A}}^{\min})^{-1} [\mathcal{G}_{\mathrm{A}} d(k)]_m, c \rangle_{\mathbb{C}^d} . \end{aligned}$$

Putting $f^\# := (L-z)^{-1} f \in \mathfrak{H}_{m+2}$ and applying Lemma 3.5, *i.e.*

$$\begin{aligned} d(k) =& \eta(\chi') + k''_\perp , \quad k''_\perp \in \mathcal{H}_{\mathrm{A}}^\perp , \quad \chi' := (\mathcal{G}_{\mathrm{A}}^{\min})^{-1} [\mathcal{G}_{\mathrm{A}} d(k)]_m \\ \Rightarrow & [\mathcal{G}_{\mathrm{A}} d(k)]_m = \mathcal{G}_{\mathrm{A}}^{\min} \chi' , \end{aligned}$$

we deduce that

$$\begin{aligned} \overline{\operatorname{ran}}(B_{\min} - z) =& \{ (L_m - z) f^\# + h_m(\chi) + k_\perp \,|\, f^\# \in \mathfrak{H}_{m+2} ; \chi \in \mathbb{C}^d ; \\ & k_\perp \in \mathcal{H}_{\mathrm{A}}^\perp ; \langle \varphi, f^\# \rangle = \hat{r}(z) \chi \} = \operatorname{ran}(B_{\min} - z) \end{aligned}$$

for $z \in \operatorname{res} L \cap \operatorname{res} \hat{\Delta}$. We remark that the functional

$$\Phi(\cdot) := (\mathcal{G}_{\mathrm{A}}^{\min})^{-1} [\mathcal{G}_{\mathrm{A}} d(\cdot)]_m \colon \mathfrak{K}_{\mathrm{A}} \to \mathbb{C}^d$$

is surjective, and that therefore $\chi' = \Phi(k)$ ranges over all $\mathbb{C}^d$ whenever k ranges over all $\mathfrak{K}_{\mathrm{A}}$. This accomplishes the proof of the lemma. □

Because $\mathfrak{N}_z(B_{\max}) \subseteq \mathcal{H}_{\mathrm{A}}^{\min}$, $z \in \mathbb{C}$, and because $\mathbb{C} \setminus \mathbb{R} \subseteq \operatorname{res} L \cap \operatorname{res} \hat{\Delta}$, it follows that

$$\mathring{\mathcal{H}}_s := \overline{\operatorname{span}} \{ \mathfrak{N}_z(B_{\max}) \,|\, z \in \mathbb{C} \setminus \mathbb{R} \} \subseteq \mathcal{H}_s \subseteq \mathcal{H}_{\mathrm{A}}^{\min} .$$

By the proof of Lemma 3.6, the orthogonal complement $\mathring{\mathcal{H}}_s^\perp$ in $\mathcal{H}_{\mathrm{A}}$ of $\mathring{\mathcal{H}}_s$ is given by

$$\mathring{\mathcal{H}}_s^\perp = \bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \operatorname{ran}(B_{\min} - z) = X [\dot{+}] \mathcal{H}_{\mathrm{A}}^\perp$$

where the subset $X \subseteq \mathcal{H}_{\mathrm{A}}^{\min}$ is defined by

$$X := \{ f + h_m(\chi) \in \mathfrak{H}_m \dot{+} \mathfrak{K}_{\mathrm{A}}^{\min} \,|\, (\forall z \in \mathbb{C} \setminus \mathbb{R}) \, \langle \varphi, (L-z)^{-1} f \rangle = \hat{r}(z) \chi \}$$

and $[\dot{+}]$ indicates the direct sum which is orthogonal with respect to the $\mathcal{H}_{\mathrm{A}}$-metric $[\cdot,\cdot]_{\mathrm{A}}$. If $X = \{0\}$, *i.e.* if (3.2) has the only solutions $f = 0$, $\chi = 0$, then $\mathring{\mathcal{H}}_s^\perp = \mathcal{H}_{\mathrm{A}}^\perp$ implies $\mathring{\mathcal{H}}_s = \mathcal{H}_s = \mathcal{H}_{\mathrm{A}}^{\min}$. □

Assuming the hypotheses in Theorem 3.3, an extension $B_\Theta \in \operatorname{Ext}(B_{\min})$ parametrized by a linear relation Θ in $\mathbb{C}^d$ is defined by

$$B_\Theta := \{ \widehat{f} \in B_{\max} \,|\, \Gamma' \widehat{f} \in \Theta \} .$$

The Krein-Naimark resolvent formula for B_Θ is given by (*cf.* [15, Theorem 4.12])

$$(B_\Theta - z)^{-1} = (B_0 - z)^{-1} + \gamma_{\Gamma'}(z) (\Theta - M_{\Gamma'}(z))^{-1} \gamma_{\Gamma'}(\overline{z})^* , \quad z \in \operatorname{res} B_0 \cap \operatorname{res} B_\Theta$$

with $\gamma_{\Gamma'}(\overline{z})^* = \Gamma'_1 (B_0 - z)^{-1}$. The self-adjoint extension $B_0 := \ker \Gamma'_0$ corresponds to the self-adjoint linear relation $\Theta = \{0\} \times \mathbb{C}^d$. The resolvent of B_0 is presented below.

PROPOSITION 3.7. *Assuming the hypotheses in Theorem 3.3 we have*

$$(B_0 - z)^{-1}(f+k) = (L_m - z)^{-1} f + h_m((\hat{\Delta} - z)^{-1}\Phi(k))$$

for $f \in \mathfrak{H}_m$, $k \in \mathfrak{K}_A$, *and* $z \in \operatorname{res} B_0 = \operatorname{res} L \cap \operatorname{res} \hat{\Delta}$.

Proof. By applying Lemma 3.5

$$B_0 = \{(f^{\#} + h_m(\chi), L_m f^{\#} + h_m(\hat{\Delta}\chi) + k_\perp) \,|\, f^{\#} \in \mathfrak{H}_{m+2}\,;\, \chi \in \mathbb{C}^d\,;\, k_\perp \in \mathcal{H}_A^\perp\}\,.$$

Thus the eigenspace

$$\mathfrak{N}_z(B_0) = \mathfrak{N}_z(L_m) \dot{+} h_m(\mathfrak{N}_z(\hat{\Delta}))\,, \quad z \in \mathbb{C}\,.$$

From here we see that the point spectrum

$$\sigma_p(B_0) = \sigma_p(L) \cup \sigma_p(\hat{\Delta})\,.$$

Then for $z \notin \sigma_p(B_0)$, the operator

$$\begin{aligned}(B_0 - z)^{-1} = &\{(f + h_m(\chi) + k_\perp, (L_m - z)^{-1} f + h_m((\hat{\Delta} - z)^{-1}\chi)) \,| \\ & f \in \operatorname{ran}(L_m - z)\,;\, \chi \in \mathbb{C}^d\,;\, k_\perp \in \mathcal{H}_A^\perp\}\end{aligned}$$

and it therefore follows that $\operatorname{res} B_0 = \operatorname{res} L \cap \operatorname{res} \hat{\Delta}$. Putting $k := h_m(\chi) + k_\perp$ we have that $\chi = \Phi(k)$, and this leads to the resolvent formula as stated. □

In view of Proposition 3.7, the compressed resolvent $P_{\mathfrak{H}_m}(B_\Theta - z)^{-1}|_{\mathfrak{H}_m}$ is given for $z \in \operatorname{res} B_0 \cap \operatorname{res} B_\Theta$ by the right hand side of (2.6), but where now M_Γ is replaced by $M_{\Gamma'}$.

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(Received August 19, 2020)

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