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REVISITING THE GRÜSS INEQUALITY

H. R. MORADI, S. FURUICHI, Z. HEYDARBEYGI AND M. SABABHEH

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Abstract. In this article, we explore the celebrated Grüss inequality, where we present a new approach using the Grüss inequality to obtain new refinements of operator means inequalities. We also present several operator Grüss-type inequalities with applications to the numerical radius and entropies.

1. Introduction

The celebrated Čebyšev's inequality [2] states that if h and g are two functions having the same monotonicity on $[a,b]$, then

$$\frac{1}{b-a}\int_a^b h(t)\,dt\,\frac{1}{b-a}\int_a^b g(t)\,dt \leqslant \frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt. \tag{1.1}$$

Reversing this inequality, Grüss inequality [11] states that, for the same f,g,

$$\frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt\,\frac{1}{b-a}\int_a^b g(t)\,dt \leqslant \frac{1}{4}(M-m)(N-n)$$

provided that there exist real numbers m, M, n, N such that

$$m \leqslant h(t) \leqslant M \quad \& \quad n \leqslant g(t) \leqslant N;\ \forall a \leqslant t \leqslant b.$$

Grüss inequality has received a considerable attention in the literature, as one can see in [1, 3, 4, 5, 15, 16].

For a complex Hilbert space $\mathscr{H}$, $\mathbb{B}(\mathscr{H})$ will denote the C^*-algebra of all bounded operators on $\mathscr{H}$. Upper case letters A,B and T will be used to denote elements in $\mathbb{B}(\mathscr{H})$. When $A \in \mathbb{B}(\mathscr{H})$, we say that A is positive if $\langle Ax,x\rangle > 0$, for all non-zero vectors $x \in \mathscr{H}$.

In this article, we are interested in obtaining operator versions of the Grüss inequality and implementing the Grüss inequality to obtain refinements of some means' inequalities, as a new approach in this direction.

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2. Scalar versions

The arithmetic-geometric mean inequality (AM-GM inequality) states that

$$\sqrt{ab} \leqslant \frac{a+b}{2}, \quad a,b > 0.$$

The term on the left is called the geometric mean, while the right term is the arithmetic mean of a and b. The weighted version of this inequality states that

$$a^{1-v}b^{v} \leqslant (1-v)a + vb, \ \forall 0 \leqslant v \leqslant 1, \ \ a,b > 0.$$

This inequality is usually referred to as Young's inequality. For simplicity, we use the notations

$$a\sharp_v b := a^{1-v}b^{v} \text{ and } a\nabla_v b = (1-v)a + vb.$$

When $v = \frac{1}{2}$, we use $\sharp$ and ∇ instead of $\sharp_{\frac{1}{2}}$ and $\nabla_{\frac{1}{2}}$, respectively. Refinements of this inequality have received a considerable attention in the literature, where many forms have been found. We refer the reader to [7, 8, 9, 13, 17] as a sample of such refinements.

In this article, we present a new approach to refine the AM-GM inequality, resulting in new forms of such refinements. This approach uses the Grüss inequality.

To better state our results, we remind the reader of the so called Heron mean, which is defined as follows:

$$F_{t,v}(a,b) = (1-t)(a\sharp_v b) + t(a\nabla_v b); \ \ 0 \leqslant t,v \leqslant 1.$$

THEOREM 2.1. *Let $a,b \geqslant 0$. If $g:[0,1] \to \mathbb{R}$ is non-decreasing on $[0,1]$ and $0 \leqslant v \leqslant 1$, then*

$$a\sharp_v b + \frac{4}{g(1)-g(0)} \int_0^1 \left(F_{t,v}(a,b) - F_{1/2,v}(a,b)\right) g(t)dt \leqslant a\nabla_v b.$$

In particular,

$$a\sharp b + \frac{4}{g(1)-g(0)} \left[\int_0^1 g(t) F_{t,1/2}(a,b)\,dt - F_{1/2,1/2}(a,b) \int_0^1 g(t)\,dt\right] \leqslant a\nabla b.$$

Proof. If $a,b > 0$, then the function $f:[0,1] \to \mathbb{R}$ defined by

$$f(t) = F_{t,v}(a,b)$$

is non-decreasing on $[0,1]$. Furthermore,

$$f(0) = a\sharp_v b \ \ \& \ \ f(1) = a\nabla_v b.$$

Assume that g is a non-decreasing function on $[0,1]$. If we write the inequality (1.1) for the functions f and g, we get

$$\int_0^1 F_{t,v}(a,b)dt \int_0^1 g(t)\,dt \leqslant \int_0^1 g(t) F_{t,v}(a,b)dt,$$

which can be written as

$$\frac{1}{2}\left(a\sharp_v b + a\nabla_v b\right)\int_0^1 g\left(t\right)dt \leqslant \int_0^1 g\left(t\right)F_{t,v}(a,b)dt.$$

This means that

$$F_{1/2,v}(a,b)\int_0^1 g(t)dt \leqslant \int_0^1 g(t)F_{t,v}(a,b)dt.$$

It follows from the Grüss inequality that

$$0 \leqslant \int_0^1 g\left(t\right)F_{t,v}(a,b)dt - F_{1/2,v}\int_0^1 g\left(t\right)dt \leqslant \frac{1}{4}\left(g\left(1\right)-g\left(0\right)\right)\left(a\nabla_v b - a\sharp_v b\right).$$

Equivalently,

$$a\sharp_v b + \frac{4}{g\left(1\right)-g\left(0\right)}\int_0^1 \left(F_{t,v}(a,b) - F_{1/2,v}(a,b)\right)g(t)dt \leqslant a\nabla_v b.$$

This proves the first inequality.

Letting $v=\frac{1}{2}$ in the first inequality yields the second inequality and completes the proof. $\square$

COROLLARY 2.1. *Let $a,b \geqslant 0$. If $g:[0,1]\to\mathbb{R}$ is non-decreasing on $[0,1]$, then*

$$\sqrt{ab} \leqslant F_{1/2,1/2}\left(a,b\right) \leqslant \frac{\int_0^1 g\left(t\right)F_{t,1/2}\left(a,b\right)dt}{\int_0^1 g\left(t\right)dt} \leqslant \frac{a+b}{2}.$$

Applying Grüss inequality, we obtain the following refinement of the AM-GM inequality, in terms of the Heinz and the logarithmic means. Recall that for two positive numbers a,b, the Heinz and logarithmic means are defined, respectively, by

$$H_t(a,b) = \frac{a\sharp_t b + b\sharp_t a}{2},\ 0\leqslant t\leqslant 1 \text{ and } L(a,b) = \frac{b-a}{\ln b - \ln a}.$$

THEOREM 2.2. *Let g be a non-decreasing function on $[1/2,1]$. Then for any $a,b>0$,*

$$a\sharp b + \frac{2}{g\left(1\right)-g\left(\frac{1}{2}\right)}\left[\int_{\frac{1}{2}}^1 g\left(t\right)H_t(a,b)dt - L(a,b)\cdot\int_{\frac{1}{2}}^1 g\left(t\right)dt\right] \leqslant a\nabla b.$$

Proof. For $x>0$, define

$$f\left(t\right) = \frac{x^t + x^{1-t}}{2},\ \ t\in\left[\frac{1}{2},1\right].$$

This function is non-decreasing on $[1/2,1]$. Furthermore,

$$f\left(\frac{1}{2}\right)=\sqrt{x} \quad \& \quad f(1)=\frac{1+x}{2}.$$

Assume that g is a non-decreasing function on $[1/2,1]$. If we write the inequality (1.1) for the functions f and g, we get

$$\int_{\frac{1}{2}}^{1}\frac{x^t+x^{1-t}}{2}dt\cdot\int_{\frac{1}{2}}^{1}g(t)\,dt\leqslant\frac{1}{2}\int_{\frac{1}{2}}^{1}g(t)\frac{x^t+x^{1-t}}{2}dt.$$

or equivalently

$$\left(\frac{x-1}{2\ln x}\right)\cdot\int_{\frac{1}{2}}^{1}g(t)\,dt\leqslant\frac{1}{2}\int_{\frac{1}{2}}^{1}g(t)\frac{x^t+x^{1-t}}{2}dt.$$

It follows from the Grüss inequality that

$$\frac{1}{2}\int_{\frac{1}{2}}^{1}g(t)\frac{x^t+x^{1-t}}{2}dt-\left(\frac{x-1}{2\ln x}\right)\cdot\int_{\frac{1}{2}}^{1}g(t)\,dt\leqslant\left(\frac{g(1)-g\left(\frac{1}{2}\right)}{4}\right)\left(\frac{1+x}{2}-\sqrt{x}\right).$$

Therefore,

$$\sqrt{x}+\frac{2}{g(1)-g\left(\frac{1}{2}\right)}\left[\int_{\frac{1}{2}}^{1}g(t)\frac{x^t+x^{1-t}}{2}dt-\left(\frac{x-1}{\ln x}\right)\cdot\int_{\frac{1}{2}}^{1}g(t)\,dt\right]\leqslant\frac{1+x}{2}.$$

Replacing x by $\frac{b}{a}$, we obtain the desired inequality. $\square$

If we take $g(t)=t$ in Theorem 3.2, we get

COROLLARY 2.2. *For any $x\geqslant 0$,*

$$\sqrt{x}+\frac{4}{\ln^2 x}\left(\frac{1}{8}(x-1)\ln x+\sqrt{x}-\frac{x+1}{2}\right)\leqslant\frac{1+x}{2}.$$

Corollary 2.2 implies the following refined arithmetic-geometric mean inequality with the logarithmic mean.

COROLLARY 2.3. *For any $a,b>0$,*

$$a\sharp b+\gamma(a,b)\cdot L(a,b)\leqslant a\nabla b,$$

where

$$\gamma(a,b):=\frac{\ln^2 b/a}{2(\ln^2 b/a+4)}\geqslant 0.$$

Proof. From Corollary 2.2, we have

$$\frac{\ln^2 x}{2(\ln^2 x+4)}\frac{x-1}{\ln x}\leqslant\frac{x+1}{2}-\sqrt{x}. \tag{2.1}$$

Replacing x by $\frac{b}{a}$ implies the desired inequality and completes the proof. □

It is interesting to compare (2.1) with the following inequality [18]:

$$\frac{\ln^2 x}{8}\sqrt{x}\leqslant\frac{x+1}{2}-\sqrt{x}. \tag{2.2}$$

However, there is no ordering between L.H.S. in (2.1) and L.H.S. in (2.2), since we have

$$\sqrt{x}\leqslant\frac{x-1}{\ln x},\quad \frac{1}{2(\ln^2 x+4)}\leqslant\frac{1}{8}$$

for $x>0$. Actually, for a small $x>0$, we have the ordering

$$\frac{1}{2(\ln^2 x+4)}\frac{x-1}{\ln x}\leqslant\frac{1}{8}\sqrt{x},$$

but we have the opposite inequality for a large $x>0$, for example $x>11288$.

3. Non-commutative versions that follow from the scalar ones

In this section, we present some non-commutative versions for the scalar inequalities we have shown earlier. The arithmetic and geometric means of two positive $A,B\in\mathbb{B}(\mathscr{H})$ are defined, respectively, by

$$A\nabla_v B=(1-v)A+vB \text{ and } A\sharp_v B=A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}},\ \ 0\leqslant v\leqslant 1.$$

Similar to the scalar case, we have the so called operator arithmetic geometric mean inequality

$$A\sharp_v B\leqslant A\nabla_v B, A,B\in\mathbb{B}(\mathscr{H}) \text{ being } \textit{positive} \text{ and } 0\leqslant v\leqslant 1.$$

Refining the operator AM-GM inequality has received a considerable interest in the literature, as one can see in [7, 9, 13, 18]. In the next result, we present a new type of such refinements, where we employ Grüss inequality. The first result, is the following operator version of Theorem 2.1, in which we still adopt the notation

$$F_{t,v}(A,B)=(1-t)(A\sharp_v B)+t(A\nabla_v B);$$

as the operator weighted Heron mean of the positive operators A,B.

THEOREM 3.1. *Let $A,B\in\mathbb{B}(\mathscr{H})$ be positive operators and let $0\leqslant v\leqslant 1$. If $g:[0,1]\to\mathbb{R}$ is non-decreasing on $[0,1]$, then*

$$A\sharp_v B+\frac{4}{g(1)-g(0)}\int_0^1\left(F_{t,v}(A,B)-F_{1/2,v}(A,B)\right)g(t)dt\leqslant A\nabla_v B.$$

Proof. Letting $a=1$ in Theorem 2.1, we have

$$b^{v}+\frac{4}{g(1)-g(0)}\int_0^1\left(\{(1-t)b^{v}+t(1-v+vb)\}-\frac{1}{2}\left(\sqrt{b}+\frac{1+b}{2}\right)\right)g(t)dt\leqslant 1-v+vb.$$

Applying a standard functional calculus argument with $b=A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, then multiplying both sides of the inequality by $A^{\frac{1}{2}}$ imply the desired inequality. □

On the other hand, an operator version of Theorem 3.2 may be stated as follows. The proof is similar to that of Theorem 3.1, hence is not included.

THEOREM 3.2. *Let $A,B\in\mathbb{B}(\mathscr{H})$ be positive and let g be a non-decreasing function on $[1/2,1]$. Then*

$$A\sharp B+\frac{2}{g(1)-g\left(\frac{1}{2}\right)}\left[\int_{\frac{1}{2}}^{1}g(t)\frac{A\sharp_t B+A\sharp_{1-t}B}{2}dt-(B-A)S_0(A|B)^{-1}A\cdot\int_{\frac{1}{2}}^{1}g(t)\,dt\right]\leqslant A\nabla B,$$

where $S_0(A|B)=A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$ is the relative operator entropy of the positive operators A,B [6].

For the next result, we define

$$Am_{t,v}B=A^{\frac{1}{2}}\left((1-v)I+v\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{t}\right)^{\frac{1}{t}}A^{\frac{1}{2}},$$

for the positive $A,B\in\mathbb{B}(\mathscr{H})$, $-1\leqslant t\leqslant 1$ and $0\leqslant v\leqslant 1$. In this result, we present a refinement of the operator AM-GM inequality, without using a functional calculus argument.

THEOREM 3.3. *Let $A,B\in\mathbb{B}(\mathscr{H})$ be two positive operators. If $g:[0,1]\to\mathbb{R}$ is non-decreasing on $[0,1]$, then*

$$A\sharp_v B+\frac{4}{g(1)-g(0)}\left[\int_0^1(Am_{t,v}B)\,g(t)\,dt-\int_0^1(Am_{t,v}B)\,dt\int_0^1 g(t)\,dt\right]\leqslant A\nabla_v B.$$

Proof. Define

$$f(t)=\langle(Am_{t,v}B)x,x\rangle,\ \text{for any } x\in\mathscr{H}.$$

Of course, f is non-decreasing on $[-1,1]$ (since $Am_{t,v}B$ is an operator mean). In particular, we have

$$f(-1)=\langle(A!_v B)x,x\rangle,\quad f(0)=\langle(A\sharp_v B)x,x\rangle,\quad f(1)=\langle(A\nabla_v B)x,x\rangle,$$

where $A!_vB=((1-v)A^{-1}+vB^{-1})^{-1}$ is the harmonic mean of A,B. From the inequality (1.1), we have

$$\int_0^1 \langle (Am_{t,v}B)x,x\rangle\,dt\int_0^1 g(t)\,dt \leqslant \int_0^1 \langle (Am_{t,v}B)x,x\rangle\,g(t)\,dt,$$

which is equivalent to

$$\left\langle \left(\int_0^1 (Am_{t,v}B)\,dt\int_0^1 g(t)\,dt\right)x,x\right\rangle \leqslant \left\langle \left(\int_0^1 (Am_{t,v}B)\,g(t)\,dt\right)x,x\right\rangle.$$

Now, Grüss inequality implies

$$\begin{aligned}&\left\langle \left[\int_0^1 (Am_{t,v}B)\,g(t)\,dt-\int_0^1 (Am_{t,v}B)\,dt\int_0^1 g(t)\,dt\right]x,x\right\rangle\\ &\leqslant \left\langle \left[\left(\frac{g(1)-g(0)}{4}\right)(A\nabla_vB-A\sharp_vB)\right]x,x\right\rangle,\end{aligned}$$

for any vector $x\in\mathscr{H}$. Therefore we obtain

$$A\sharp_vB+\frac{4}{g(1)-g(0)}\left[\int_0^1 (Am_{t,v}B)\,g(t)\,dt-\int_0^1 (Am_{t,v}B)\,dt\int_0^1 g(t)\,dt\right]\leqslant A\nabla_vB.$$

Therefore the desire inequality is obtained. $\square$

On the other hand, a refinement of the operator geometric-harmonic mean inequality can be stated as follows. The proof is similar to the above arguments, and hence we omit it.

THEOREM 3.4. *Let $A,B\in\mathbb{B}(\mathscr{H})$ be two positive operators. If $g:[-1,0]\to\mathbb{R}$ is non-decreasing on $[-1,0]$, then*

$$A!_vB+\frac{4}{g(0)-g(-1)}\left[\int_{-1}^0 (Am_{t,v}B)\,g(t)\,dt-\int_{-1}^0 (Am_{t,v}B)\,dt\int_{-1}^0 g(t)\,dt\right]\leqslant A\sharp_vB$$

We conclude this section by presenting the following application towards relative operator entropies.

THEOREM 3.5. *Let $A,B\in\mathbb{B}(\mathscr{H})$ be positive and $0<s<1$. Then*

$$S_0(A|B)+2\int_0^1 (2t-1)S_{st}(A|B)dt\leqslant S_s(A|B),$$

where $S_p(A|B):=A^{1/2}\ln_p\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$ is Tsallis relative operator entropy [10] and $S_0(A|B)=\lim\limits_{p\to 0}S_p(A|B)=A^{1/2}\log\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$ is relative operator entropy.

Proof. Define

$$f(t) = \frac{x^{ts}-1}{ts}, \;\; x>0, \;\; 0 \leqslant s \leqslant 1, \;\; t \in [0,1].$$

Then

$$f(0) = \log x \text{ and } f(1) = \frac{x^s-1}{s}.$$

Now, from the Grüss inequality

$$\int_0^1 \frac{x^{ts}-1}{ts} g(t)\,dt - \int_0^1 \frac{x^{ts}-1}{ts} dt \int_0^1 g(t)\,dt \leqslant \left(\frac{g(1)-g(0)}{4}\right)\left(\frac{x^s-1}{s} - \log x\right).$$

Namely,

$$\log x + \frac{4}{g(1)-g(0)}\left[\int_0^1 \frac{x^{ts}-1}{ts} g(t)\,dt - \int_0^1 \frac{x^{ts}-1}{ts} dt \int_0^1 g(t)\,dt\right] \leqslant \frac{x^s-1}{s}.$$

If we set $g(t) := 2t$, then the above inequality is written by

$$\log x + 2\int_0^1 \frac{(2t-1)(x^{st}-1)}{st} dt \leqslant \ln_s x,$$

where $\ln_s x := \frac{x^s-1}{s}$. Applying functional calculus argument in the above inequality implies

$$S_0(A|B) + 2\int_0^1 (2t-1) S_{st}(A|B)\,dt \leqslant S_s(A|B),$$

where $S_p(A|B) := A^{1/2} \ln_p \left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$ is Tsallis relative operator entropy and $S_0(A|B) = \lim\limits_{p \to 0} S_p(A|B) = A^{1/2} \log \left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$ is relative operator entropy. This completes the proof. $\square$

Theorem 3.5 gives a refinement of $S_0(A|B) \leqslant S_s(A|B)$ shown in [10, Proposition 3.1].

4. Sharpening Grüss inequality and covariance versions

We conclude this article by presenting some covariance inequalities that are of Grüss type, with an application to the numerical radius.

THEOREM 4.1. *Let $T \in \mathbb{B}(\mathscr{H})$ and $x \in \mathscr{H}$ be a unit vector. Then*

$$|\langle |T|\,|T^*|x,x\rangle - \langle |T|x,x\rangle \langle |T^*|x,x\rangle| \leqslant \||T|x\| \, \||T^*|x\| - |\langle Tx,x\rangle|^2.$$

Proof. Let $A,B \in \mathbb{B}(\mathscr{H})$ be two positive operators and $x \in \mathscr{H}$ be a unit vector. Then

$$\begin{aligned}
|\langle ABx,x\rangle - \langle Ax,x\rangle \langle Bx,x\rangle| &= |\langle (B-\langle Bx,x\rangle I)x, (A-\langle Ax,x\rangle I)x\rangle| \\
&\leqslant \|(A-\langle Ax,x\rangle I)x\| \, \|(B-\langle Bx,x\rangle I)x\| \\
&= \left(\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right)^{\frac{1}{2}} \left(\left\langle B^2x,x\right\rangle - \langle Bx,x\rangle^2\right)^{\frac{1}{2}} \qquad (4.1) \\
&\leqslant \sqrt{\langle A^2x,x\rangle \langle B^2x,x\rangle} - \langle Ax,x\rangle \langle Bx,x\rangle, \qquad (4.2)
\end{aligned}$$

where (4.2) follows from the inequality $\left(a^2-b^2\right)\left(c^2-d^2\right) \leqslant (ac-bd)^2$, $a,b,c,d \in \mathbb{R}$. Notice that (4.1) is meaningful, since for any self-adjoint operator $X \in \mathbb{B}(\mathscr{H})$, we have

$$\langle Xx,x\rangle^2 \leqslant \left\langle X^2x,x\right\rangle.$$

Therefore,

$$\langle Ax,x\rangle \langle Bx,x\rangle + |\langle ABx,x\rangle - \langle Ax,x\rangle \langle Bx,x\rangle| \leqslant \sqrt{\langle A^2x,x\rangle \langle B^2x,x\rangle}. \qquad (4.3)$$

Now, replacing A and B by $|T|$ and $|T^*|$, respectively, then we get

$$\langle |T|x,x\rangle \langle |T^*|x,x\rangle + |\langle |T|\,|T^*|x,x\rangle - \langle |T|x,x\rangle \langle |T^*|x,x\rangle| \leqslant \sqrt{\left\langle |T|^2x,x\right\rangle \left\langle |T^*|^2x,x\right\rangle}.$$

On the other hand, since (see e.g., [12, pp. 75–76])

$$|\langle Tx,x\rangle| \leqslant \sqrt{\langle |T|x,x\rangle \langle |T^*|x,x\rangle},$$

we infer that

$$|\langle Tx,x\rangle|^2 + |\langle |T|\,|T^*|x,x\rangle - \langle |T|x,x\rangle \langle |T^*|x,x\rangle| \leqslant \sqrt{\left\langle |T|^2x,x\right\rangle \left\langle |T^*|^2x,x\right\rangle},$$

as desired $\square$

As an application, we present the following numerical radius inequality that refines the celebrated Kittaneh result in [14]. The notation $\omega(T)$ denotes the numerical radius of the operator T.

COROLLARY 4.1. *Let $T \in \mathbb{B}(\mathscr{H})$. Then*

$$\omega^2(T) + \inf_{\substack{x\in\mathscr{H} \\ \|x\|=1}} \{|\langle |T|\,|T^*|x,x\rangle - \langle |T|x,x\rangle \langle |T^*|x,x\rangle|\} \leqslant \frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\|.$$

Proof. Applying the arithmetic-geometric mean inequality, Theorem 4.1 implies

$$|\langle Tx,x\rangle|^2 + |\langle |T|\,|T^*|x,x\rangle - \langle |T|x,x\rangle \langle |T^*|x,x\rangle| \leqslant \left\langle \left(\frac{|T|^2+|T^*|^2}{2}\right)x,x \right\rangle.$$

Consequently, by taking supremum over all unit vector $x \in \mathscr{H}$, we get

$$\omega^2(A) + \inf_{\substack{x\in\mathscr{H}\\ \|x\|=1}} \{|\langle |T|\,|T^*|x,x\rangle - \langle |T|x,x\rangle\langle |T^*|x,x\rangle|\} \leqslant \frac{1}{2}\left\||T|^2+|T^*|^2\right\|.$$

This completes the proof. $\square$

REMARK 4.1. From the inequality (4.3), we obtain the covariance inequality

$$\langle Ax,x\rangle\langle Bx,x\rangle - |\langle ABx,x\rangle| \leqslant \sqrt{\langle A^2x,x\rangle\langle B^2x,x\rangle - \langle Ax,x\rangle\langle Bx,x\rangle},$$

for the positive operators $A,B \in \mathbb{B}(\mathscr{H})$. Thus,

$$\langle Ax,x\rangle\langle Bx,x\rangle \leqslant \frac{\sqrt{\langle A^2x,x\rangle\langle B^2x,x\rangle} + |\langle ABx,x\rangle|}{2},$$

which implies

$$\begin{aligned}\langle Ax,x\rangle^2\langle Bx,x\rangle^2 &\leqslant \left(\frac{\sqrt{\langle A^2x,x\rangle\langle B^2x,x\rangle} + |\langle ABx,x\rangle|}{2}\right)^2\\ &\leqslant \frac{\langle A^2x,x\rangle\langle B^2x,x\rangle + |\langle ABx,x\rangle|^2}{2}\\ &\leqslant \langle A^2x,x\rangle\langle B^2x,x\rangle.\end{aligned}$$

This provides two refining terms of the celebrated inequality

$$\langle Ax,x\rangle^2\langle Bx,x\rangle^2 \leqslant \langle A^2x,x\rangle\langle B^2x,x\rangle.$$

We conclude this article by presenting some covariance inequalities similar to Remark 4.1, but in a more elaborated form. First, a scalar inequality.

LEMMA 4.1. *Let $a,b,c,d > 0$. Then*

$$\frac{1}{2}\frac{\left(a^2d^2-b^2c^2\right)^2}{a^2d^2+b^2c^2} + \left(a^2-b^2\right)\left(c^2-d^2\right) \leqslant (ac-bd)^2.$$

Proof. Since

$$\left(\frac{a+b}{2}\right)^2 - ab = \left(\frac{a+b}{2} - \sqrt{ab}\right)\left(\frac{a+b}{2} + \sqrt{ab}\right),$$

we have

$$\frac{\left(\frac{a+b}{2}\right)^2 - ab}{\frac{a+b}{2} + \sqrt{ab}} = \frac{a+b}{2} - \sqrt{ab}.$$

Equivalently,

$$\left(\frac{\frac{a+b}{2}+\sqrt{ab}}{2}\right)^{-1}\frac{\left(\frac{a+b}{2}\right)^2-ab}{2}=\frac{a+b}{2}-\sqrt{ab}.$$

Now, by applying the arithmetic-geometric mean inequality, we obtain

$$\frac{1}{4}\frac{(a-b)^2}{a+b}\leqslant\frac{a+b}{2}-\sqrt{ab}\leqslant\frac{1}{8}\frac{(a-b)^2}{\sqrt{ab}}.$$

Rearranging the terms, we get

$$\frac{1}{2}\frac{\left(a^2d^2-b^2c^2\right)^2}{a^2d^2+b^2c^2}+\left(a^2-b^2\right)\left(c^2-d^2\right)\leqslant(ac-bd)^2,$$

as desired. □

THEOREM 4.2. *Let $A,B\in\mathbb{B}(\mathscr{H})$ be positive operators such that $mI\leqslant A\leqslant MI$ $nI\leqslant B\leqslant NI$, for some positive scalars m,M,n,N. Then for any unit vector $x\in\mathscr{H}$,*

$$\begin{aligned}&|\langle ABx,x\rangle-\langle Ax,x\rangle\langle Bx,x\rangle|\\&\leqslant\frac{(M-m)(N-n)}{4}-\left(\sqrt{\mathscr{C}(A,x)\mathscr{C}(B,x)}+\frac{\left((M-m)^2\mathscr{C}(B,x)-(N-n)^2\mathscr{C}(A,x)\right)^2}{8(M-m)^2\mathscr{C}(B,x)+(N-n)^2\mathscr{C}(A,x)}\right),\end{aligned}$$

where

$$\mathscr{C}(A,x)=\langle(M-A)(A-m)x,x\rangle\ \text{and}\ \mathscr{C}(B,x)=\langle(N-B)(B-n)x,x\rangle.$$

Proof. It has been shown in (4.1) that

$$|\langle ABx,x\rangle-\langle Ax,x\rangle\langle Bx,x\rangle|\leqslant\left(\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right)\left(\left\langle B^2x,x\right\rangle-\langle Bx,x\rangle^2\right).$$

By the arithmetic-geometric mean inequality, we have

$$\begin{aligned}&\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\\&=(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m)-\langle(MI-A)(A-mI)x,x\rangle\\&\leqslant\left(\frac{M-m}{2}\right)^2-\langle(MI-A)(A-mI)x,x\rangle,\end{aligned}\tag{4.4}$$

and similarly

$$\left\langle B^2x,x\right\rangle-\langle Bx,x\rangle^2\leqslant\left(\frac{N-n}{2}\right)^2-\langle(NI-B)(B-nI)x,x\rangle.$$

Now, by applying Lemma 4.1, we get

$$\begin{aligned}
&|\langle ABx,x\rangle-\langle Ax,x\rangle\langle Bx,x\rangle|\\
&\leqslant\sqrt{\left(\left(\frac{M-m}{2}\right)^2-\mathscr{C}(A,x)\right)\left(\left(\frac{N-n}{2}\right)^2-\mathscr{C}(B,x)\right)}\\
&\leqslant\frac{(M-m)(N-n)}{4}-\left(\sqrt{\mathscr{C}(A,x)\mathscr{C}(B,x)}+\frac{\left((M-m)^2\mathscr{C}(B,x)-(N-n)^2\mathscr{C}(A,x)\right)^2}{8\left((M-m)^2\mathscr{C}(B,x)-(N-n)^2\mathscr{C}(A,x)\right)}\right).
\end{aligned}$$

This completes the proof of the theorem. □

REMARK 4.2. Since

$$(NI-B)(B-nI)=\left(\frac{N-n}{2}\right)^2I-\left|B-\frac{N+n}{2}I\right|^2,$$

and

$$(MI-A)(A-mI)=\left(\frac{M-m}{2}\right)^2I-\left|A-\frac{M+m}{2}I\right|^2,$$

we infer from (4.4) that

$$\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\leqslant\left\langle\left|A-\frac{M+m}{2}I\right|^2x,x\right\rangle$$

and

$$\left\langle B^2x,x\right\rangle-\langle Bx,x\rangle^2\leqslant\left\langle\left|B-\frac{N+n}{2}I\right|^2x,x\right\rangle.$$

This in turns implies that

$$|\langle ABx,x\rangle-\langle Ax,x\rangle\langle Bx,x\rangle|\leqslant\left\|A-\frac{M+m}{2}I\right\|\left\|B-\frac{N+n}{2}I\right\|.$$

Since $mI\leqslant A\leqslant MI$ and $nI\leqslant B\leqslant NI$, then

$$\left|\left\langle\left(A-\frac{M+m}{2}I\right)x,x\right\rangle\right|\leqslant\frac{M-m}{2},$$

and

$$\left|\left\langle\left(B-\frac{N+n}{2}I\right)x,x\right\rangle\right|\leqslant\frac{N-n}{2}.$$

The above relations imply

$$\left\|A-\frac{M+m}{2}I\right\|\leqslant\frac{M-m}{2},$$

and

$$\left\|B-\frac{N+n}{2}I\right\| \leqslant \frac{N-n}{2}.$$

Consequently,

$$\begin{aligned}|\langle ABx,x\rangle - \langle Ax,x\rangle \langle Bx,x\rangle| &\leqslant \left\|A-\frac{M+m}{2}I\right\| \left\|B-\frac{N+n}{2}I\right\| \\ &\leqslant \frac{(M-m)(N-n)}{4}.\end{aligned}$$

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H. R. Moradi
Department of Mathematics
Payame Noor University (PNU)
P.O. Box 19395-4697, Tehran, Iran
e-mail: hrmoradi@mshdiau.ac.ir

S. Furuichi
Department of Information Science
College of Humanities and Sciences, Nihon University
3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan
e-mail: furuichi@chs.nihon-u.ac.jp

Z. Heydarbeygi
Department of Mathematics
Payame Noor University (PNU)
P.O. Box 19395-4697, Tehran, Iran
e-mail: zheydarbeygi@yahoo.com

M. Sababheh
Department of basic sciences
Princess Sumaya University for Technology
Amman 11941, Jordan
e-mail: sababheh@psut.edu.jo