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PATH–CONNECTED CLOSURE OF UNITARY ORBITS

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Dedicated to Lyra

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Abstract. If $\mathscr{A}$ and $\mathscr{B}$ are unital C*-algebras and $\pi : \mathscr{A} \to \mathscr{B}$ is a unital $*$-homomorphism, then $\mathscr{U}_{\mathscr{B}}(\pi)^{-}$ is the set of all $*$-homomorphisms from $\mathscr{A}$ to $\mathscr{B}$ that are approximately (unitarily) equivalent to π. We address the question of when $\mathscr{U}_{\mathscr{B}}(\pi)^{-}$ is path-connected with respect to the topology of pointwise norm convergence. When $\mathscr{A}$ is singly generated and $\mathscr{B} = B\left(\ell^2\right)$, an affirmative answer was given in [4]; we extend this to the case when $\mathscr{A}$ is separable. We also give an affirmative answer when $\mathscr{B}$ is a von Neumann algebra and $\mathscr{A}$ is AF or homogeneous; if $\mathscr{B}$ is finite $\mathscr{A}$ need only be ASH.

1. Introduction

In [4] D. Hadwin proved that the norm closure of the unitary orbit of an operator in $B\left(\ell^2\right)$ is path-connected. In this paper we address the problem of extending this result to representations of separable C*-algebras.

Throughout this paper $\mathscr{A}$ will be a unital separable C*-algebra. If $\mathscr{B}$ is a unital C*-algebra, we define $\operatorname{Rep}(\mathscr{A},\mathscr{B})$ as the set of all unital $*$-homomorphisms from $\mathscr{A}$ to $\mathscr{B}$ with the topology of pointwise norm convergence. Suppose $\{a_1, a_2, \ldots\}$ is a norm dense subset of the closed unit ball of $\mathscr{A}$. We define a metric $d = d_{\mathscr{A},\mathscr{B}}$ by

$$d(\pi,\rho) = \sum_{n=1}^{\infty} \frac{1}{2^n} \|\pi(a_n) - \rho(a_n)\| .$$

Clearly, d makes $\operatorname{Rep}(\mathscr{A},\mathscr{B})$ into a complete metric space. When $\mathscr{B}$ is finite-dimensional, $\operatorname{Rep}(\mathscr{A},\mathscr{B})$ is compact.

Let $\mathscr{U}_{\mathscr{B}}$ denote the group of unitary elements of $\mathscr{B}$. If $\pi \in \operatorname{Rep}(\mathscr{A},\mathscr{B})$, we define the *unitary orbit* $\mathscr{U}_{\mathscr{B}}(\pi)$ of π by

$$\mathscr{U}_{\mathscr{B}}(\pi) = \{U^*\pi(\cdot)U : U \in \mathscr{U}_B\}.$$

If $T \in \mathscr{B}$ we define the unitary orbit $\mathscr{U}_{\mathscr{B}}(T)$ of T by

$$\mathscr{U}_{\mathscr{B}}(T) = \{U^*TU : U \in \mathscr{U}_{\mathscr{B}}\}.$$

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It is clear that $\mathscr{U}_{\mathscr{B}}(T)$ corresponds to $\mathscr{U}_{\mathscr{B}}(\pi)$ when π is the identity representation of the identity representation of $C^*(T)$.

In this paper we address the problem of when $\mathscr{U}_{\mathscr{B}}(\pi)^-$ is path-connected in $\mathrm{Rep}(\mathscr{A},\mathscr{B})$. In Section 2 we discuss special paths in $\mathscr{U}_{\mathscr{B}}(\pi)^-$. In Section 3 we provide an affirmative answer (Theorem 3) for the case when $\mathscr{A}$ is separable and $\mathscr{B} = B(\ell^2)$. We reduce the separable case to the singly generated case by tensoring with the algebra $\mathscr{K}(\ell^2)$ of compact operators on ℓ^2. In Section 4 we give an affirmative answer (Theorem 5) when $\mathscr{A}$ is AF and $\mathscr{B}$ has the property that $\mathscr{U}_{p\mathscr{B}p}$ is connected for every projection $p \in \mathscr{B}$. We also give an affirmative answer (Theorem 6) when there is an *LF* C*-algebra $\mathscr{D}$ such that $\mathscr{A} \subset \mathscr{D} \subset \mathscr{A}^{\#\#}$, and $\mathscr{B}$ is an arbitrary finite von Neumann algebra. In section 5 we give an affirmative answer (Theorem 7) when $\mathscr{A}$ is abelian (or homogeneous) and $\mathscr{B}$ is an arbitrary von Neumann algebra.

2. Connectedness of $\mathscr{U}_{\mathscr{B}}$ and special paths

An *internal path* in $\mathscr{U}_{\mathscr{B}}(\pi)^-$ joining π to ρ is a continuous map $\gamma\colon [0,1] \to \mathscr{U}_{\mathscr{B}}(\pi)^-$ such that $\gamma(0) = \pi$, $\gamma(1) = \rho$ and $\gamma(t) \in \mathscr{U}_{\mathscr{B}}(\pi)$ whenever $0 \leqslant t < 1$. A *strong internal path* from π to $\rho \in \mathscr{U}_{\mathscr{B}}(\pi)^-$ is a continuous map $\gamma\colon [0,1) \to \mathscr{U}_{\mathscr{B}}$ such that

$$\lim_{t\to 1^-} \gamma(t)^* \pi() \gamma(t) = \rho.$$

In [4, Theorem 3.9] the first author proved that $\mathscr{U}_{\mathscr{B}}(T)^-$ is always path connected when $\mathscr{B} = B(\ell^2)$. Actually a slightly stronger result was proved.

THEOREM 1. [4, Theorem 3.9] *Suppose $X \in B(\ell^2)$ and $Y \in \mathscr{U}_{B(\ell^2)}(X)^-$. Then there is a W such that*

1. *W is unitarily equivalent to $W \oplus W \oplus \cdots$,*
2. *$X \oplus W$ is unitarily equivalent to $Y \oplus W$,*
3. *If $C \in B(\ell^2)$ is unitarily equivalent to $X \oplus W$, then*
   - *(a) $C \in \mathscr{U}_{B(\ell^2)}(X)^- = \mathscr{U}_{B(\ell^2)}(Y)^-$,*
   - *(b) there is a strong internal path in $\mathscr{U}_{B(\ell^2)}(X)^-$ from X to C, and*
   - *(c) there is a strong internal path in $\mathscr{U}_{B(\ell^2)}(Y)^-$ from Y to C.*

There is no reason, a priori, that $\mathscr{U}_{\mathscr{B}}(\pi)$ is even connected. It is well-known that if P and Q are projections in a unital C*-algebra $\mathscr{B}$ and $\|P-Q\| < 1$, then P and Q are unitarily equivalent [8]. It was proved in [3] that two unital representations π, ρ of a finite-dimensional C*-algebra $\mathscr{A}$ are unitarily equivalent if and only if $\pi(p)$ is unitarily equivalent to $\rho(p)$ for every minimal projection $p \in \mathscr{A}$.

If $\mathscr{U}_{\mathscr{B}}$ is connected, then every $\mathscr{U}_{\mathscr{B}}(\pi)$ must be connected. If $x \in \mathscr{U}_{\mathscr{B}}$ and $\|1-x\| < 1$, then $(-\infty,0] \cap \sigma(x) = \varnothing$, so $A(x) = -i\log(x) \in \mathscr{B}$, $A(x) = A(x)^*$, and $x = e^{iA(x)}$. (Here log represents the principal branch of the logarithm.) Since $t \mapsto e^{i(1-t)A(x)}$ is a path in $\mathscr{U}_{\mathscr{B}}$ from x to 1, we see that $\{x \in \mathscr{U}_{\mathscr{B}} : \|1-x\| < 1\}$ is contained in the path component W of 1 in $\mathscr{U}_{\mathscr{B}}$. Since $W = \cup uW$ such that $u \in W$, we see that W is open in $\mathscr{U}_{\mathscr{B}}$. Thus $\mathscr{U}_{\mathscr{B}}$ is connected if and only if it is path-connected. This means that if $\mathscr{U}_{\mathscr{B}}$ is connected, then $\mathscr{U}_{\mathscr{B}}(\pi)$ is path-connected.

LEMMA 1. *If $\mathscr{A}$ is finite-dimensional, then for every $\mathscr{B}$ and every $\pi \in Rep(\mathscr{A}, \mathscr{B})$, $\mathscr{U}_{\mathscr{B}}(\pi)$ is closed.*

Proof. It follows from [3, Theorem 2 (4)] that if $\rho \in \mathscr{U}_{\mathscr{B}}(\pi)^{-}$, then $\rho \in \mathscr{U}_{\mathscr{B}}(\pi)$. $\square$

EXAMPLE 1. *B. Blackadar* [1, 4.4] *showed that in $\mathscr{B} = \mathbb{M}_2(C(S^3))$ there are two projections P, Q that are unitarily equivalent, but are not homotopy equivalent. Thus $\mathscr{U}_{\mathscr{B}}(P) = \mathscr{U}_{\mathscr{B}}(P)^{-}$ is not path-connected. This implies that $\mathscr{U}_{\mathscr{B}}$ is not connected.*

We say that a unital C*-algebra $\mathscr{B}$ has *property UC* if $\mathscr{U}_{\mathscr{B}}$ is connected. The algebra $\mathscr{B}$ has *property HUC* if, for every projection $P \in \mathscr{B}$, $P\mathscr{B}P$ has property UC. We say that $\mathscr{B}$ is *matricially stable* if and only if, for every $n \in \mathbb{N}$, $\mathscr{B}$ is isomorphic to $\mathbb{M}_n(\mathscr{B})$.

LEMMA 2. *The following are true:*

1. *Every von Neumann algebra has property HUC.*
2. *A direct limit of unital C*-algebras with property HUC has property HUC.*
3. *Every unital AF algebra has property HUC.*
4. *If $\mathscr{A}$ is a unital C*-algebra and, for every $n \in \mathbb{N}$, $\mathbb{M}_n(\mathscr{A})$ has property UC, then $K_1(\mathscr{A}) = 0$.*
5. *If $\mathscr{B}$ is matricially stable, then $\mathscr{B}$ has property UC if and only if $K_1(\mathscr{B}) = 0$.*

Proof. (1). In a von Neumann algebra $\mathscr{A}$ every unitary U can be written $U = e^{iA}$ with $A = A^*$, and the path $g(t) = e^{(1-t)iA}$ connects U to 1 in $\mathscr{U}_{\mathscr{A}}$. Thus $\mathscr{A}$ has property UC. But $P\mathscr{A}P$ is a von Neumann algebra for every projection $P \in \mathscr{A}$. Thus $\mathscr{A}$ has property HUC.

(2). Suppose $\{\mathscr{A}_\lambda : \lambda \in \Lambda\}$ is an increasingly directed family of unital C*-subalgebras of a unital C*-subalgebra $\mathscr{A}$ with property UC, and $\mathscr{A} = [\cup_{\lambda \in \Lambda} \mathscr{A}_\lambda]^{-}$. Let E be the connected component of $\mathscr{U}_{\mathscr{A}}$ that contains 1. Suppose $U \in \mathscr{U}_{\mathscr{A}}$ and $\varepsilon > 0$. Then there is a $\lambda \in \Lambda$ and a unitary $V \in \mathscr{A}_\lambda$ such that $\|U - V\| < \varepsilon$. Since $\mathscr{A}_\lambda$ has property UC, there is a path in $\mathscr{U}_{\mathscr{A}_\lambda}$ joining V to 1, implying $V \in E$. Since E is closed, we see that $U \in E$.

Next suppose each $\mathscr{A}_\lambda$ has property HUC and $P \in \mathscr{A}$ is a projection. Then there is a $\lambda_0 \in \Lambda$ and a projection $Q \in \mathscr{A}_{\lambda_0}$ such that $\|P - Q\| < 1$, which implies there is a unitary $W \in \mathscr{A}$ such that $P = W^* Q W$. Hence

$$P\mathscr{A}P = W^* Q W \mathscr{A} W^* Q W = W^* (Q\mathscr{A}Q) W.$$

Thus $P\mathscr{A}P$ is isomorphic to

$$Q\mathscr{A}Q = \left[\cup_{\lambda \geqslant \lambda_0} Q\mathscr{A}_\lambda Q\right]^-.$$

We see, by the previous paragraph, that $P\mathscr{A}P$ has property UC. Thus $\mathscr{A}$ has property HUC.

(3). This follows from (1) and (2).

(4). This follows from the definition of $K_1(\mathscr{A})$.

(5). This follows from (4). $\square$

3. $B(\ell^2)$

In this section we extend Theorem 1 to the case where the single operator is replaced with a representation of a separable C*-algebra. The key idea is a result of C. Olsen and W. Zame [7] that if $\mathscr{A}$ is a separable C*-algebra, then $\mathscr{A} \otimes \mathscr{K}(\ell^2)$ is singly generated. This gives us a general technique for relating the separable case to the singly generated case.

Suppose $\mathscr{A}$ is a unital C*-algebra. Let $\mathscr{A}^\dagger$ denote the unitization of $\mathscr{A} \otimes \mathscr{K}(\ell^2)$. If $\pi \in \mathrm{Rep}(\mathscr{A}, \mathscr{B})$ we define $\pi^\dagger : \mathscr{A}^\dagger \to \mathscr{B}^\dagger$ by

$$\pi^\dagger(\lambda 1 + (a_{ij})) = \lambda 1 + (\pi(a_{ij})).$$

Let $\mathscr{B}^{\maltese}$ be the C*-algebra generated by $\mathscr{B}^\dagger$ and $\{diag(a, a, \ldots) : a \in \mathscr{A}\}$.

THEOREM 2. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are unital C*-algebras and $\pi, \rho \in Rep(\mathscr{A}, \mathscr{B})$. Then*

1. *The map $\rho \mapsto \rho^\dagger$ from $Rep(\mathscr{A}, \mathscr{B})$ to $Rep(\mathscr{A}^\dagger, \mathscr{B}^\dagger)$ is continuous.*
2. *If $\pi, \rho \in Rep(\mathscr{A}, \mathscr{B})$, then*

$$\rho \in \mathscr{U}_{\mathscr{B}}(\pi)^- \textit{ if and only if } \rho^\dagger \in \mathscr{U}_{\mathscr{B}^\dagger}(\pi^\dagger)^-.$$

3. *If $\rho \in \mathscr{U}_{\mathscr{B}}(\pi)^-$ and there is an internal path in $\mathscr{U}(\pi)^-$ joining π to ρ, then there is an internal path in $\mathscr{U}_{\mathscr{B}^{\maltese}}(\pi^\dagger)^-$ joining $\pi^\dagger$ to $\rho^\dagger$.*
4. *If*
   (a) *$\mathscr{B}^\dagger \subset \mathscr{E}$ and $\mathscr{E}$ is a C*-algebra with $e_{11}\mathscr{E}e_{11} = e_{11}\mathscr{B}^\dagger e_{11}$,*
   (b) *$\rho_1 \in \mathscr{U}_{\mathscr{E}}(\pi^\dagger)^-$,*

(c) For every $a \in \mathscr{A}$,

$$\rho_1 (diag(a,0,0,\ldots)) = diag(\rho(a),0,0,\ldots)$$

(d) $\mathscr{U}_{\mathscr{B}}$ *is connected, and*

(e) there is a strong internal path in $\mathscr{U}_{\mathscr{E}}\left(\pi^{\dagger}\right)^{-}$ *from* $\pi^{\dagger}$ *to* ρ_1,

then there is a strong internal path in $\mathscr{U}_{\mathscr{B}}(\pi)^{-}$ *from* π *to* ρ.

Proof. (1). This is obvious.

(2). Suppose $\rho \in \mathscr{U}_{\mathscr{B}}(\pi)^{-}$. Then there is a sequence $\{U_n\}$ in $\mathscr{U}_{\mathscr{B}}$ such that, for every $a \in \mathscr{A}$,

$$\lim_{n\to\infty} \|U_n \pi(a) U_n^* - \rho(a)\| = 0.$$

For each positive integer n, let $W_n = diag(U_n,\ldots,U_n,1,1,1,\ldots)$ in $\mathscr{B}^{\dagger}$ (with U_n repeated n times). Since

$$\left\{T \in \mathscr{A}^{\dagger} : \lim_{n\to\infty} \|W_n \pi(T) W_n^* - \rho(T)\| = 0\right\}$$

is a unital subalgebra containing the operators $(A_{ij}) \in \mathscr{A}^{\dagger}$ such that,

$$\{(i,j) \in \mathbb{N}\times\mathbb{N} : (i,j) \neq (0,0)\}$$

is finite, we see that $\rho^{\dagger} \in \mathscr{U}_{\mathscr{B}^{\dagger}}\left(\pi^{\dagger}\right)^{-}$.

Conversely, suppose $\rho^{\dagger} \in \mathscr{U}_{\mathscr{B}^{\dagger}}\left(\pi^{\dagger}\right)^{-}$. Then there is a sequence $\{V_n\}$ in $\mathscr{B}^{\dagger}$ such that, for every $T \in \mathscr{A}^{\dagger}$,

$$\lim_{n\to\infty} \left\|V_n \pi^{\dagger}(T) V_n^* - \rho^{\dagger}(T)\right\| = 0.$$

Since $\pi^{\dagger}(e_{11}) = \rho^{\dagger}(e_{11}) = e_{11}$, we see that

$$\lim_{n\to\infty} \|V_n e_{11} - e_{11} V_n\| = \lim_{n\to\infty} \left\|V_n \pi^{\dagger}(e_{11}) V_n^* - \rho^{\dagger}(e_{11})\right\| = 0.$$

Hence

$$\left\|V_n - \left[(e_{11} V_n e_{11}) + e_{11}^{\perp} V_n e_{11}^{\perp}\right]\right\| \to 0.$$

Since V_n is unitary,

$$\lim_{n\to\infty} \left\|(e_{11} V_n e_{11})^* (e_{11} V_n e_{11}) - e_{11}\right\|$$

$$= \lim_{n\to\infty} \left\|(e_{11} V_n e_{11})(e_{11} V_n e_{11})^* - e_{11}\right\| = 0.$$

This implies that, eventually $e_{11} V_n e_{11}$ is invertible in $e_{11}\mathscr{B}^{\dagger} e_{11}$. Thus there is a sequence $\{W_n\}$ in $\mathscr{U}_{\mathscr{B}}$, namely (for sufficiently large n),

$$W_n = (e_{11} V_n e_{11}) \left[(e_{11} V_n e_{11})(e_{11} V_n e_{11})^*\right]^{-1/2},$$

such that

$$\lim_{n\to\infty}\left\|W_n - e_{11}V_n e_{11}|_{\mathrm{ran}(e_{11})}\right\| = 0.$$

Thus, for every $a\in\mathscr{A}$,

$$\lim_{n\to\infty}\|W_n\pi(a)W_n^* - \rho(a)\| = 0.$$

Thus $\rho\in\mathscr{U}_{\mathscr{B}}(\pi)^-$.

(3). Suppose there is an internal path $\gamma:[0,1]\to\mathscr{U}(\pi)^-$ joining π to ρ. For $0\leqslant t<1$ write $\gamma(t)=U_t\pi()U_t^*$ with $U_t\in\mathscr{U}_{\mathscr{B}}$. For each $0\leqslant t<1$ let $V_t=diag(U_t,U_t,\ldots)\in\mathscr{U}_{\mathscr{B}^\maltese}$ and let $\Gamma(t)=V_t\pi^\dagger()V_t^*$. Then, for every $T\in\mathscr{A}^\dagger$,

$$\lim_{t\to1^-}\left\|V_t\pi^\dagger(T)V_t^* - \rho^\dagger(T)\right\| = 0.$$

(4). Suppose $\Gamma:[0,1)\to\mathscr{U}_{\mathscr{E}}$ is continuous, and, for every $T\in\mathscr{A}^\dagger$,

$$\lim_{t\to1^-}\left\|\Gamma(t)\pi^\dagger(T)\Gamma(t)^* - \rho_1(T)\right\| = 0.$$

Since $\rho_1(e_{11})=\rho^\dagger(e_{11})=e_{11}$, we conclude that

$$\lim_{t\to1^-}\|\Gamma(t)e_{11}-e_{11}\Gamma(t)\| = \lim_{t\to1^-}\left\|\Gamma(t)\pi^\dagger(e_{11})\Gamma(t)^* - \rho_1(e_{11})\right\| = 0.$$

Since $\Gamma(t)$ is unitary, there is a $t_0\in[0,1)$ such that, whenever $t_0\leqslant t<1$, we have $C_t=e_{11}\Gamma(t)e_{11}$ is invertible in $\mathscr{B}$ and if

$$U_t = C_t\left[C_t^*C_t\right]^{-1/2},$$

then $U_t\in\mathscr{U}_{\mathscr{B}}$ and

$$\lim_{t\to1^-}\|C_t-U_t\| = 0.$$

Since $\mathscr{U}_{\mathscr{A}}$ is connected, there is a continuous map $t\mapsto U_t\in\mathscr{U}_{\mathscr{A}}$ for $0\leqslant t\leqslant t_0$ so that $U_0=1$. If, for every $a\in\mathscr{A}$, we consider $T_a=diag(a,0,0,\ldots)$, it is easily seen that

$$\lim_{t\to1^-}\|U_t\pi(a)U_t^* - \rho(a)\| = 0. \quad \square$$

THEOREM 3. *Suppose $\mathscr{A}$ is a separable unital C*-algebra and $\pi\in Rep\left(\mathscr{A},B\left(\ell^2\right)\right)$. Then $\mathscr{U}_{B(\ell^2)}(\pi)^-$ is path-connected.*

Proof. Suppose $\rho\in\mathscr{U}_{B(\ell^2)}(\pi)^-$. Then, by Theorem 2, $\rho^\dagger\in\mathscr{U}_{B(\ell^2)^\dagger}\left(\pi^\dagger\right)^-$. But $B\left(\ell^2\right)^\dagger\subset B\left(\ell^2\oplus\ell^2\oplus\cdots\right)=\mathscr{E}$. Also, by [7] there is an operator $T\in\mathscr{A}^\dagger$ such that $\mathscr{A}^\dagger=C^*(T)$. Thus $\rho(T)\in\mathscr{U}_{\mathscr{E}}(\pi(T))^-$. Apply Theorem 1 to $X=\pi^\dagger(T)$ and $Y=\rho^\dagger(T)$ to find W in $\mathscr{E}$ and a strong internal paths from $\pi^\dagger(T)\oplus W$ in $\mathscr{U}_{\mathscr{E}}(\pi(T))^-$ and in $\mathscr{U}_{\mathscr{E}}(\rho(T))^-$ from $\rho^\dagger(T)$ to $\pi^\dagger(T)\oplus W$. There is a representation δ_0 of $C^*(T)$ such that $\delta_0(T)=W$, and if $\delta(A)=A\oplus\delta_0(A)$, we have $\delta(T)=T\oplus W$. Since e_{11}

and $\delta(e_{11}) = e_{11} \oplus \delta_0(e_{11})$ are projections with infinite rank and infinite corank, there is a unitary operator V such that $V^*\delta(e_{11})V = e_{11}$ and $V^*TV \in \mathscr{E}$. Let $C = V^*\delta(T)V$ and $\rho_1() = V^*\delta()V$. It follows that there is a $\sigma \in \mathrm{Rep}\left(\mathscr{A}, B\left(\ell^2\right)\right)$ such that, for every $a \in \mathscr{A}$,

$$\rho_1(diag(a,0,0,\cdots)) = diag(\sigma(a),0,0,\cdots).$$

Since there is an internal path in $\mathscr{U}_{\mathscr{E}}\left(\pi^\dagger(T)\right)^-$ from $\pi^\dagger(T)$ to $\rho_1(T)$, there is a strong internal path in $\mathscr{U}_{\mathscr{E}}\left(\pi^\dagger\right)^-$ from $\pi^\dagger$ to ρ_1. It follows from part (4) of Theorem 2 that there is a strong internal path in $\mathscr{U}_{B\left(\ell^2\right)}(\pi)^-$ from π to σ. Similarly, there is a strong internal path in $\mathscr{U}_{B\left(\ell^2\right)}(\rho)^-$ from ρ to σ. Thus there is a path in $\mathscr{U}_{B\left(\ell^2\right)}(\pi)^- = \mathscr{U}_{B\left(\ell^2\right)}(\rho)^-$ from π to ρ. $\square$

4. AF algebras

LEMMA 3. *Suppose $1 \in \mathscr{A} \subset \mathscr{D}$ are separable unital C*-algebras, $\mathscr{B}$ is a unital C*-algebra and $\pi, \rho \in Rep(\mathscr{D}, \mathscr{B})$, and suppose $V, W \in \mathscr{U}_{\mathscr{B}}$ such that*

1. *for every $x \in \mathscr{D}$,*
$$W^*\rho(x)W = \pi(x),$$
2. *for every $x \in \mathscr{A}$,*
$$V^*\rho(x)V = \pi(x),$$
3. *$\mathscr{U}_{\mathscr{B}\cap\rho(\mathscr{A})'}$ is connected.*

Then there is a path $t \mapsto U_t$ of unitary operators in $\mathscr{B}$ such that $U_0 = V$, $U_1 = W$, and for every $t \in [0,1]$ and every $x \in \mathscr{A}$,

$$U_t^*\rho(x)U_t = \pi(x).$$

Proof. We know that, for every $x \in \mathscr{A}$,

$$W^*\rho(x)W = V^*\rho(x)V.$$

Thus $VW^* = X \in \rho(\mathscr{A})' \cap \mathscr{B}$. Thus $W = X^*V$. Since $\mathscr{U}_{\rho(\mathscr{A})'\cap\mathscr{B}}$ is path connected, there is a path $t \mapsto X_t$ of unitary elements in $\rho(\mathscr{A})' \cap \mathscr{B}$ such that $X_0 = 1$ and $X_1 = X$. For $t \in [0,1]$ let $U_t = X_t^*V$. Then U_t is a path in $\mathscr{U}_{\mathscr{B}}$, $U_0 = V$ and $U_1 = X^*V = W$. Moreover, for each $t \in [0,1]$ and each $x \in \mathscr{A}$,

$$U_t^*\rho(x)U_t = V^*X_t\rho(x)X_t^*V = V^*\rho(x)V = \pi(x). \quad \square$$

THEOREM 4. *Suppose $\mathscr{A}_1 \subset \mathscr{A}_2 \subset \cdots \subset \mathscr{A}$ and $\mathscr{A} = [\cup_{n\in\mathbb{N}}\mathscr{A}_n]^-$ is separable. Suppose $\pi, \rho \in Rep(\mathscr{A}, \mathscr{B})$ such that, for every $n \in \mathbb{N}$,*

1. *$\rho|_{\mathscr{A}_n} \in \mathscr{U}_{\mathscr{B}}(\pi|_{\mathscr{A}_n})$,*

2. $\mathscr{U}_{\rho(\mathscr{A}_n)'\cap\mathscr{B}}$ is connected.

Then there is a strong internal path from π to ρ.

Proof. For each $n\in\mathbb{N}$, choose $U_n\in\mathscr{U}_{\mathscr{B}}$ such that, for every $a\in\mathscr{A}_n$,

$$U_n^*\rho(a)U_n=\pi(a).$$

It follows from Lemma 3 that we can define a path $t\mapsto U_t$ from $[n,n+1]$ so that for $n\leqslant t\leqslant n+1$ and $a\in\mathscr{A}_n$, we have

$$U_t^*\rho(a)U_t=\pi(a).$$

Thus the map $t\mapsto U_t$ is continuous, and, for every $a\in\cup_{n\in\mathbb{N}}\mathscr{A}_n$ we have

$$\lim_{t\to+\infty}\|U_t^*\rho(a)U_t-\pi(a)\|=0.$$

Hence, if we define $\pi_t(\cdot)=U_t^*\rho(\cdot)U_t$ for $t\in[0,\infty)$ and $\pi_\infty=\rho$, we have a strong internal path in $\mathscr{U}_{\mathscr{B}}(\pi)^-$ from π to ρ. □

THEOREM 5. *Suppose $\mathscr{A}$ is a separable unital AF C*-algebra, $\mathscr{B}$ is a C*-algebra with property HUC, and $\pi\in Rep(\mathscr{A},\mathscr{B})$. Then $\mathscr{U}_{\mathscr{B}}(\pi)^-$ is path-connected.*

Proof. We can assume that $\ker\pi=0$, since $\mathscr{A}/\ker\rho$ is a separable unital AF algebra. Since $\mathscr{A}$ is unital and AF, there is a sequence $\{\mathscr{A}_n\}$ of unital finite-dimensional C*-subalgebras

$$1\in\mathscr{A}_1\subset\mathscr{A}_2\subset\cdots$$

such that

$$\left[\bigcup_{n=1}^{\infty}\mathscr{A}_n\right]^-=\mathscr{A}.$$

Suppose $\rho\in\mathscr{U}_{\mathscr{M}}(\pi)^-$. Since each $\mathscr{A}_n$ is finite-dimensional, where approximate equivalence is the same as unitary equivalence, we have $\rho|_{\mathscr{A}_n}\in\mathscr{U}_{\mathscr{B}}(\pi|_{\mathscr{A}_n})$ for each $n\in\mathbb{N}$.

Fix $n\in\mathbb{N}$ and write $\mathscr{A}_n$ as $\mathbb{M}_{s_1}(\mathbb{C})\oplus\cdots\oplus\mathbb{M}_{s_t}(\mathbb{C})$ and, for $1\leqslant k\leqslant t$, let $\{e_{ij,k}:1\leqslant i,j\leqslant s_k\}$ be the system of matrix units for $\mathbb{M}_{s_k}(\mathbb{C})$. It is easily seen that $\rho(\mathscr{A}_n)'\cap\mathscr{B}$ is the set of all

$$\sum_{k=1}^{t}\sum_{j=1}^{s_k}\rho(e_{j1},k)\rho(e_{11,k})x\rho(e_{11,k})\rho(e_{ij,k})$$

for $x\in\mathscr{B}$. It follows that $\rho(\mathscr{A}_n)'\cap\mathscr{B}$ is isomorphic to

$$\sum_{1\leqslant k\leqslant t}^{\oplus}\rho(e_{11,k})\mathscr{B}\rho(e_{11,k}).$$

Since $\mathscr{B}$ has property HUC, we see that $\rho(\mathscr{A}_n)'\cap\mathscr{B}$ has property UC. The desired conclusion now follows from Theorem 4. □

COROLLARY 1. *If $\mathscr{A}$ is a separable unital AF C*-algebra and $\mathscr{B}$ is either an AF C*-algebra or a von Neumann algebra, then, for every $\rho \in Rep(\mathscr{A},\mathscr{B})$, $\mathscr{U}_{\mathscr{B}}(\rho)^{-}$ is path-connected.*

A separable C*-algebra is *homogeneous* if it is a finite direct sum of algebras of the form $\mathbb{M}_n(C(X))$, where X is a compact metric space. A unital C*-algebra is *subhomogeneous* if it is a unital subalgebra of a homogeneous C*-algebra. Every subhomogeneous von Neumann algebra is homogeneous; in particular, if $\mathscr{A}$ is subhomogeneous, then the second dual $\mathscr{A}^{\#\#}$ of $\mathscr{A}$ is homogeneous. A C*-algebra is *approximately subhomogeneous* (ASH) if it is a direct limit of subhomogeneous C*-algebras.

A (possibly nonseparable) C*-algebra $\mathscr{B}$ is LF if, for every finite subset $F \subset \mathscr{B}$ and every $\varepsilon > 0$ there is a finite-dimensional C*-algebra $\mathscr{D}$ of $\mathscr{B}$ such that, for every $b \in F$, $\operatorname{dist}(b,\mathscr{D}) < \varepsilon$. Every separable unital C*-subalgebra of a LF C*-algebra is contained in a separable AF subalgebra. See [2] for details.

We are interested in a more general property. We say that a unital C*-algebra $\mathscr{A}$ is *strongly LF-embeddable* if there is an LF C*-algebra $\mathscr{D}$ such that $\mathscr{A} \subset \mathscr{D} \subset \mathscr{A}^{\#\#}$. It is easily shown that an ASH algebra is strongly LF-embeddable, i.e., if $\{\mathscr{A}_\lambda\}$ is an increasingly directed family of subhomogeneous C*-algebras and $\mathscr{A} = (\cup_\lambda \mathscr{A}_\lambda)^{-\|\|}$, then $\mathscr{A} \subset \left(\cup_\lambda \mathscr{A}_\lambda^{\#\#}\right)^{-\|\|} \subset \mathscr{A}^{\#\#}$. The proof of the next theorem relies on results in [5].

THEOREM 6. *Suppose $\mathscr{A}$ is a separable strongly LF embeddable C*-algebra and $\mathscr{M}$ is a finite von Neumann algebra. Then, for every $\pi \in Rep(\mathscr{A},\mathscr{M})$, $\mathscr{U}_{\mathscr{M}}(\pi)^{-}$ is path connected.*

Proof. Suppose $\rho \in \mathscr{U}_{\mathscr{M}}(\pi)^{-}$. It follows that there are weak*-weak* continuous unital $*$-homomorphisms $\hat{\pi},\hat{\rho} : \mathscr{A}^{\#\#} \to \mathscr{M}$ such that $\hat{\pi}|_{\mathscr{A}} = \pi$ and $\hat{\rho}|_{\mathscr{A}} = \rho$. Since $\mathscr{A}$ is strongly LF embeddable, there is a separable unital AF C*-algebra $\mathscr{D}$ such that

$$\mathscr{A} \subset \mathscr{D} \subset \mathscr{A}^{\#\#}.$$

It follows from [5, Theorem 2] that $\hat{\rho}|_{\mathscr{D}} \in \mathscr{U}_{\mathscr{M}}(\hat{\pi}|_{\mathscr{D}})^{-}$. We know from Theorem 5 that $\mathscr{U}_{\mathscr{M}}(\hat{\pi}|_{\mathscr{D}})^{-}$ is path connected. Thus there is a path in $\mathscr{U}_{\mathscr{M}}(\hat{\pi}|_{\mathscr{D}})^{-}$ from $\hat{\pi}|_{\mathscr{D}}$ to $\hat{\rho}|_{\mathscr{D}}$. Restricting to $\mathscr{A}$, we obtain a path in $\mathscr{U}_{\mathscr{M}}(\pi)^{-}$ from π to ρ. $\square$

5. Abelian algebras

Suppose $\mathscr{M}$ is a von Neumann algebra and $T \in \mathscr{M}$. In [3] H. Ding and D. Hadwin defined $\mathscr{M}$-rank(T) to be the Murray von Neumann equivalence class of the orthogonal projection $\mathfrak{R}(T)$ onto the closure of the range of T. We say $\mathscr{M}$-rank$(S) \leqslant$ $\mathscr{M}$-rank(T) if and only if there is a projection $P \in \mathscr{M}$ such that $P \leqslant \mathfrak{R}(T)$ and P is Murray von Neumann equivalent to $\mathfrak{R}(S)$. They proved that if a separable unital C*-algebra is a direct limit of homogeneous algebras, and $\mathscr{M}$ acts on a separable Hilbert space, then for all $\pi,\rho \in \operatorname{Rep}(\mathscr{A},\mathscr{M})$, $\rho \in \mathscr{U}_{\mathscr{M}}(\pi)^{-}$ if and only if, for every $x \in \mathscr{A}$,

$$\mathscr{M}\text{-rank}(\pi(x)) = \mathscr{M}\text{-rank}(\rho(x)).$$

A key ingredient of the proof of this result was a sequential semicontinuity of $\mathscr{M}$-rank with respect to the *-SOT that was proved when $\mathscr{M}$ is a von Neumann algebra acting on a separable Hilbert space [3, Theorem 1]. We extend this to the general case.

LEMMA 4. *Suppose $\mathscr{M}$ is a von Neumann algebra, $A,B \in \mathscr{M}$ and, for each $n \in \mathbb{N}$, $B_n \in \mathscr{M}$ and $\mathscr{M}$-rank$(B_n) \leqslant \mathscr{M}$-rank$(A)$. If $B_n \to B$ is the $*$-SOT, then $\mathscr{M}$-rank$(B) \leqslant \mathscr{M}$-rank$(A)$.*

Proof. Let $P_n = \mathfrak{R}(B_n)$, $Q = \mathfrak{R}(A)$, and, for each $n \in \mathbb{N}$, choose a partial isometry $V_n \in \mathscr{M}$ such that $V_n^* V_n = P_n$ and $V_n V_n^* \leqslant Q$. Let

$$\mathscr{N} = W^*(\{A, B, B_1, V_1, B_2, V_2, \ldots\}).$$

Clearly, we have, for every $n \in \mathbb{N}$, that

$$\mathscr{N}\text{-rank}(B_n) \leqslant \mathscr{N}\text{-rank}(A).$$

Because $\mathscr{N}$ is countably generated, by [10, Corollary 2.4] we may write

$$\mathscr{N} = \sum_{i \in I}^{\oplus} \mathscr{N}_i$$

with each $\mathscr{N}_i$ acting on a separable Hilbert space.

Write

$$A = \sum_{i\in I}^{\oplus} A_i, \quad B = \sum_{i\in I}^{\oplus} B_i, \quad B_n = \sum_{i\in I}^{\oplus} B_{n,i}, \quad V_n = \sum_{i\in I}^{\oplus} V_{n,i}.$$

Since $\mathfrak{R}(A) = \sum_{i\in I}^{\oplus} \mathfrak{R}(A_i)$ and $\mathfrak{R}(B) = \sum_{i\in I}^{\oplus} \mathfrak{R}(B_{n,i})$, for each $i \in I$, $\mathscr{N}_i$-rank$(B_{n,i}) \leqslant \mathscr{N}_i$-rank$(A_i)$ and the limit in the $*$-*SOT* of $B_{n,i}$ is B_i. Thus, by [3, Theorem 1], for each $i \in I$,

$$\mathscr{N}_i\text{-rank}(B_i) \leqslant \mathscr{N}_i\text{-rank}(A_i).$$

Thus, for each $i \in I$, there is a partial isometry $W_i \in \mathscr{N}_i$ such that

$$W_i^* W_i = \mathfrak{R}(B_i) \text{ and } W_i W_i^* \leqslant \mathfrak{R}(A_i).$$

Then $W = \sum_{i\in I}^{\oplus} W_i$ is a partial isometry in $\mathscr{N}$ such that

$$W^* W = \mathfrak{R}(B) \text{ and } W W^* \leqslant \mathfrak{R}(A).$$

Since we also have $W \in \mathscr{M}$, we conclude $\mathscr{M}$-rank$(B) \leqslant \mathscr{M}$-rank$(A)$. $\square$

COROLLARY 2. *If $\mathscr{A}$ is a unital C*-algebra, $\mathscr{M}$ is a von Neumann algebra and $\pi \in Rep(\mathscr{A}, \mathscr{M})$ and $\rho \in \mathscr{U}_{\mathscr{M}}(\pi)^-$, then, for every $a \in \mathscr{A}$,*

$$\mathscr{M}\text{-}rank(\pi(a)) = \mathscr{M}\text{-}rank(\rho(a)).$$

Proof. Suppose $a \in \mathscr{A}$. There is a sequence $\{U_n\}$ in $\mathscr{U}_{\mathscr{M}}$ such that

$$\lim_{n\to\infty} \|U_n^* \pi(A) U_n - \rho(A)\| = \lim_{n\to\infty} \|\pi(a) - U_n \rho(a) U_n^*\| = 0.$$

Also $\mathscr{M}$-rank$(U_n^* \pi(a) U_n) = \mathscr{M}$-rank$(\pi(a))$ and $\mathscr{M}$-rank$(U_n \rho(a) U_n^*) = \mathscr{M}$-rank$(\rho(a))$ for each $n \in \mathbb{N}$. Thus, by Lemma 4,

$$\mathscr{M}\text{-rank}(\rho(a)) \leqslant \mathscr{M}\text{-rank}(\pi(a)) \text{ and } \mathscr{M}\text{-rank}(\pi(a)) \leqslant \mathscr{M}\text{-rank}(\rho(a)). \quad \square$$

REMARK 1. *Corollary 2 can also be proved without Lemma 4, but instead using Theorem 1.3(2) from [9], which states that two normal operators S,T in a von Neumann algebra are approximately equivalent if and only if, for every open subset $U \subset \mathbb{C}$, we have $\chi_U(S)$ and $\chi_U(T)$ are Murray von Neumann equivalent. Since $\mathscr{M}$-rank$(\pi(a))$ (resp., $\mathscr{M}$-rank$(\rho(a))$) is the Murray von Neumann equivalence class of $\chi_{(0,\infty)}\left(\pi(a)^* \pi(a)\right)$ (resp., $\chi_{(0,\infty)}\left(\rho(a)^* \rho(a)\right)$), Corollary is an immediate consequence.*

Suppose $\mathscr{A}$ is a unital C*-algebra and $\mathscr{M}$ is a von Neumann algebra and $\pi : \mathscr{A} \to \mathscr{M}$ is a unital $*$-homomorphism. Then there is a unique $*$-homomorphism $\hat{\pi} : \mathscr{A}^{\#\#} \to \mathscr{M}$ that is weak*-weak* continuous (see [6]).

LEMMA 5. *Suppose (X,d) is a compact metric space, $\mathscr{M}$ is a σ-finite von Neumann algebra, and $\pi, \rho : C(X) \to \mathscr{M}$, $\rho \in \mathscr{U}_{\mathscr{M}}(\pi)^-$. Then there is a sequence $\mathscr{F}_1, \mathscr{F}_2, \ldots$ of finite disjoint collections of nonempty Borel sets such that*

1. $\sum_{E\in\mathscr{F}_n} \hat{\pi}(\chi_E) = \sum_{E\in\mathscr{F}_n} \hat{\rho}(\chi_E) = 1$,
2. $\{\hat{\pi}(\chi_E) : E \in \mathscr{F}_n\} \subset sp(\{\hat{\pi}(\chi_F) : F \in \mathscr{F}_{n+1}\})$ *and* $\{\hat{\rho}(\chi_E) : E \in \mathscr{F}_n\} \subset sp(\{\hat{\rho}(\chi_F) : F \in \mathscr{F}_{n+1}\})$,
3. *For every $E \in \mathscr{F}_n$, and*
$$diam(E) < 1/n.$$
4. *For every $E \in \cup_{n\in\mathbb{N}} \mathscr{F}_n$ $\hat{\pi}(\chi_E)$ and $\hat{\rho}(\chi_E)$ are Murray von Neumann equivalent.*

Proof. Let Bor(X) be the C*-algebra with the supremum norm. We then have

$$C(X) \subset \mathrm{Bor}(X) \subset C(X)^{\#\#}$$

and $\hat{\pi}|_{\mathrm{Bor}(X)}$, $\hat{\rho}|_{\mathrm{Bor}(X)}$ are unital $*$-homomorphisms.

Let $\Sigma = \left\{U \subset X : U \text{ is open and } \hat{\pi}\left(\chi_{\bar{U}\setminus U}\right) = \hat{\rho}\left(\chi_{\bar{U}\setminus U}\right) = 0\right\}$. It is easily shown that if $U,V \in \Sigma$, then $U \setminus \bar{V}$, $U \cup V$, $U \cap V \in \Sigma$. Moreover, if $a \in X$ and $S(a,r) = \{x \in X : d(a,x) = r\}$ for all $r > 0$, it follows from the fact that $\mathscr{M}$ is σ-finite that if $E_a = \left\{r \in (0,\infty) : \hat{\pi}\left(\chi_{S(a,r)}\right) = \hat{\rho}\left(\chi_{S(a,r)}\right) = 0\right\}$, then $(0,\infty) \setminus E_a$ is countable.

We can assume that diam$(X) < 1$ and we can let $\mathscr{F}_1 = \{X\}$.

Suppose $n \in \mathbb{N}$ and $\mathscr{F}_n$ has been defined.

For each $a \in X$, there is an $r_a \in E_a \cap \left(0, \frac{1}{2(n+1)}\right)$. Since X is compact and $\{\text{ball}(a, r_a) : a \in X\}$ is an open cover with sets in Σ, there is a finite subcover $\{U_1, \ldots, U_s\}$. We let $V_1 = U_1$, and $V_k = U_k \setminus \cup_{1 \leqslant j < k} \bar{U}_j$ for $1 < k \leqslant s$. Then $\{V_1, \ldots V_s\}$ is a disjoint family of open sets in Σ with union V such that

$$\hat{\pi}(\chi_V) = \hat{\rho}(\chi_V) = 1.$$

We now let

$$\mathscr{F}_{n+1} = \left\{V_j \cap W : 1 \leqslant j \leqslant s, W \in \mathscr{F}_n, V_j \cap W \neq \varnothing\right\}.$$

If $U \subset X$ is open and nonempty, then there is a continuous $f : X \to [0,1]$ such that $f(x) = 0$ if and only if $x \in X \setminus U$. Thus the sequence $f^{1/n} \uparrow \chi_U$, which means

$$f^{1/n} \to \chi_U$$

weak* in $C(X)^{\#\#}$. Thus $\pi(f)^{1/n} \uparrow \hat{\pi}(\chi_U)$ and $\rho(f)^{1/n} \uparrow \hat{\rho}(\chi_U)$ in the weak* topology. Thus $\hat{\pi}(\chi_U)$ is the projection onto the closure of the range of $\pi(f)$ and $\hat{\rho}(\chi_U)$ is the projection onto the closure of the range of $\rho(f)$. It follows from Corollary 2 that $\hat{\pi}(\chi_U)$ and $\hat{\rho}(\chi_U)$ are Murray von Neumann equivalent. $\square$

THEOREM 7. *Suppose $\mathscr{A}$ is a separable unital commutative C*-algebra and $\mathscr{M}$ is a von Neumann algebra. If $\pi \in Rep(\mathscr{A}, \mathscr{M})$ then $\mathscr{U}_{\mathscr{B}}(\pi)^-$ is path-connected. In fact, for every $\rho \in \mathscr{U}_{\mathscr{M}}(\pi)^-$ there is a strong internal path from π to ρ.*

Proof. Suppose $\rho \in \mathscr{U}_{\mathscr{M}}(\pi)^-$. Since $\mathscr{A}$ is separable, there is a sequence $\{U_n\} \in \mathscr{U}_{\mathscr{M}}$ such that, for every $a \in \mathscr{A}$,

$$\lim_{n\to\infty} \|U_n^* \pi(a) U_n - \rho(a)\| = 0.$$

Let $\mathscr{N} = W^*(\pi(\mathscr{A}) \cup \rho(\mathscr{A}) \cup \{U_1, U_2, \ldots\})$. Then $\mathscr{N}$ is a countably generated von Neumann algebra, and $\pi, \rho : \mathscr{A} \to \mathscr{N}$. Hence we can write

$$\mathscr{N} = \sum_{i \in I}^{\oplus} \mathscr{N}_i,$$

where each $\mathscr{N}_i$ acts on a separable Hilbert space, and we can write

$$\pi = \sum_{i\in I}^{\oplus} \pi_i \text{ and } \rho = \sum_{i\in I}^{\oplus} \rho_i.$$

We also have

$$\hat{\pi} = \sum_{i\in I}^{\oplus} \hat{\pi}_i \text{ and } \hat{\rho} = \sum_{i\in I}^{\oplus} \hat{\rho}_i.$$

For each $i \in I$, we can choose a sequence $\mathscr{F}_{n,i}$ of families of nonempty open subsets as in Lemma 5. Since, for each $i \in I$ and each $n \in N$ and each $E \in \mathscr{F}_{n,i}$ we know $\hat{\pi}_i(\chi_E)$ and $\hat{\rho}_i(\chi_E)$ are Murray von Neumann equivalent in $\mathscr{N}_i$ and since

$$\sum_{E\in\mathscr{F}_n} \hat{\pi}_i(\chi_E) = \sum_{E\in\mathscr{F}_n} \hat{\rho}_i(\chi_E) = 1,$$

there is a unitary $U_{n,i} \in \mathcal{N}_i$ such that

$$U_{n,i}^* \hat{\pi}_i(\chi_E) U_{n,i} = \hat{\rho}_i(\chi_E)$$

for every $E \in \mathcal{F}_{n,i}$. For each $n \in \mathbb{N}$, let $U_n = \sum_{i\in I}^{\oplus} U_{n,i}$ for each $i \in I$, and let $\mathcal{D}_n = \sum_{i\in I}^{\oplus} \mathrm{sp}(\{\hat{\pi}_i(\chi_E) : E \in \mathcal{F}_{n,i}\})$. Since $U_n U_{n+1}^* \in \mathcal{D}_n'$, we know from the proof of Lemma 3 that the map $n \mapsto U_n$ on $\mathbb{N}$ extends to a continuous map $t \mapsto U_t = \sum_{i\in I}^{\oplus} U_{t,i}$ such that $U_0 = 1$, and such that, for every $n \in \mathbb{N}$, for every $i \in I$, every $n \leqslant t < \infty$, and every $E \in \mathcal{F}_{n,i}$

$$U_{t,i}^* \hat{\pi}_i(\chi_E) U_{t,i} = U_{n,i}^* \hat{\pi}_i(\chi_E) U_{n,i} = \hat{\rho}(\chi_E).$$

Suppose $f \in C(X)$ and $\varepsilon > 0$. Since f is uniformly continuous, there is a positive integer n_0 such that, if $x, y \in X$ and $d(x,y) < 1/n_0$, then $|f(x) - f(y)| < \varepsilon/2$.

For each $i \in I$ and all $E \in \mathcal{F}_{n_0,i}$ we choose $x_{i,n_0,E} \in E$. Since $\operatorname{diam}(E) < 1/n_0$, we then have

$$\left\| \left[f - f\left(x_{n_0,i,E}\right) \right] \chi_E \right\| < \varepsilon/2,$$

so

$$\left\| \pi_i(f) - \sum_{E \in \mathcal{F}_{n_{0,i}}} f\left(x_{n_0,i,E}\right) \hat{\pi}_i(\chi_E) \right\| \leqslant \varepsilon/2,$$

and

$$\left\| \rho_i(f) - \sum_{E \in \mathcal{F}_{n_{0,i}}} f\left(x_{n_0,i,E}\right) \hat{\rho}_i(\chi_E) \right\| \leqslant \varepsilon/2.$$

Thus, for $t \geqslant n_0$, we have

$$\begin{aligned}
\|U_t^* \pi(f) U_t - \rho(f)\| &= \sup_{i\in I} \left\| U_{t,i}^* \pi_i(f) U_{t,i} - \rho_i(f) \right\| \\
&\leqslant \sup_{i\in I} \left\| U_{t,i}^* \left[\pi_i(f) - \sum_{E \in \mathcal{F}_{n_{0,i}}} f\left(x_{n_0,i,E}\right) \hat{\pi}_i(\chi_E) \right] U_{t,i} \right\| \\
&\quad + \sup_{i\in I} \left\| \sum_{E \in \mathcal{F}_{n_{0,i}}} f\left(x_{n_0,i,E}\right) \left[U_{t,i}^* \hat{\pi}_i(\chi_E) U_{t,i} - \hat{\rho}_i(\chi_E) \right] \right\| \\
&\quad + \sup_{i\in I} \left\| \sum_{E \in \mathcal{F}_{n_{0,i}}} f\left(x_{n_0,i,E}\right) \hat{\rho}_i(\chi_E) - \rho_i(f) \right\| \\
&\leqslant \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Thus, the map $t \mapsto U_t$ is continuous on $[1,\infty)$, and, for every $f \in C(X)$,

$$\lim_{t\to\infty} \|U_t \pi(f) U_t^* - \rho(f)\| = 0. \quad \square$$

COROLLARY 3. *Suppose $\mathcal{A}$ is a separable unital homogeneous C*-algebra and $\mathcal{M}$ is a von Neumann algebra. If $\pi \in Rep(\mathcal{A}, \mathcal{M})$ then $\mathcal{U}_{\mathcal{M}}(\pi)^-$ is path-connected. In fact, for every $\rho \in \mathcal{U}_{\mathcal{M}}(\pi)^-$ there is a strong internal path from π to ρ.*

Proof. We give the proof when $\mathscr{A} = \mathbb{M}_n(C(X))$ for some compact metric space X. If $\rho \in \mathscr{U}_{\mathscr{M}}(\pi)^-$. In the obvious way we have $\mathbb{M}_n(\mathbb{C}) \subset \mathbb{M}_n(C(X))$. Since

$$\rho|_{\mathbb{M}_n(\mathbb{C})} \in \mathscr{U}_{\mathscr{M}}\left(\pi|_{\mathbb{M}_n(\mathbb{C})}\right)^-,$$

it follows from [3] that $\pi|_{\mathbb{M}_n(\mathbb{C})}$ and $\rho|_{\mathbb{M}_n(\mathbb{C})}$ are unitarily equivalent in $\mathscr{M}$. Since $\mathscr{U}_{\mathscr{M}}$ is path-connected, there is a path in $\mathscr{U}_{\mathscr{M}}(\pi)$ joining π to a representation whose restriction $\mathbb{M}_n(\mathbb{C})$ coincides with $\rho|_{\mathbb{M}_n(\mathbb{C})}$. Hence we can assume that $\pi|_{\mathbb{M}_n(\mathbb{C})} = \rho|_{\mathbb{M}_n(\mathbb{C})}$. Since $\pi(\mathbb{M}_n(\mathbb{C}))$ is an isomorphic copy of $\mathbb{M}_n(\mathbb{C})$, so there is a von Neumann algebra $\mathscr{D}$ such that $\mathscr{M} = \mathbb{M}_n(\mathscr{D})$ and the map π from $\mathbb{M}_n(\mathbb{C}) \subset \mathbb{M}_n(C(X))$ to $\mathbb{M}_n(\mathbb{C}) \subset \mathbb{M}_n(\mathscr{D})$ is the identity map. In this case there are unital $*$-homomorphisms $\sigma_\pi, \sigma_\rho : C(X) \to \mathscr{D}$ such that, for every $A = (f_{ij}) \in \mathbb{M}_n(C(X))$,

$$\pi(A) = (\sigma_\pi(f_{ij})) \text{ and } \pi(A) = \left(\sigma_\rho(f_{ij})\right).$$

It is clear that $\sigma_\rho \in \mathscr{U}_{\mathscr{D}}(\sigma_\pi)$. The rest follows from Theorem 7. $\square$

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